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К. Debicki, Some properties of generalized Pickands constants, *Теория вероятн. и ее примен.*, 2005, том 50, выпуск 2, 396–404

DOI: <https://doi.org/10.4213/tvp118>

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Поступила в редакцию
5.VI.2004

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SOME PROPERTIES OF GENERALIZED PICKANDS CONSTANTS¹⁾

Изучаются свойства обобщенных констант Пикандса $\mathcal{H}_\eta$, которые возникают в теории экстремальных значений гауссовских процессов и определяются следующим образом:

$$\mathcal{H}_\eta = \lim_{T \rightarrow \infty} \frac{\mathcal{H}_\eta(T)}{T},$$

где $\mathcal{H}_\eta(T) = \mathbf{E} \exp(\max_{t \in [0, T]} (\sqrt{2} \eta(t) - D\eta(t)))$ и $\eta(t)$ — центрированный гауссовский процесс со стационарными приращениями.

Даны оценки скорости сходимости $\mathcal{H}_\eta(T)/T$ к $\mathcal{H}_\eta$ и доказано, что если $\eta_{(n)}(t)$ слабо сходится в $C([0, \infty))$ к $\eta(t)$, то при некоторых неограниченных условиях $\lim_{n \rightarrow \infty} \mathcal{H}_{\eta_{(n)}} = \mathcal{H}_\eta$.

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¹⁾ This work was partially supported by KBN grant 5 P03A 021 20 (2001–2003) and the CWI project P1201.

В качестве применения доказывается, что функция $\Upsilon(\alpha) = \mathcal{H}_{B_{\alpha/2}}$ непрерывна на $(0, 2]$, где $B_{\alpha/2}(t)$ — дробное броуновское движение с параметром Хэрста $\alpha/2$.

Ключевые слова и фразы: точная асимптотика, экстремали, дробное броуновское движение, гауссовский процесс, обобщенные константы Пикандса.

1. Introduction. Pickands constants $\mathcal{H}_{B_{\alpha/2}}$ play an important role in the asymptotic of supremum of Gaussian stochastic processes (see [5], [6]). Recall that $\mathcal{H}_{B_{\alpha/2}}$ is defined by the following limit:

$$\mathcal{H}_{B_{\alpha/2}} = \lim_{T \rightarrow \infty} \frac{\mathcal{H}_{B_{\alpha/2}}(T)}{T}, \quad (1.1)$$

where

$$\mathcal{H}_{B_{\alpha/2}}(T) = \mathbf{E} \exp \left(\max_{t \in [0, T]} \sqrt{2} B_{\alpha/2}(t) - \mathbf{D} B_{\alpha/2}(t) \right)$$

and $B_{\alpha/2}(t)$ is a fractional Brownian motion with Hurst parameter $\alpha/2$, that is, a centered Gaussian process with stationary increments, continuous sample paths, $B_{\alpha/2}(0) = 0$ and variance function $\mathbf{D} B_{\alpha/2}(t) = t^\alpha$ ($\alpha \in (0, 2]$).

It turns out that the notion of $\mathcal{H}_{B_{\alpha/2}}$ has a natural extension (see [2]). Let $\eta(t)$ be a centered Gaussian process with stationary increments, a.s. continuous sample paths, $\eta(0) = 0$ and such that the variance function $\mathbf{D}\eta(t) = \sigma_\eta^2(t)$ satisfies the following conditions.

C1. $\sigma_\eta^2(t) \in C^1([0, \infty))$ is strictly increasing and there exists $\varepsilon > 0$ such that

$$\limsup_{t \rightarrow \infty} \frac{t \dot{\sigma}_\eta^2(t)}{\sigma_\eta^2(t)} \leq \varepsilon. \quad (1.2)$$

C2. $\sigma_\eta^2(t)$ is regularly varying at 0 with index $\alpha_0 \in (0, 2]$ and $\sigma_\eta^2(t)$ is regularly varying at ∞ with index $\alpha_\infty \in (0, 2)$.

In the paper we use the notation $\dot{\sigma}^2(t)$ or $\ddot{\sigma}^2(t)$ for the derivatives of $\sigma^2(t)$. Moreover, in order to short the notation, we will write that $\eta(t) \in \mathbf{C1-C2}$ provided that $\eta(t)$ is a centered Gaussian stochastic process with stationary increments and variance function that satisfies **C1-C2**.

By the *generalized Pickands constant* $\mathcal{H}_\eta$ we mean

$$\mathcal{H}_\eta = \lim_{T \rightarrow \infty} \frac{\mathcal{H}_\eta(T)}{T}, \quad (1.3)$$

where

$$\mathcal{H}_\eta(T) = \mathbf{E} \exp \left(\max_{t \in [0, T]} \sqrt{2} \eta(t) - \mathbf{D}\eta(t) \right)$$

and $\eta(t) \in \mathbf{C1-C2}$. In [2] it was proved that if $\eta(t) \in \mathbf{C1-C2}$, then $\mathcal{H}_\eta$ is well defined, finite, and positive. Meanwhile it turned out that generalized Pickands constants appear in the exact asymptotics of supremum of some classes of Gaussian processes (see [2], [3]).

From the theoretical point of view and in spite of recent attempts to find numerical approximations of $\mathcal{H}_\eta$, there is a great need of analyzing properties of $\mathcal{H}_\eta$. In the case of classical Pickands constants the effort to compute $\mathcal{H}_{B_{\alpha/2}}$ was made by many authors, but only few fragmentary results were obtained. Namely the exact value of $\mathcal{H}_{B_{\alpha/2}}$ is known only if $\alpha = 1$ and $\alpha = 2$ (see, for example, [6] or [1]). Recently Shao [7] and Dębicki, Michna, and Rolski [3] obtained some bounds of $\mathcal{H}_{B_{\alpha/2}}$. Upper and lower bounds for generalized Pickands constants were given in [2] and [3].

In this paper we focus on continuity and limit properties of $\mathcal{H}_\eta$. After introducing basic notation (Section 2), we give some estimates of the rate of convergence of $\mathcal{H}_\eta(T)/T$ to $\mathcal{H}_\eta$ as $T \rightarrow \infty$ (Section 3). In particular, using Theorem 3.1, we prove that the asymptotic rate of convergence is almost linear (Corollary 3.1).

Theorem 3.1 enables us to analyze some properties of a sequence of generalized Pickands constants $\mathcal{H}_{\eta(n)}$, where $\eta(n)(t) \in \mathbf{C1-C2}$ and weakly converges in $C([0, \infty))$

to a Gaussian process $\eta(t) \in \mathbf{C1-C2}$. Namely, under some mild conditions on the variance functions of $\eta_{(n)}(t)$ we prove that $\lim_{n \rightarrow \infty} \mathcal{H}_{\eta_{(n)}} = \mathcal{H}_\eta$ (Theorem 4.1).

Theorem 4.1 opens a way to the analysis of properties of the function

$$\Upsilon(\alpha) = \mathcal{H}_{B_{\alpha/2}}, \quad \alpha \in (0, 2].$$

In particular, in Corollary 4.1 we prove that $\Upsilon(\alpha)$ is continuous as the function of $\alpha \in (0, 2]$. This contradicts numerical simulations carried out in [4]. Moreover, in Corollary 4.2 we obtain some limit properties of $\mathcal{H}_\eta$ for $\eta(t) = \int_0^t Z(s) ds$, where $Z(s)$ is a centered stationary Gaussian process with continuous covariance function $R(t)$ and $\eta(t) \in \mathbf{C1-C2}$.

2. Notation and preliminary results. We write $\{\eta_u(t), u > 0\}$ for the family of centered Gaussian processes $\{\eta_u(t): t \geq 0\}$ ($u > 0$), where $\eta_u(t) = \eta(u+t)$ and $\eta(t) \in \mathbf{C1-C2}$. This family will play the central role in the technique of proofs presented in this article.

By the attached bar we always denote the standardized process, that is, $\bar{\eta}(t) = \eta(t)/\sigma_\eta(t)$.

Before we state the main results, we need to point out some properties of family $\{\eta_u(t), u > 0\}$.

Proposition 2.1. *If $\eta(t) \in \mathbf{C1-C2}$, then*

$$(i) \quad \sup_{s, t \in J(u)} \left| \frac{1 - \text{cov}(\bar{\eta}_u(t), \bar{\eta}_u(s))}{\sigma_\eta^2(|t-s|)/(2\sigma_\eta^2(u))} - 1 \right| \rightarrow 0$$

as $u \rightarrow \infty$, where $J(u) = [-\Delta(u), \Delta(u)]$ and $\Delta(u)$ is such that $\lim_{u \rightarrow \infty} \Delta(u)/u = 0$;

(ii) *there exists $\Lambda > 1$ such that*

$$\sup_{t > 0} \frac{\sigma_\eta^2(t)}{\sigma_\eta^2(\Lambda t)} \leq \frac{1}{2}. \quad (2.1)$$

P r o o f. (i) is a special case of Lemma 2.1 in [2]. (ii) is a consequence of the assumption that $\sigma_\eta^2(t)$ is regularly varying at 0 and at ∞ and the fact that $\sigma_\eta^2(t)$ is strictly increasing.

The following constants will play an important role in further analysis. Let

$$\mathcal{H}_{\eta, \eta}(T) = \mathbf{E} \exp \left(\max_{(t_1, t_2) \in [0, T]^2} \left(\eta_{(1)}(t_1) + \eta_{(2)}(t_2) - \frac{1}{2} \left(\sigma_{\eta_{(1)}}^2(t_1) + \sigma_{\eta_{(2)}}^2(t_2) \right) \right) \right), \quad (2.2)$$

where $\eta_{(1)}(t)$, $\eta_{(2)}(t)$ are independent copies of a centered Gaussian stochastic process $\eta(t)$ such that $\eta(t) \in \mathbf{C1-C2}$. Analogously we introduce $\eta_{(1);u}(t)$, $\eta_{(2);u}(t)$ as independent copies of $\eta_u(t)$.

The following proposition is a consequence of Theorem 2.1 in [2].

Proposition 2.2. *Let $n(u)$ be such that $\lim_{u \rightarrow \infty} \sqrt{2} \sigma_\eta(u)/n(u) = 1$. Then*

$$\mathbf{P} \left\{ \sup_{(t_1, t_2) \in [0, T]^2} \frac{1}{\sqrt{2}} (\bar{\eta}_{(1);u}(t_1) + \bar{\eta}_{(2);u}(t_2)) > n(u) \right\} = \mathcal{H}_{\eta, \eta}(T) \Psi(n(u))(1 + o(1))$$

as $u \rightarrow \infty$.

Following the same argumentation as presented in the proof of Corollary D.1 in [6] (see also [2]) we recall some basic properties of $\mathcal{H}_\eta(\cdot)$ and $\mathcal{H}_{\eta, \eta}(\cdot)$.

Proposition 2.3. *If $\eta(t) \in \mathbf{C1-C2}$, then*

- (i) $\mathcal{H}_\eta(T) \leq T \mathcal{H}_\eta(1)$ for $T \in \mathbf{N}$;
- (ii) $\mathcal{H}_{\eta, \eta}(T) \leq T^2 \mathcal{H}_{\eta, \eta}(1)$ for $T \in \mathbf{N}$;
- (iii) $\mathcal{H}_\eta(\cdot)$ is subadditive.

3. Estimates of rate of convergence to $\mathcal{H}_\eta$. In this section we find some estimates of the rate of convergence of $\mathcal{H}_\eta(T)/T$ to $\mathcal{H}_\eta$ as $T \rightarrow \infty$. The main theorem of this section is the following upper bound.

Theorem 3.1. *Let $\eta(t) \in \mathbf{C1-C2}$, and let $f: \mathbf{R}^+ \rightarrow \mathbf{R}^+$ be such that $f(x) < x$ for each $x \geq 1$. If $\Lambda > 0$ is such that $\sup_{t>0} \sigma_\eta^2(t)/\sigma_\eta^2(\Lambda t) \leq \frac{1}{2}$, then for each $T \in \mathbf{N}$*

$$\left| \mathcal{H}_\eta - \frac{\mathcal{H}_\eta(T)}{T} \right| \leq 2 \frac{\mathcal{H}_\eta(f(T))}{T} + 2 \frac{\mathcal{H}_{\eta,\eta}(\Lambda^5 T)}{T} \exp\left(-\frac{\sigma_\eta^2(f(T))}{8}\right) + 2 \frac{\mathcal{H}_{\eta,\eta}(\Lambda^5 T)}{T} \sum_{i=1}^{\infty} \exp\left(-\frac{\sigma_\eta^2(iT)}{8}\right). \quad (3.1)$$

The complete proof of Theorem 3.1 is presented in Section 5.

As a consequence of Theorem 3.1 we obtain the following corollary.

Corollary 3.1. *If $\eta(t) \in \mathbf{C1-C2}$ and $\alpha < 1$, then*

$$\lim_{T \rightarrow \infty} \left| \frac{\mathcal{H}_\eta(T)}{T} - \mathcal{H}_\eta \right| T^\alpha = 0. \quad (3.2)$$

P r o o f. Since $\eta(t) \in \mathbf{C1-C2}$, there exists $\Lambda > 0$ such that (2.1) is satisfied. Let $\alpha < 1$. We take $\varepsilon > 0$ such that $\alpha + \varepsilon < 1$ and $f(x) = x^{1-\alpha-\varepsilon}$. Moreover, let T_0 be such that $\min(\Lambda^5 T, f(T)) > 1$ for $T > T_0$. From Proposition 2.3 we have

$$\mathcal{H}_{\eta,\eta}(\Lambda^5 T) \leq \mathcal{H}_{\eta,\eta}(1)(\Lambda^5 T + 1)^2, \quad \mathcal{H}_\eta(f(T)) \leq \mathcal{H}_\eta(1)(f(T) + 1)$$

for $T > T_0$. Thus, using Theorem 3.1, for $T > T_0$

$$\left| \mathcal{H}_\eta - \frac{\mathcal{H}_\eta(T)}{T} \right| \leq 2 \mathcal{H}_\eta(1) \frac{T^{1-\alpha-\varepsilon} + 1}{T} + 2 \mathcal{H}_{\eta,\eta}(1) \frac{(\Lambda^5 T + 1)^2}{T} \exp\left(-\frac{\sigma_\eta^2(T^{1-\alpha-\varepsilon})}{8}\right) + 2 \mathcal{H}_{\eta,\eta}(1) \frac{(\Lambda^5 T + 1)^2}{T} \sum_{i=1}^{\infty} \exp\left(-\frac{\sigma_\eta^2(iT)}{8}\right).$$

Since

$$\frac{(\Lambda^5 T + 1)^2}{T} \exp\left(-\frac{\sigma_\eta^2(T^{1-\alpha-\varepsilon})}{8}\right) + \frac{(\Lambda^5 T + 1)^2}{T} \sum_{i=1}^{\infty} \exp\left(-\frac{\sigma_\eta^2(iT)}{8}\right) = o(T^{-\alpha-\varepsilon})$$

as $T \rightarrow \infty$, we have $|\mathcal{H}_\eta - \mathcal{H}_\eta(T)/T| T^\alpha \rightarrow 0$ as $T \rightarrow \infty$. This completes the proof.

4. A convergence theorem. In this section we present a theorem that enables us to analyze properties of a sequence of generalized Pickands constants for weakly converging Gaussian processes.

Let $\zeta(t) \in \mathbf{C1-C2}$, and let $\{\zeta_{(n)}(t)\}_{n=1}^\infty$ be a sequence of centered Gaussian processes with stationary increments such that $\zeta_{(n)}(t) \in \mathbf{C1-C2}$ for each $n \in \mathbf{N}$ and the following conditions are satisfied.

W1. The sequence $\{\zeta_{(n)}(t)\}_{n=1}^\infty$ weakly converges in $C([0, \infty))$ to $\zeta(t)$.

W2. There exists $\Lambda > 0$ such that $\sup_{t>0} \sigma_{\zeta_{(n)}}^2(t)/\sigma_{\zeta_{(n)}}^2(\Lambda t) \leq \frac{1}{2}$ for each $n \in \mathbf{N}$.

In order to short the notation, if $\zeta(t)$ and $\{\zeta_{(n)}(t)\}_{n=1}^\infty$ satisfy **W1-W2**, we will write $(\{\zeta_{(n)}\}, \zeta) \in \mathbf{W1-W2}$.

Let

$$\phi_{\zeta,T}(x) = \mathbf{E} \exp\left(x \sup_{t \in [0,T]} (\sqrt{2} \zeta(t) - \sigma_\zeta^2(t))\right)$$

be the moment generating function of random variable $\sup_{t \in [0,T]} (\sqrt{2} \zeta(t) - \sigma_\zeta^2(t))$. Note that $\mathcal{H}_\zeta(T) = \phi_{\zeta,T}(1)$.

Theorem 4.1. *If $(\{\zeta_{(n)}\}, \zeta) \in \mathbf{W1-W2}$, then*

$$\lim_{n \rightarrow \infty} \mathcal{H}_{\zeta_{(n)}} = \mathcal{H}_\zeta.$$

P r o o f. Since the functional $\sup_{t \in [0, T]} g(t)$ is continuous in uniform metric, we have from **W1** that

$$\lim_{n \rightarrow \infty} \phi_{\zeta(n), T}(1) = \phi_{\zeta, T}(1)$$

for each $T > 0$ (for details see also the proof of Lemma D.1 in [6]) and hence

$$\lim_{n \rightarrow \infty} \mathcal{H}_{\zeta(n)}(T) = \mathcal{H}_{\zeta}(T) \quad (4.1)$$

for each $T > 0$. Thus in order to complete the proof it suffices to show that $\mathcal{H}_{\zeta(n)}(T)/T \rightarrow \mathcal{H}_{\zeta(n)}$, as $T \rightarrow \infty$, uniformly with respect to n .

Due to **W2** there exists a universal constant $\Lambda > 1$ such that (2.1) is satisfied for $\zeta(n)(t)$, $n \geq 1$. Hence taking $f(x) = \sqrt{x}$ in Theorem 3.1 we have

$$\begin{aligned} \left| \mathcal{H}_{\zeta(n)} - \frac{\mathcal{H}_{\zeta(n)}(T)}{T} \right| &\leq 2 \frac{\mathcal{H}_{\zeta(n)}(\sqrt{T})}{T} + 2 \frac{\mathcal{H}_{\zeta(n), \zeta(n)}(\Lambda^5 T)}{T} \exp\left(-\frac{\sigma_{\zeta(n)}^2(\sqrt{T})}{8}\right) \\ &\quad + 2 \frac{\mathcal{H}_{\zeta(n), \zeta(n)}(\Lambda^5 T)}{T} \sum_{i=1}^{\infty} \exp\left(-\frac{\sigma_{\zeta(n)}^2(iT)}{8}\right) \end{aligned} \quad (4.2)$$

for $T \in \mathbb{N}$. From Proposition 2.3 we get

$$\mathcal{H}_{\zeta(n)}(\sqrt{T}) \leq (\sqrt{T} + 1) \mathcal{H}_{\zeta(n)}(1), \quad \mathcal{H}_{\zeta(n), \zeta(n)}(\Lambda^5 T) \leq (\Lambda^5 T + 1)^2 \mathcal{H}_{\zeta(n), \zeta(n)}(1).$$

Combining it with the fact that $\lim_{n \rightarrow \infty} \mathcal{H}_{\zeta(n)}(1) = \mathcal{H}_{\zeta}(1)$ and $\lim_{n \rightarrow \infty} \mathcal{H}_{\zeta(n), \zeta(n)}(1) = \mathcal{H}_{\zeta, \zeta}(1)$ we obtain

$$\mathcal{H}_{\zeta(n)}(\sqrt{T}) \leq 2(\sqrt{T} + 1) \mathcal{H}_{\zeta}(1), \quad (4.3)$$

$$\mathcal{H}_{\zeta(n), \zeta(n)}(\Lambda^5 T) \leq 2(\Lambda^5 T + 1)^2 \mathcal{H}_{\zeta, \zeta}(1) \quad (4.4)$$

for sufficiently large n . Moreover, by **W2** we have

$$\sigma_{\zeta(n)}^2(\sqrt{T}) \geq \frac{1}{2} T^{\log_{\Lambda}(2)/2} \sigma_{\zeta(n)}^2(1) \quad \text{and} \quad \sigma_{\zeta(n)}^2(iT) \geq \frac{1}{2} (iT)^{\log_{\Lambda}(2)} \sigma_{\zeta(n)}^2(1).$$

Now, using that $\lim_{n \rightarrow \infty} \sigma_{\zeta(n)}^2(1) = \sigma_{\zeta}^2(1)$, we obtain that

$$\sigma_{\zeta(n)}^2(\sqrt{T}) \geq T^{\frac{\log_{\Lambda}(2)}{2}} \sigma_{\zeta}^2(1) \quad \text{and} \quad \sigma_{\zeta(n)}^2(iT) \geq (iT)^{\log_{\Lambda}(2)} \sigma_{\zeta}^2(1)$$

for sufficiently large n , which implies that

$$\exp\left(-\frac{\sigma_{\zeta(n)}^2(\sqrt{T})}{8}\right) \leq \exp\left(-T^{\log_{\Lambda}(2)/2} \frac{\sigma_{\zeta}^2(1)}{8}\right), \quad (4.5)$$

$$\sum_{i=1}^{\infty} \exp\left(-\frac{\sigma_{\zeta(n)}^2(iT)}{8}\right) \leq \sum_{i=1}^{\infty} \exp\left(-(iT)^{\log_{\Lambda}(2)} \frac{\sigma_{\zeta}^2(1)}{8}\right) \quad (4.6)$$

for sufficiently large n .

Combining (4.2) with (4.3)–(4.6) we obtain

$$\begin{aligned} \left| \mathcal{H}_{\zeta(n)} - \frac{\mathcal{H}_{\zeta(n)}(T)}{T} \right| &\leq 4 \frac{(\sqrt{T} + 1) \mathcal{H}_{\zeta}(1)}{T} \\ &\quad + 4 \frac{(\Lambda^5 T + 1)^2 \mathcal{H}_{\zeta, \zeta}(1)}{T} \exp\left(-T^{\log_{\Lambda}(2)/2} \frac{\sigma_{\zeta}^2(1)}{8}\right) \\ &\quad + 4 \frac{(\Lambda^5 T + 1)^2 \mathcal{H}_{\zeta, \zeta}(1)}{T} \sum_{i=1}^{\infty} \exp\left(-(iT)^{\log_{\Lambda}(2)} \frac{\sigma_{\zeta}^2(1)}{8}\right) \end{aligned}$$

for $T \in \mathbb{N}$ and sufficiently large n . Hence we proved that $\mathcal{H}_{\zeta(n)}(T)/T \rightarrow \mathcal{H}_{\zeta(n)}$, as $T \rightarrow \infty$, uniformly with respect to n . Thus

$$\lim_{n \rightarrow \infty} \mathcal{H}_{\zeta(n)} = \lim_{n \rightarrow \infty} \lim_{T \rightarrow \infty} \frac{\mathcal{H}_{\zeta(n)}(T)}{T} = \lim_{T \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{\mathcal{H}_{\zeta(n)}(T)}{T} = \lim_{T \rightarrow \infty} \frac{\mathcal{H}_{\zeta}(T)}{T} = \mathcal{H}_{\zeta}.$$

This completes the proof.

Theorem 4.1 enables us to obtain the following result, which contradicts simulations carried out in [4].

Corollary 4.1. *Function $\Upsilon(\alpha) = \mathcal{H}_{B_{\alpha/2}}$ is continuous on $(0, 2]$.*

P r o o f. It suffices to prove that if $\alpha_n \rightarrow \alpha$, then $\lim_{n \rightarrow \infty} \mathcal{H}_{B_{\alpha_n/2}} = \mathcal{H}_{B_{\alpha/2}}$, where $\alpha_n, \alpha \in (0, 2]$.

Since for sufficiently large n we have $(\{B_{\alpha_n/2}\}, B_{\alpha/2}) \in \mathbf{W1-W2}$ with $\Lambda = 2^{2/\alpha}$, then applying Theorem 4.1 we obtain the thesis.

Corollary 4.2. *If $\eta(t) = \int_0^t Z(s) ds$, where $Z(s)$ is a stationary centered Gaussian process with a continuous covariance function $\text{cov}(Z(s+t), Z(s)) = R(t)$ and $\eta(t) \in \mathbf{C1-C2}$, then*

$$\lim_{c \rightarrow \infty} \frac{\mathcal{H}_{c\eta}}{c} = \sqrt{\frac{R(0)}{\pi}}.$$

P r o o f. Let $\eta_{(c)}(t) = \int_0^t Z(s/c) ds$ and note that for each $c > 0$ we have $c\eta(t) = \eta_{(c)}(ct)$. Thus

$$\begin{aligned} \mathcal{H}_{c\eta} &= \lim_{T \rightarrow \infty} \frac{1}{T} \mathbf{E} \exp \left(\max_{t \in [0, T]} \sqrt{2} c\eta(t) - \sigma_{c\eta}^2(t) \right) \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \mathbf{E} \exp \left(\max_{t \in [0, T]} \sqrt{2} \eta_{(c)}(ct) - \sigma_{\eta_{(c)}}^2(ct) \right) \\ &= c \lim_{T \rightarrow \infty} \frac{1}{cT} \mathbf{E} \exp \left(\max_{t \in [0, cT]} \sqrt{2} \eta_{(c)}(t) - \sigma_{\eta_{(c)}}^2(t) \right) \\ &= c \lim_{T \rightarrow \infty} \frac{1}{T} \mathbf{E} \exp \left(\max_{t \in [0, T]} \sqrt{2} \eta_{(c)}(t) - \sigma_{\eta_{(c)}}^2(t) \right) = c\mathcal{H}_{\eta_{(c)}}. \end{aligned}$$

On the other hand, using that $\sigma_{\eta_{(c)}}^2(t) = c^2 \sigma_{\eta}^2(t/c)$, we have

$$\frac{\sigma_{\eta_{(c)}}^2(t)}{\sigma_{\eta_{(c)}}^2(\Lambda t)} = \frac{c^2 \sigma_{\eta}^2(t/c)}{c^2 \sigma_{\eta}^2(\Lambda t/c)} = \frac{\sigma_{\eta}^2(t/c)}{\sigma_{\eta}^2(\Lambda t/c)}.$$

Thus, if $\Lambda > 0$ is such that (2.1) is satisfied for $\eta(t)$, then (2.1) is satisfied with the same Λ also for each $\eta_{(c)}(t)$.

Since $\eta_{(c)}(t)$ converges weakly in $C([0, \infty))$ to the process $\sqrt{R(0)}B_1(t) = \sqrt{R(0)}\mathcal{N}$ as $c \rightarrow \infty$, where $\mathcal{N}$ has the standard normal law, from Theorem 4.1 we conclude that

$$\lim_{c \rightarrow \infty} \frac{\mathcal{H}_{c\eta}}{c} = \lim_{c \rightarrow \infty} \mathcal{H}_{\eta_{(c)}} = \mathcal{H}_{\sqrt{R(0)}B_1},$$

which combined with the fact that $\mathcal{H}_{\sqrt{R(0)}B_1} = \sqrt{R(0)}/\pi$ completes the proof.

5. Proof of Theorem 3.1. Before presenting the proof of Theorem 3.1, we need the following lemma, which slightly extends Lemma D.1 in [6], giving the exact form of the constants before the exponent.

Lemma 5.1. *If $\eta(t) \in \mathbf{C1-C2}$ and $\Lambda > 0$ is such that $\sup_{t>0} \sigma_{\eta}^2(t)/\sigma_{\eta}^2(\Lambda t) \leq \frac{1}{2}$, then for $\Delta(u)$ such that $\lim_{u \rightarrow \infty} \Delta(u)/u = 0$ and for each $T > 0$, $\delta > 0$*

$$\begin{aligned} &\mathbf{P} \left\{ \sup_{s \in [0, T]} \bar{\eta}_u(s) > \sqrt{2} \sigma_{\eta}(u); \sup_{t \in [\delta+T, \delta+2T]} \bar{\eta}_u(t) > \sqrt{2} \sigma_{\eta}(u) \right\} \\ &\leq \mathcal{H}_{\eta, \eta}(\Lambda^5 T) \exp \left(-\frac{\sigma_{\eta}^2(\delta)}{8} \right) \Psi \left(\sqrt{2} \sigma_{\eta}(u) \right) (1 + o(1)) \end{aligned} \quad (5.1)$$

as $u \rightarrow \infty$. Inequality (5.1) holds uniformly with respect to u for $\delta \leq \Delta(u) - 2T$.

P r o o f. Since the idea of the proof is analogous to the proof of Lemma 6.3 in [6] then we present only the main steps of the argumentation. Consider the Gaussian field $\mathbf{Y}_u(s, t) = \bar{\eta}_u(s) + \bar{\eta}_u(t)$ and let $A_0 = [0, T]$, $A_{\delta+T} = [\delta + T, \delta + 2T]$ for $0 \leq \delta \leq \Delta(u) - 2T$. We have

$$\begin{aligned} & \mathbf{P} \left\{ \sup_{t \in [0, T]} \bar{\eta}_u(t) > \sqrt{2} \sigma_\eta(u); \sup_{t \in [\delta + T, \delta + 2T]} \bar{\eta}_u(t) > \sqrt{2} \sigma_\eta(u) \right\} \\ & \leq \mathbf{P} \left\{ \sup_{(s, t) \in A_0 \times A_{\delta+T}} \mathbf{Y}_u(s, t) > 2\sqrt{2} \sigma_\eta(u) \right\}. \end{aligned} \quad (5.2)$$

Note that for each $s \in A_0$, $t \in A_{\delta+T}$ and sufficiently large u

$$\mathbf{D}\mathbf{Y}_u(s, t) \geq 4 - 2 \frac{\sigma_\eta^2(|t - s|)}{\sigma_\eta^2(u)} \geq 2, \quad (5.3)$$

$$\mathbf{D}\mathbf{Y}_u(s, t) \leq 4 - \frac{\sigma_\eta^2(|t - s|)}{2\sigma_\eta^2(u)} \leq 4 - \frac{\sigma_\eta^2(\delta)}{2\sigma_\eta^2(u)}. \quad (5.4)$$

Let $\bar{\mathbf{Y}}_u(s, t) = \mathbf{Y}_u(s, t) / \sqrt{\mathbf{D}\mathbf{Y}_u(s, t)}$ and observe that

$$\begin{aligned} & \mathbf{P} \left\{ \sup_{(s, t) \in A_0 \times A_{\delta+T}} \mathbf{Y}_u(s, t) > \sqrt{2} \sigma_\eta(u) \right\} \\ & \leq \mathbf{P} \left\{ \sup_{(s, t) \in A_0 \times A_{\delta+T}} \bar{\mathbf{Y}}_u(s, t) > \frac{2\sqrt{2} \sigma_\eta(u)}{\sqrt{4 - \sigma_\eta^2(\delta)/(2\sigma_\eta^2(u))}} \right\}. \end{aligned} \quad (5.5)$$

Moreover, for each $s, s_1 \in A_0$ and $t, t_1 \in A_{\delta+T}$

$$\begin{aligned} \mathbf{E}(\bar{\mathbf{Y}}_u(s, t) - \bar{\mathbf{Y}}_u(s_1, t_1))^2 & \leq \frac{4}{\mathbf{D}\mathbf{Y}_u(s, t)} \mathbf{E}(\mathbf{Y}_u(s, t) - \mathbf{Y}_u(s_1, t_1))^2 \\ & \leq 4 \left(\mathbf{E}(\bar{\eta}_u(s) - \bar{\eta}_u(s_1))^2 + \mathbf{E}(\bar{\eta}_u(t) - \bar{\eta}_u(t_1))^2 \right) \\ & = 8 \left((1 - \text{cov}(\bar{\eta}_u(s), \bar{\eta}_u(s_1))) + (1 - \text{cov}(\bar{\eta}_u(t), \bar{\eta}_u(t_1))) \right) \\ & \leq 16 \left(\frac{\sigma_\eta^2(|s - s_1|)}{2\sigma_\eta^2(u)} + \frac{\sigma_\eta^2(|t - t_1|)}{2\sigma_\eta^2(u)} \right) \end{aligned} \quad (5.6)$$

$$\begin{aligned} & \leq \frac{1}{2} \left(\frac{\sigma_\eta^2(\Lambda_0|s - s_1|)}{2\sigma_\eta^2(u)} + \frac{\sigma_\eta^2(\Lambda_0|t - t_1|)}{2\sigma_\eta^2(u)} \right) \\ & \leq (1 - \text{cov}(\bar{\eta}_u(\Lambda_0 s), \bar{\eta}_u(\Lambda_0 s_1))) + (1 - \text{cov}(\bar{\eta}_u(\Lambda_0 t), \bar{\eta}_u(\Lambda_0 t_1))) \\ & = \frac{1}{2} \left(\mathbf{E}(\bar{\eta}_u(\Lambda_0 s) - \bar{\eta}_u(\Lambda_0 s_1))^2 + \mathbf{E}(\bar{\eta}_u(\Lambda_0 t) - \bar{\eta}_u(\Lambda_0 t_1))^2 \right), \end{aligned} \quad (5.7)$$

where (5.6) follows from Proposition 2.1 and (5.7) with $\Lambda_0 = \Lambda^5$ follows from (2.1). Hence for $\bar{\eta}_{(1);u}(t), \bar{\eta}_{(2);u}(t)$ being independent copies of the process $\bar{\eta}_u(t)$, the covariance function of the process $(\bar{\eta}_{(1);u}(\Lambda_0 s) + \bar{\eta}_{(2);u}(\Lambda_0 t)) / \sqrt{2}$ is dominated by the covariance function of $\bar{\mathbf{Y}}_u(s, t)$. Thus from the Slepian inequality (see [6, Theorem C.1])

$$\begin{aligned} & \mathbf{P} \left\{ \sup_{(s, t) \in A_0 \times A_{\delta+T}} \bar{\mathbf{Y}}_u(s, t) > \frac{\sqrt{2} \sigma_\eta(u)}{\sqrt{4 - \sigma_\eta^2(\delta)/(2\sigma_\eta^2(u))}} \right\} \\ & \leq \mathbf{P} \left\{ \sup_{(s, t) \in A_0^2} \frac{1}{\sqrt{2}} (\eta_{(1);u}(\Lambda_0 s) + \eta_{(2);u}(\Lambda_0 t)) > \frac{\sqrt{2} \sigma_\eta(u)}{\sqrt{4 - \sigma_\eta^2(\delta)/(2\sigma_\eta^2(u))}} \right\} \end{aligned} \quad (5.8)$$

$$= \mathcal{H}_{\eta, \eta}(\Lambda^5 T) \Psi \left(\frac{\sqrt{2} \sigma_\eta(u)}{\sqrt{4 - \sigma_\eta^2(\delta)/(2\sigma_\eta^2(u))}} \right) (1 + o(1)) \quad (5.9)$$

$$= \mathcal{H}_{\eta, \eta}(\Lambda^5 T) \exp \left(- \frac{\sigma_\eta^2(\delta)}{8} \right) \Psi(\sqrt{2} \sigma_\eta(u)) (1 + o(1)), \quad (5.10)$$

where (5.8) holds uniformly with respect to u for $\delta \leq \Delta(u) - 2T$ and (5.9) follows from Proposition 2.2. Thus the assertion of Lemma 5.1 follows by combining (5.2) and (5.5) with (5.10).

P r o o f o f T h e o r e m 3.1. The idea of the proof is based on the comparison of upper and lower bound of $\mathbf{P}\{\sup_{t \in [0, \sigma_\eta(u)]} \bar{\eta}_u(t) > \sqrt{2} \sigma_\eta(u)\}$.

Let T, S be given. We introduce $A_i = [iS, (i+1)S]$, $B_i = [iT, (i+1)T]$ and let $N_u^{(S)} = [\sigma_\eta(u)/S]$, $N_u^{(T)} = [\sigma_\eta(u)/T] - 1$. Note that from the Bonferroni inequality and Proposition 2.2

$$\begin{aligned} \mathbf{P}\left\{\sup_{t \in [0, \sigma_\eta(u)]} \bar{\eta}_u(t) > \sqrt{2} \sigma_\eta(u)\right\} &\leq \sum_{i=0}^{N_u^{(S)}} \mathbf{P}\left\{\sup_{t \in A_i} \bar{\eta}_u(t) > \sqrt{2} \sigma_\eta(u)\right\} \\ &= N_u^{(S)} \mathcal{H}_\eta(S) \Psi(\sqrt{2} \sigma_\eta(u))(1 + o(1)) \end{aligned}$$

as $u \rightarrow \infty$. Thus for each $S > 0$

$$\limsup_{u \rightarrow \infty} \frac{\mathbf{P}\{\sup_{t \in [0, \sigma_\eta(u)]} \bar{\eta}_u(t) > \sqrt{2} \sigma_\eta(u)\}}{\sigma_\eta(u) \Psi(\sqrt{2} \sigma_\eta(u))} \leq \frac{\mathcal{H}_\eta(S)}{S}. \quad (5.11)$$

On the other hand, again using the Bonferroni inequality, we have for each $T \geq 1$

$$\begin{aligned} \mathbf{P}\left\{\sup_{t \in [0, \sigma_\eta(u)]} \bar{\eta}_u(t) > \sqrt{2} \sigma_\eta(u)\right\} &\geq \sum_{i=0}^{N_u^{(T)}} \mathbf{P}\left\{\sup_{t \in B_i} \bar{\eta}_u(t) > \sqrt{2} \sigma_\eta(u)\right\} \\ &\quad - 2 \sum_{0 \leq i < j \leq N_u^{(T)}} \mathbf{P}\left\{\sup_{t \in B_i} \bar{\eta}_u(t) > \sqrt{2} \sigma_\eta(u); \sup_{s \in B_j} \bar{\eta}_u(s) > \sqrt{2} \sigma_\eta(u)\right\}. \end{aligned} \quad (5.12)$$

We split the sum in line (5.12) into the following sub-sums:

$$\begin{aligned} &2 \sum_{0 \leq i < j \leq N_u^{(T)}} \mathbf{P}\left\{\sup_{t \in B_i} \bar{\eta}_u(t) > \sqrt{2} \sigma_\eta(u); \sup_{s \in B_j} \bar{\eta}_u(s) > \sqrt{2} \sigma_\eta(u)\right\} \\ &= 2 \sum_{i+1=j} \mathbf{P}\left\{\sup_{t \in B_i} \bar{\eta}_u(t) > \sqrt{2} \sigma_\eta(u); \sup_{s \in B_j} \bar{\eta}_u(s) > \sqrt{2} \sigma_\eta(u)\right\} \\ &\quad + 2 \sum_{1 < j-i \leq N_u^{(T)}} \mathbf{P}\left\{\sup_{t \in B_i} \bar{\eta}_u(t) > \sqrt{2} \sigma_\eta(u); \sup_{s \in B_j} \bar{\eta}_u(s) > \sqrt{2} \sigma_\eta(u)\right\} = I_1 + I_2. \end{aligned}$$

Note that for I_2 , using Lemma 5.1, we have

$$I_2 \leq 2 N_u^{(T)} \mathcal{H}_{\eta, \eta}(\Lambda^5 T) \sum_{k=1}^{\infty} \exp\left(-\frac{\sigma_\eta^2(kT)}{8}\right) \Psi(\sqrt{2} \sigma_\eta(u))(1 + o(1)) \quad (5.13)$$

as $u \rightarrow \infty$. Moreover, for $f: \mathbf{R}^+ \rightarrow \mathbf{R}^+$ such that $f(x) < x$ for each $x \geq 1$,

$$\begin{aligned} &\mathbf{P}\left\{\sup_{t \in B_i} \bar{\eta}_u(t) > \sqrt{2} \sigma_\eta(u); \sup_{s \in B_{i+1}} \bar{\eta}_u(s) > \sqrt{2} \sigma_\eta(u)\right\} \\ &\leq \mathbf{P}\left\{\sup_{t \in B_i} \bar{\eta}_u(t) > \sqrt{2} \sigma_\eta(u); \sup_{s-f(T) \in B_{i+1}} \bar{\eta}_u(s) > \sqrt{2} \sigma_\eta(u)\right\} \\ &\quad + \mathbf{P}\left\{\sup_{t \in [(i+1)T, (i+1)T+f(T)]} \bar{\eta}_u(t) > \sqrt{2} \sigma_\eta(u)\right\}. \end{aligned}$$

Thus for I_1 we have

$$\begin{aligned} I_1 &\leq 2 N_u^{(T)} \mathcal{H}_{\eta, \eta}(\Lambda^5 T) \exp\left(-\frac{\sigma_\eta^2(f(T))}{8}\right) \Psi(\sqrt{2} \sigma_\eta(u))(1 + o(1)) \\ &\quad + 2 N_u^{(T)} \mathcal{H}_\eta(f(T)) \Psi(\sqrt{2} \sigma_\eta(u))(1 + o(1)). \end{aligned} \quad (5.14)$$

Combining (5.13) with (5.14) we obtain for $T \geq 1$

$$\begin{aligned} & \liminf_{u \rightarrow \infty} \frac{\mathbf{P}\{\sup_{t \in [0, \sigma_\eta(u)]} \bar{\eta}_u(t) > \sqrt{2} \sigma_\eta(u)\}}{\sigma_\eta(u) \Psi(\sqrt{2} \sigma_\eta(u))} \\ & \geq \frac{\mathcal{H}_\eta(T)}{T} - 2 \frac{\mathcal{H}_{\eta, \eta}(\Lambda^5 T)}{T} \exp\left(-\frac{\sigma_\eta^2(f(T))}{8}\right) - 2 \frac{\mathcal{H}_\eta(f(T))}{T} \\ & \quad - 2 \frac{\mathcal{H}_{\eta, \eta}(\Lambda^5 T)}{T} \sum_{k=1}^{\infty} \exp\left(-\frac{\sigma_\eta^2(kT)}{8}\right). \end{aligned} \quad (5.15)$$

Hence, using subadditivity of $\mathcal{H}_\eta(\cdot)$, comparing (5.15) with (5.11) and taking $S \rightarrow \infty$ we have that for each $T \geq 1$

$$\begin{aligned} \mathcal{H}_\eta & \geq \frac{\mathcal{H}_\eta(T)}{T} - 2 \frac{\mathcal{H}_{\eta, \eta}(\Lambda^5 T)}{T} \exp\left(-\frac{\sigma_\eta^2(f(T))}{8}\right) - 2 \frac{\mathcal{H}_\eta(f(T))}{T} \\ & \quad - 2 \frac{\mathcal{H}_{\eta, \eta}(\Lambda^5 T)}{T} \sum_{k=1}^{\infty} \exp\left(-\frac{\sigma_\eta^2(kT)}{8}\right). \end{aligned} \quad (5.16)$$

Moreover, due to the fact that $\mathcal{H}_\eta(\cdot)$ is subadditive, we have $\mathcal{H}_\eta(T)/T - \mathcal{H}_\eta \geq 0$, which combined with (5.16) completes the proof.

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Поступила в редакцию
20.VIII.2002

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A LIMIT PROPERTY OF THE GEOMETRIC DISTRIBUTION

Пусть случайные величины X^* и X имеют дискретное распределение на множестве целых неотрицательных чисел и

$$\mathbf{P}\{X = k\} = c \sum_{j=k}^{\infty} \mathbf{P}\{X^* = j\}, \quad k = 0, 1, 2, \dots,$$

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