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EXIT LAWS AND EXCESSIVE FUNCTIONS FOR SUPERPROCESSES¹⁾

Пусть ξ есть марковский процесс с переходной функцией $p(r, x; t, dy)$, а X — соответствующий суперпроцесс Дюсона–Ватанабэ (т.е. суперпроцесс с характеристикой ветвления $\psi(u) = \gamma u^2$). Обозначим через $\mathcal{P}$ переходную функцию X и положим

$$p_n(r, x; t, dy) = \prod_{i=1}^n p(r, x_i; t, dy_i).$$

Любому закону выхода ℓ для p_n соответствует закон выхода L_ℓ для $\mathcal{P}$ такой, что $L_\ell^t(\mu)$ есть полином степени n по μ со старшим членом $\langle \ell^t, \mu^n \rangle$ для любого t . Любой полиномиальный закон выхода для $\mathcal{P}$ допускает единственное представление в виде $L_{\ell_1} + \dots + L_{\ell_n}$, где ℓ_k есть закон выхода для p_k .

Ключевые слова и фразы: марковский процесс, суперпроцесс Дюсона–Ватанабэ, полиномиальный закон выхода для $\mathcal{P}$.

Dedicated to Yu. V. Prokhorov on his 70th birthday

1. Action of the Dawson–Watanabe semigroup on polynomials

1.1. The Dawson–Watanabe superprocess. With every measurable space $(E, \mathcal{B})$ another measurable space $(\mathcal{M}, \mathcal{B}_{\mathcal{M}})$ is associated: $\mathcal{M}$ is the set of all finite measures on $(E, \mathcal{B})$ and $\mathcal{B}_{\mathcal{M}}$ is the σ -algebra in $\mathcal{M}$ generated by the functions $F(\mu) = \mu(B)$ with $B \in \mathcal{B}$. To every Markov transition function p in $(E, \mathcal{B})$ there corresponds a Markov transition function $\mathcal{P}$ in $(\mathcal{M}, \mathcal{B}_{\mathcal{M}})$ such that, for every $f \in p\mathcal{B}$,²⁾

$$\int_{\mathcal{M}} \mathcal{P}(r, \mu; t, d\nu) e^{-\langle f, \nu \rangle} = e^{-\langle u, \nu \rangle}, \quad (1.1)$$

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²⁾ This writing means that f is a $\mathcal{B}$ -measurable function from E to $[0, \infty]$. We write $f \in p\mathcal{B}$ if $f \in p\mathcal{B}$ and f is bounded.

where

$$u(r, x) + \gamma \int_r^t p(r, x; s, dy) u^2(s, y) ds = \int_E p(r, x; t, dy) f(y) \quad (1.2)$$

(γ is a positive constant). If p is the transition function of a right Markov process $\xi = (\xi_t, \Pi_{r,x})$, then $\mathcal{P}$ is the transition function of a right Markov process $X = (X_t, P_{r,\mu})$.³⁾ We call X the Dawson–Watanabe superprocess. The Markov semigroups of ξ and X are denoted T_t^r and $\mathcal{T}_t^r$.

Both transition functions p and $\mathcal{P}$ are defined on the same time interval I . Without any loss of generality we can assume that $I = \mathbf{R}_+$.

1.2. Martingale characterization of X . We use the following property of the Dawson–Watanabe superprocess: for all $r < u \in \mathbf{R}_+$, $\mu \in \mathcal{M}$ and $\varphi \in \mathbf{p}\mathcal{B}$,

$$\langle T_u^t \varphi, X_t \rangle = \int X_t(dx) p(t, x; u, dy) \varphi(y) \quad (1.3)$$

is a continuous square integrable martingale on (r, u) relative to $P_{r,\mu}$ with the quadratic variation $\gamma \langle (T_u^t \varphi)^2, X_t \rangle dt$ (see, e.g., [2]; in a slightly different form this was proved earlier in [5]).

1.3. Polynomials on the space $\mathcal{M}$. Denote by S^k the cone of finite positive measurable symmetric functions on the product space $(E^k, \mathcal{B}^k)$ of k replicas of $(E, \mathcal{B})$. To every $\mu \in \mathcal{M}$ there corresponds a measure

$$\mu^k(dx_1, \dots, dx_k) = \mu(dx_1) \cdots \mu(dx_k)$$

on $(E^k, \mathcal{B}^k)$. We deal with functions on $\mathcal{M}$ of the form

$$F(\mu) = \sum_1^n \langle f_k, \mu^k \rangle \quad \text{with } f_k \in S^k. \quad (1.4)$$

We call them polynomials in μ . Note that the coefficients f_k are uniquely determined by F . Indeed, $k! f_k(x_1, \dots, x_k)$ is the coefficient at $\lambda_1 \cdots \lambda_k$ in the polynomial $F(\lambda_1 \delta_{x_1} + \cdots + \lambda_k \delta_{x_k})$.

Operators $B_k: S^k \rightarrow S^{k-1}$ are defined by the formula

$$B_k f(x_1, \dots, x_k) = \frac{k\gamma}{2} \sum_{\sigma} f(x_{\sigma_1}, \dots, x_{\sigma_k}), \quad (1.5)$$

where σ runs over all monotone increasing mappings from $\{1, \dots, k\}$ onto $\{1, \dots, k-1\}$. For instance,

$$B_3 f(x_1, x_2) = \frac{3\gamma}{2} [f(x_1, x_1, x_2) + f(x_1, x_2, x_2)].$$

Formula

$$p_k(r, x; t, dy) = \prod_1^k p(r, x_i; t, dy_i) \quad (1.6)$$

defines a transition function in $(E^k, \mathcal{B}^k)$. The corresponding semigroup $T_k(r, t)$ preserves cone S^k .

1.4. Action of $\mathcal{T}_t^r$ on polynomials.

Theorem 1.1. For every $r < t$ and every $f \in S^k$

$$P_{r,\mu} \langle f, X_t^k \rangle = \sum_{j=1}^k \langle R_{jk}(r, t) f, \mu^j \rangle, \quad (1.7)$$

where

$$R_{kk}(r, t) = T_k(r, t), \quad R_{jk}(r, t) = \int_r^t R_{j,k-1}(r, s) B_k T_k(s, t) ds \quad \text{for } j < k. \quad (1.8)$$

³⁾ The definition of a right process in the nonhomogeneous setting and the construction of X can be found, e.g., in [3].

P r o o f. Formula (1.7) follows by induction in k from the equation

$$P_{r,\mu}\langle f, X_t^k \rangle = \langle T_k(r, t) f, \mu^k \rangle + P_{r,\mu} \int_r^t \langle B_k T_k(s, t) f, X_s^{k-1} \rangle ds. \quad (1.9)$$

To establish (1.9) we note that if η and $\tilde{\eta}$ are two measures on $(E^k, \mathcal{B}^k)$ and if

$$\int f d\eta = \int f d\tilde{\eta} \quad (1.10)$$

for all f of the form

$$f(x) = \varphi(x_1) \cdots \varphi(x_k) \quad \text{with } \varphi \in \mathcal{P}\mathcal{B}, \quad (1.11)$$

then (1.10) holds for all $f \in S^k$. Therefore it is sufficient to prove (1.9) for functions of the form (1.11). For these functions

$$\langle f, X_t^k \rangle = \langle \varphi, X_t \rangle^k. \quad (1.12)$$

Fix t and put $h^s = T_t^s \varphi$. Consider the martingale $Y_s = \langle h^s, X_s \rangle$ on the interval (r, t) . As we know, its quadratic variation is equal to $\gamma(\langle h^s \rangle^2, X_s) ds$ and, by the Itô formula,

$$Y_t^k = Y_r^k + \int_r^t k Y_s^{k-1} dY_s + \frac{1}{2} k(k-1) \int_r^t Y_s^{k-2} \gamma(\langle h^s \rangle^2, X_s) ds.$$

Therefore,

$$P_{r,\mu} Y_t^k = P_{r,\mu} Y_r^k + \frac{1}{2} \gamma k(k-1) P_{r,\mu} \int_r^t Y_s^{k-2} \langle \langle h^s \rangle^2, X_s \rangle ds$$

which implies (1.9). Theorem 1.1 is proved.

1.5. Properties of R_{jk} . It follows from (1.8) that, for all $j < k$,

$$R_{jk}(r, t) = \int_{r < s_j < s_{j+1} < \cdots < s_{j+k-1} < t} T_j(r, s_j) B_{j+1} T_{j+1}(s_j, s_{j+1}) \cdots B_k T_k(s_{k-1}, t) ds_j \cdots ds_{k-1}. \quad (1.13)$$

For instance,

$$R_{j,j+2}(r, t) = \int_{r < s_j < s_{j+1} < t} T_j(r, s_j) B_{j+1} T_{j+1}(s_j, s_{j+1}) B_{j+2} T_{j+2}(s_{j+1}, t) ds_j ds_{j+1}.$$

Formula (1.3) implies that, for all $r < s < t$, $i \leq j \leq k$,

$$\sum_{j=i}^k R_{ij}(r, s) R_{jk}(s, t) = R_{ik}(r, t). \quad (1.14)$$

(The semigroup property of operators $R_{ii}(r, t) = T_i(r, t)$ is a particular case of (1.14) corresponding to $i = j = k$.)

1.6. Time-homogeneous setting. In this setting, it is convenient to assume that $I = \mathbf{R}$. If the transition function p is stationary (that is, if $p(r, x; t, dy) = p_{t-r}(x, dy)$), then the Dawson-Watanabe superprocess has also a stationary transition function $\mathcal{P}_t(\mu, d\nu)$. The operator

$$G_\lambda^k = \int_0^\infty e^{-\lambda t} T_k(t) dt \quad (1.15)$$

is the resolvent of the semigroup $T_k(t) = T_k(r, t+r)$. We denote resolvent corresponding to $\mathcal{P}$ by $\mathcal{G}_\lambda$.

Theorem 1.2. *In the time-homogeneous case,*

$$\int_{\mathcal{M}} \mathcal{P}_t(\mu, d\nu) \langle f, \nu^k \rangle = \sum_{j=1}^k \langle R_{jk}(t) f, \mu^j \rangle, \quad (1.16)$$

where

$$R_{kk}(t) = T_k(t), \quad R_{jk}(t) = \int_0^t R_{j,k-1}(s) B_k T_k(t-s) ds \quad \text{for } j < k. \quad (1.17)$$

If

$$F(\nu) = \langle f, \nu^k \rangle \quad \text{with } f \in S^k, \quad (1.18)$$

then

$$\mathcal{G}_\lambda F(\mu) = \sum_{j=1}^k \langle G_\lambda^j B_{j+1} G_\lambda^{j+1} \dots B_k G_\lambda^k f, \mu^j \rangle. \quad (1.19)$$

P r o o f. Formulae (1.16)–(1.17) follow from (1.7)–(1.8). Clearly, they imply (1.18)–(1.19).

2. Exit laws

2.1. Construction of polynomial $\mathcal{P}$ -exit laws. Suppose that p is a Markov transition function in a measurable space $(E, \mathcal{B})$. Let $0 < \beta \leq \infty$ and let a function $\ell^t \in p\mathcal{B}$ not identically equal to ∞ be given for every $t \in (0, \beta)$. We say that ℓ is a p -exit law at time β and we write $\ell \in \mathcal{L}_\beta(p)$ if $T_r^t \ell^t = \ell^r$ for all $r < t < \beta$.

Theorem 2.1. To every $\ell \in \mathcal{L}_\beta(p_n)$ there corresponds a $\mathcal{P}$ -exit law

$$L_\ell^t(\mu) = \langle \ell^t, \mu^n \rangle + \sum_{j=1}^{n-1} \int_t^\beta \langle R_{j,n-1}(t, u) B_n \ell^u, \mu^j \rangle du. \quad (2.1)$$

If $L \in \mathcal{L}_\beta(\mathcal{P})$ and if, for every t , L^t has the form (1.4), then

$$L = \sum_{j=1}^n L_{\ell_j}, \quad (2.2)$$

where $\ell_j \in \mathcal{L}_\beta(p_j)$.

P r o o f. 1°. By using (1.7), we establish that

$$L^t(\mu) = \sum_{k=1}^n \langle f_k^t, \mu^k \rangle \quad \text{with } f_k^t \in S^k \quad (2.3)$$

satisfies the condition

$$\mathcal{G}_t^r L^t = L^r \quad (2.4)$$

if and only if

$$f_k^r = \sum_{j=k}^n R_{kj}(r, t) f_j^t \quad \text{for } k = 1, \dots, n. \quad (2.5)$$

2°. Function (2.1) has the form (2.3) with

$$f_n^t = \ell^t, \quad f_j^t = \int_t^\beta R_{j,n-1}(t, u) B_n \ell^u du \quad \text{for } j < n. \quad (2.6)$$

We claim that these function satisfy (2.5). This is obvious for $k = n$. If $k < n$, then, by (2.6),

$$\sum_{j=k}^n R_{kj}(r, t) f_j^t = R_{kn}(r, t) f_n^t + \sum_{j=k}^{n-1} \int_t^\beta R_{kj}(r, t) R_{j,n-1}(t, u) B_n \ell^u du. \quad (2.7)$$

By (1.8), the first term in the right-hand side of (2.7) is equal to

$$\int_r^t R_{k,n-1}(r, s) B_n T_n(s, t) f_n^t ds = \int_r^t R_{k,n-1}(r, u) B_n \ell^u du. \quad (2.8)$$

By (1.14), the second term is equal to

$$\int_t^\beta R_{k,n-1}(r, u) B_n \ell^u du. \quad (2.9)$$

3°. By 1°, if L given by (2.3) is a $\mathcal{P}$ -exit law, then the coefficients f_k^t satisfy conditions (2.5). By (1.8), this implies

$$\begin{aligned} f_n^r &= T_n(r, t) f_n^t, \\ f_k^r &= T_k(r, t) f_k^t + \sum_{j=k+1}^n \int_r^t R_{k,j-1}(r, s) B_j T_j(s, t) f_j^t ds \quad \text{for } k < n. \end{aligned} \quad (2.10)$$

We conclude from this formula that $T_k(r, t) f_k^t \leq f_k^r$ for all k and all $r < t < \beta$. Therefore $T_k(r, t) f_k^t$ is monotone decreasing in t , a limit $\ell_k^r = \lim_{t \rightarrow \beta} T_k(r, t) f_k^t$ exists and it does not exceed $T_k(r, t) f_k^t$ for all k .

By passing to the limit in formula $T_k(r, s) T_k(s, t) f_k^t = T_k(r, t) f_k^t$, we prove that $\ell_k \in \mathcal{L}_\beta(p_k)$. (The dominated convergence theorem is applicable since $T_k(s, t) f_k^t \leq f_k^s$ and $T_k(r, s) f_k^s \leq f_k^r < \infty$.)

Since $\ell_j^r \leq T_j(s, t) f_j^t$, formula (2.10) implies

$$\begin{aligned} \int_r^t R_{k,j-1}(r, s) B_j \ell_j^s ds &\leq \int_r^t R_{k,j-1}(r, s) B_j T_j(s, t) f_j^t ds \leq f_k^r \\ \text{for all } r < t < \beta \quad \text{and all } k < j &\leq n. \end{aligned} \quad (2.11)$$

Therefore,

$$\int_r^\beta R_{k,j-1} B_j \ell_j^s ds \leq f_k^r < \infty. \quad (2.12)$$

By passing to the limit in (2.10), we get

$$f_n^r = \ell_n^r, \quad f_k^r = \ell_k^r + \sum_{j=k+1}^n \int_r^\beta R_{k,j-1}(r, s) B_j \ell_j^s ds \quad \text{for } k < n. \quad (2.13)$$

By (2.3), (2.10) and (2.13),

$$L^t(\mu) = \sum_{k=1}^n \langle f_k^t, \mu^k \rangle = \sum_{k=1}^n \langle \ell_k^t, \mu^k \rangle + \sum_{1 \leq k < j \leq n} \int_t^\beta \langle R_{k,j-1}(t, s) B_j \ell_j^s, \mu^k \rangle ds \quad (2.14)$$

which implies (2.2). Theorem 2.1 is proved.

2.2. Examples. Fix $\beta < \infty$. To every function $f \in S$ there corresponds $\ell_{\beta, f}$ given by the formula

$$\ell_{\beta, f}^t(x) = \int \prod_1^n p(t, x_i; \beta, dy_i) f(y_1, \dots, y_n). \quad (2.15)$$

We say that m is a *reference measure* for p if the measure $p(r, x; t, \cdot)$ is absolutely continuous with respect to m for all $r < t$ and for all $x \in E$. By [4], the density function ρ can be chosen to satisfy the equation

$$\int \rho(r, x; s, y) m(dy) \rho(s, y; t, z) = \rho(r, x; t, z) \quad (2.16)$$

for all $r < s < t$ and for all $x, z \in E$. To every symmetric measure η on $(E^n, \mathcal{B}^n)$ there corresponds $\ell_{\beta, \eta} \in \mathcal{L}_\beta(p_n)$ defined by the formula

$$\ell_{\beta, \eta}^t(x) = \int \prod_1^n \rho(t, x_i; \beta, y_i) \eta(dy_1, \dots, dy_n). \quad (2.17)$$

We denote by $L_{\beta, f}$ and by $L_{\beta, \eta}$ the $\mathcal{P}$ -exit laws which correspond to $\ell_{\beta, f}$ and to $\ell_{\beta, \eta}$ by Theorem 2.1. Clearly, $L(\beta, \eta) = L(\beta, f)$ if

$$\eta(dy_1, \dots, dy_n) = f(y_1, \dots, y_n) m(dy_1) \cdots m(dy_n).$$

3. Excessive functions

Let $T(t)$ be the semigroup corresponding to a stationary transition function p . We say that a function $h \in p\mathcal{B}$ is p -excessive and we write $h \in \mathcal{H}(p)$ if $h(x) < \infty$ for some x , $T(t)h \leq h$ for all t and $T(t)h \rightarrow h$ as $t \rightarrow 0$. A function $h \in \mathcal{H}(p)$ is called *purely excessive* if $T(t)h \rightarrow 0$ as $t \rightarrow \infty$ and it is called *invariant* if $T(t)h = h$ for all t . We denote the set of all purely excessive functions by $\mathcal{H}_{\text{pur}}(p)$ and the set of all invariant functions by $\mathcal{H}_{\text{inv}}(p)$.

Every $h \in \mathcal{H}(p)$ has a unique representation $h = h_1 + h_2$, where $h_1 \in \mathcal{H}_{\text{inv}}(p)$ and $h_2 \in \mathcal{H}_{\text{pur}}(p)$.

Note that $\mathcal{H}_{\text{inv}}(p)$ coincides with the class of t -independent p -exit laws at ∞ .

To every $\ell \in \mathcal{L}_0(p)$ there corresponds a function

$$h = \int_{-\infty}^0 \ell^t dt \quad (3.1)$$

which is purely excessive unless it is equal to ∞ identically. The set of all purely excessive functions h of the form (3.1) will be denoted by $\mathcal{H}_{\text{pur}}^0(p)$.

If there exists a reference measure for p , then, by [1] (Section 7.6), $\mathcal{H}_{\text{pur}}(p) = \mathcal{H}_{\text{pur}}^0(p)$.

Theorem 3.1. *To every $h \in \mathcal{H}_{\text{pur}}^0(p_n)$ there corresponds $H_h \in \mathcal{H}_{\text{pur}}(\mathcal{P})$ defined by the formula*

$$H_h(\mu) = \langle h, \mu^n \rangle + \sum_{j=1}^{n-1} \langle G^j B_{j+1} G^{j+1} \dots B_n h, \mu^j \rangle, \quad (3.2)$$

where G^j is given by (1.15) with $\lambda = 0$.

P r o o f. Let ℓ correspond to h by (3.1) and let $L \in \mathcal{L}_0(\mathcal{P})$ correspond to ℓ by Theorem 2.1. Then

$$H_h(\mu) = \int_{-\infty}^0 L^t(\mu) dt \quad (3.3)$$

belongs to $\mathcal{H}_{\text{pur}}(p)$. Formula (3.2) follows from (3.3), (2.1), (1.9), (1.17).

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