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Combining (5.13) with (5.14) we obtain for $T \geq 1$

$$\begin{aligned} & \liminf_{u \rightarrow \infty} \frac{\mathbf{P}\{\sup_{t \in [0, \sigma_\eta(u)]} \bar{\eta}_u(t) > \sqrt{2} \sigma_\eta(u)\}}{\sigma_\eta(u) \Psi(\sqrt{2} \sigma_\eta(u))} \\ & \geq \frac{\mathcal{H}_\eta(T)}{T} - 2 \frac{\mathcal{H}_{\eta, \eta}(\Lambda^5 T)}{T} \exp\left(-\frac{\sigma_\eta^2(f(T))}{8}\right) - 2 \frac{\mathcal{H}_\eta(f(T))}{T} \\ & \quad - 2 \frac{\mathcal{H}_{\eta, \eta}(\Lambda^5 T)}{T} \sum_{k=1}^{\infty} \exp\left(-\frac{\sigma_\eta^2(kT)}{8}\right). \end{aligned} \quad (5.15)$$

Hence, using subadditivity of $\mathcal{H}_\eta(\cdot)$, comparing (5.15) with (5.11) and taking $S \rightarrow \infty$ we have that for each $T \geq 1$

$$\begin{aligned} \mathcal{H}_\eta & \geq \frac{\mathcal{H}_\eta(T)}{T} - 2 \frac{\mathcal{H}_{\eta, \eta}(\Lambda^5 T)}{T} \exp\left(-\frac{\sigma_\eta^2(f(T))}{8}\right) - 2 \frac{\mathcal{H}_\eta(f(T))}{T} \\ & \quad - 2 \frac{\mathcal{H}_{\eta, \eta}(\Lambda^5 T)}{T} \sum_{k=1}^{\infty} \exp\left(-\frac{\sigma_\eta^2(kT)}{8}\right). \end{aligned} \quad (5.16)$$

Moreover, due to the fact that $\mathcal{H}_\eta(\cdot)$ is subadditive, we have $\mathcal{H}_\eta(T)/T - \mathcal{H}_\eta \geq 0$, which combined with (5.16) completes the proof.

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A LIMIT PROPERTY OF THE GEOMETRIC DISTRIBUTION

Пусть случайные величины X^* и X имеют дискретное распределение на множестве целых неотрицательных чисел и

$$\mathbf{P}\{X = k\} = c \sum_{j=k}^{\infty} \mathbf{P}\{X^* = j\}, \quad k = 0, 1, 2, \dots,$$

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где c — соответствующая константа. Изучаются повторные суммирование такого вида. Предельное распределение оказывается геометрическим для широкого класса распределений X^* .

Ключевые слова и фразы: дискретные распределения, распределения частичных сумм, геометрическое распределение.

1. Introduction. Let X^* (parent) and X (descendant) be random variables defined on the nonnegative integers with the probability mass functions $\{P_j^*\}_{j=0}^\infty$ and $\{P_j\}_{j=0}^\infty$, respectively. Let all the moments of X^* be finite. Let us denote by μ^* the mean of X^* . The partial summation

$$P_k = c \sum_{j=k}^{\infty} P_j^*, \quad k = 0, 1, 2, \dots,$$

has been introduced in [1]. In [5] there are many pairs of parent and descendant distributions. The summation process is generalized in [2], where also relations between probability generating functions and moments of parent and descendant distributions are analyzed. Due to [6] the necessary and sufficient condition for the parent and descendant distributions to be identical under the partial summation is to be the geometric distribution. In [3] one can find a correspondence between the Katz family of distributions (see [4]) and a wide class of partial summations.

According to [2],

$$c = \frac{1}{1 + \mu^*}$$

and

$$G(t) = c \frac{1 - tG^*(t)}{1 - t},$$

$G^*(t)$ and $G(t)$ being the probability generating functions of X^* and X , respectively.

2. Repeated summations. Let $X^{(1)}, \dots, X^{(n)}, \dots$ be random variables on the nonnegative integers with the probability mass functions $\{P_j^{(1)}\}_{j=0}^\infty, \dots, \{P_j^{(n)}\}_{j=0}^\infty$, respectively, generated from X^* by the summations

$$P_k^{(1)} = c_1 \sum_{j=k}^{\infty} P_j^*, \quad k = 0, 1, 2, \dots,$$

$$P_k^{(n)} = c_n \sum_{j=k}^{\infty} P_j^{(n-1)}, \quad k = 0, 1, 2, \dots, \quad n = 2, 3, \dots$$

$c_1, \dots, c_n, \dots$ being proper constants.

Lemma 1. *The generated probabilities satisfy the relation*

$$P_k^{(n)} = \prod_{s=1}^n c_s \sum_{j=0}^{\infty} \binom{n+j-1}{j} P_{k+j}^*, \quad (1)$$

$k = 0, 1, 2, \dots; n = 1, 2, \dots$

P r o o f. For $n = 1$ it is obvious. Assume that (1) holds for a certain $n > 1$. Then

$$\begin{aligned} P_k^{(n+1)} &= \prod_{s=1}^{n+1} c_s \sum_{j=k}^{\infty} P_j^{(n)} = \prod_{s=1}^{n+1} c_s \sum_{l=k}^{\infty} \sum_{j=0}^{\infty} \binom{n+j-1}{j} P_{l+j}^* \\ &= \prod_{s=1}^{n+1} c_s \sum_{j=0}^{\infty} P_{k+j}^* \sum_{l=0}^j \binom{n+l-1}{l} = \prod_{s=1}^{n+1} c_s \sum_{j=0}^{\infty} \binom{n+j}{j} P_{k+j}^*. \end{aligned}$$

Lemma 1 is proved.

Let $G_1(t), \dots, G_n(t), \dots$ be probability generating functions of the distributions $\{P_j^{(1)}\}_{j=0}^\infty, \dots, \{P_j^{(n)}\}_{j=0}^\infty, \dots$, respectively.

Lemma 2. *The probability generating functions of the generated distributions satisfy the relation*

$$G_n(t) = \sum_{j=0}^{n-1} \frac{(-1)^j t^j \prod_{l=0}^j c_{n-l}}{(1-t)^{j+1}} + \frac{(-1)^n t^n \prod_{j=1}^n c_j}{(1-t)^n} G^*(t). \quad (2)$$

P r o o f. According to [2],

$$G_1(t) = c_1 \frac{1 - tG^*(t)}{1 - t}.$$

Assume that (2) holds for a certain $n > 1$. Then

$$\begin{aligned} G_{n+1}(t) &= c_{n+1} \frac{1 - tG_n(t)}{1 - t} \\ &= \frac{c_{n+1}}{1 - t} \left[1 - t \left(\sum_{j=0}^{n-1} \frac{(-1)^j t^j \prod_{l=0}^j c_{n-l}}{(1-t)^{j+1}} + \frac{(-1)^n t^n \prod_{j=1}^n c_j}{(1-t)^n} G^*(t) \right) \right] \\ &= \frac{c_{n+1}}{1 - t} \left[1 - \left(\sum_{j=0}^{n-1} \frac{(-1)^j t^{j+1} \prod_{l=0}^j c_{n-l}}{(1-t)^{j+1}} + \frac{(-1)^{n+1} t^{n+1} \prod_{j=1}^n c_j}{(1-t)^n} G^*(t) \right) \right] \\ &= \sum_{j=0}^n \frac{(-1)^j t^j \prod_{l=0}^j c_{n-l+1}}{(1-t)^{j+1}} + \frac{(-1)^{n+1} t^{n+1} \prod_{j=1}^{n+1} c_j}{(1-t)^{n+1}} G^*(t). \end{aligned}$$

Lemma 2 is proved.

Lemma 3. *If $\lim_{k \rightarrow \infty} P_{k+1}^*/P_k^* = q$ exists and $0 \leq q < 1$, then $\{c_j\}_{j=0}^\infty$ converges and $\lim_{n \rightarrow \infty} c_n = 1 - q$.*

P r o o f. From the definition of the analyzed summation it is obvious that $c_n = P_0^{(n)}$, $n = 1, 2, \dots$. We obtain (using Lemma 1)

$$\frac{c_{n+1}}{c_n} = \frac{P_0^{(n+1)}}{P_0^{(n)}} = \frac{\prod_{s=0}^{n+1} c_s \sum_{j=0}^\infty \binom{n+j}{j} P_j^*}{\prod_{s=0}^n c_s \sum_{j=0}^\infty \binom{n+j-1}{j} P_j^*} = c_{n+1} \frac{\sum_{j=0}^\infty \binom{n+j}{j} P_j^*}{\sum_{j=0}^\infty \binom{n+j-1}{j} P_j^*}.$$

Consequently,

$$c_n = \frac{\sum_{j=0}^\infty \binom{n+j-1}{j} P_j^*}{\sum_{j=0}^\infty \binom{n+j}{j} P_j^*} = 1 - \frac{P_1^* + (n+1)P_2^* + \binom{n+2}{2}P_3^* + \binom{n+3}{3}P_4^* + \dots}{P_0^* + (n+1)P_1^* + \binom{n+2}{2}P_2^* + \binom{n+3}{3}P_3^* + \dots}.$$

Provided that $\lim_{k \rightarrow \infty} P_{k+1}^*/P_k^*$ exists, it holds

$$\forall \varepsilon > 0 \quad \exists k_0 \in \mathbf{N} \quad \forall k \geq k_0 \quad q - \varepsilon < \frac{P_{k+1}^*}{P_k^*} < q + \varepsilon.$$

Let us denote $\delta_{1,n} = P_1^* + (n+1)P_2^* + \dots + \binom{n+k_0-1}{k_0-1}P_{k_0}^*$ and $\delta_{2,n} = P_0^* + (n+1)P_1^* + \dots + \binom{n+k_0-1}{k_0-1}P_{k_0-1}^*$. We obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\delta_{1,n} + \binom{n+k_0}{k_0}P_{k_0+1}^* + \binom{n+k_0+1}{k_0+1}P_{k_0+2}^* + \dots}{\delta_{2,n} + \binom{n+k_0}{k_0}P_{k_0}^* + \binom{n+k_0+1}{k_0+1}P_{k_0+1}^* + \dots} \\ \leq \lim_{n \rightarrow \infty} \frac{\delta_{1,n} + (q + \varepsilon)[\binom{n+k_0}{k_0}P_{k_0}^* + \binom{n+k_0+1}{k_0+1}P_{k_0+1}^* + \dots]}{\delta_{2,n} + \binom{n+k_0}{k_0}P_{k_0}^* + \binom{n+k_0+1}{k_0+1}P_{k_0+1}^* + \dots} \\ = \lim_{n \rightarrow \infty} \frac{\delta_{1,n}/\binom{n+k_0}{k_0} + (q + \varepsilon)(P_{k_0}^* + \frac{n+k_0+1}{k_0+1}P_{k_0+1}^* + \dots)}{\delta_{2,n}/\binom{n+k_0}{k_0} + P_{k_0}^* + \frac{n+k_0+1}{k_0+1}P_{k_0+1}^* + \dots} = q + \varepsilon. \end{aligned}$$

The opposite inequality for $q - \varepsilon$ can be proved in the same way. Hence

$$\forall \varepsilon > 0 \quad 1 - (q + \varepsilon) \leq \lim_{n \rightarrow \infty} c_n \leq 1 - (q - \varepsilon) \implies \lim_{n \rightarrow \infty} c_n = 1 - q.$$

Lemma 3 is proved.

Theorem 1. Let

$$\lim_{k \rightarrow \infty} \left(\frac{P_{k+1}^*}{P_k^*} \right) = q$$

exist and let $0 \leq q < 1$. Then

$$\lim_{n \rightarrow \infty} P_x^{(n)} = (1 - q) q^x,$$

i.e., the limit distribution is geometric with the parameter q (if $0 < q < 1$) or deterministic (if $q = 0$).

P r o o f. According to Lemma 2 we have

$$G_n(t) = \sum_{j=0}^{n-1} (-1)^j \frac{t^j \prod_{x=0}^j c_{n-x}}{(1-t)^{j+1}} + (-1)^n \frac{t^n \prod_{j=1}^n c_j}{(1-t)^n} G^*(t).$$

It holds

$$P_k^{(n)} = \frac{G_n^{(k)}(t)}{k!} \Big|_{t=0}.$$

where $G_n^{(k)}$ is the k -th derivative of the probability generating function. In what follows $x_{(n)}$ is the n -th descending factorial of the number x , i.e., $x_{(n)} = x(x-1) \cdots (x-n+1)$, x being a real number and n a nonnegative integer. We obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{k!} \left[\frac{(-1)^n t^n \prod_{j=1}^n c_j}{(1-t)^n} G^*(t) \right]^{(k)} \Big|_{t=0} \\ = \frac{1}{k!} \lim_{n \rightarrow \infty} (-1)^n \prod_{j=1}^n c_j \sum_{j=0}^k \binom{k}{j} [t^n G^*(t)]^{(j)} [(1-t)^{-n}]^{(k-j)} \Big|_{t=0} \\ = \frac{1}{k!} \lim_{n \rightarrow \infty} (-1)^n \prod_{j=1}^n c_j \sum_{j=0}^k \binom{k}{j} (-n)_{(k-j)} (1-t)^{-n-k+j} \\ \times \sum_{l=0}^j \binom{j}{l} (n)_{(l)} t^{n-l} (G^*(t))^{(j-l)} \Big|_{t=0} = 0 \end{aligned}$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{k!} \left(\sum_{j=0}^{n-1} \frac{(-1)^j t^j \prod_{l=0}^j c_{n-l}}{(1-t)^{j+1}} \right)^{(k)} \Big|_{t=0} \\ = \frac{1}{k!} \lim_{n \rightarrow \infty} \sum_{j=0}^{n-1} (-1)^j \prod_{l=0}^j c_{n-l} \sum_{m=0}^k \binom{k}{m} (t^j)^{(m)} [(1-t)^{-j-1}]^{(k-m)} \Big|_{t=0} \\ = \frac{1}{k!} \lim_{n \rightarrow \infty} \sum_{j=0}^{n-1} (-1)^j \prod_{l=0}^j c_{n-l} \sum_{m=0}^k \binom{k}{m} j_{(m)} t^{j-m} (j+1)^{(k-m)} (1-t)^{-k-j+m-1} \Big|_{t=0} \\ = \lim_{n \rightarrow \infty} \sum_{j=0}^k (-1)^j \binom{k}{j} \prod_{l=0}^j c_{n-l} = \sum_{j=0}^k (-1)^j \binom{k}{j} (1-q)^{j+1} = (1-q) q^k. \end{aligned}$$

Theorem 1 is proved.

R e m a r k 1. If the parent distribution has a finite number of nonzero probabilities, then the limit distribution is deterministic. In this case it holds (provided P_x^* is the last nonzero term)

$$\lim_{n \rightarrow \infty} P_0^{(n)} = \lim_{n \rightarrow \infty} c_n = 1 - \lim_{n \rightarrow \infty} \frac{P_1^* + (n+1)P_2^* + \cdots + \binom{n+x-1}{x-1} P_x^*}{P_0^* + (n+1)P_1^* + \cdots + \binom{n+x}{x} P_x^*} = 1.$$

3. Examples. The notation «distribution $A \rightarrow$ distribution B » will mean that the distribution A converges to the distribution B with respect to the summations analyzed in this paper.

Distributions with a finite number of nonzero probabilities $\rightarrow$ deterministic distribution (d.)

Distributions with the infinite number of nonzero probabilities:

Katz family ($P_{n+1}^* = (\alpha + \beta n) P_n^* / (1 + n)$, see [4]) $\rightarrow$ geometric (β) d.,

Poisson d. $\rightarrow$ deterministic d.,

negative binomial (k, p) d. $\rightarrow$ geometric ($1 - p$) d.,

2-shifted Flory d. (see [4]) $\rightarrow$ geometric ($\frac{1}{2}$) d.,

Ord family ($P_x^* - P_{x-1}^* = (a - x) P_{x-1}^* / (b_0 + b_1 x + b_2 x(x - 1))$, see [4]) $\rightarrow$ geometric $((b_1 - 1)/b_1)$ d. if $b_1 > 1, b_2 = 0$ and $P_x^* = 0$ for $x < 0$,

logarithmic (θ) d. $\rightarrow$ geometric (θ) d.

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