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$c = b - a, m_1 = n_3 - n_2, m_2 = n_3 - n_1, m_3 = n_3$. С помощью доказанных выше лемм легко понять, что происходит с самим процессом $(R_n)_{n \leq n_3}$ и его частями при такой инверсии.

Так, из леммы 5 следует, что сам процесс $(X_n)_{n \leq m_3}$ является простейшим симметричным случайным блужданием, остановленным в момент l_{b+} , а его отрезок $(X_n)_{n \leq m_1}$ есть случайное блуждание до момента l_{c+} . Согласно лемме 5', процесс $(c - X_{n+m_1})_{n \leq m_2 - m_1}$ является положительным процессом Бесселя, рассматриваемым до момента r_c . А процесс $(X_{n+m_2})_{n \leq m_3 - m_2}$ есть положительный процесс Бесселя, остановленный в момент l_{b+} , что следует из леммы 3''. Тем самым доказана следующая теорема.

Теорема 3 (дискретное разложение Вильямса). *Зафиксируем некоторое целое число $b > 0$ и зададим следующие четыре независимых случайных элемента: случайная величина s с дискретным равномерным распределением на отрезке $[0, b]$; простейшее симметричное случайное блуждание $(S_n)_{n=0,1,\dots}$; два положительных дискретных процесса Бесселя $(R_n^{+,1})_{n=0,1,\dots}$ и $(R_n^{+,2})_{n=0,1,\dots}$.*

Определим моменты времени m_1, m_2 и m_3 по формулам

$$m_1 = l_{c+}(S) := \inf\{n \mid S_n = c + 1\} - 1, \quad m_2 - m_1 = r_c(R_n^{+,1}) := \sup\{n \mid R_n^{+,1} = c\},$$

$$m_3 - m_2 = r_{b+}(R_n^{+,2}) := \inf\{n \mid R_n^{+,2} = b + 1\} - 1.$$

Тогда процесс $(X_n)_{n \leq m_3}$, определенный по формуле

$$X_n := \begin{cases} S_n, & 0 \leq n \leq m_1, \\ c - R_{n-m_1}^{+,1}, & m_1 \leq n \leq m_2, \\ R_{n-m_2}^{+,2}, & m_2 \leq n \leq m_3, \end{cases}$$

имеет распределение простейшего симметричного случайного блуждания, остановленного в момент t_{b+} .

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JAJTE R.*

POINTWISE ERGODIC THEOREM FOR UNBOUNDED OPERATORS IN L_2

Приводится достаточное условие справедливости усиленного закона больших чисел для траекторий нормального неограниченного оператора. Это условие формулируется в терминах спектральной меры. Чтобы рассмотреть случай неограниченных операторов, мы переходим от инструментов классической арифметики (Чезаро) к методам суммирования Бореля.

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Ключевые слова и фразы: усиленный закон больших чисел, индивидуальная эргодическая теорема, неограниченный нормальный оператор, спектральная мера, суммирование по Борелю, сходимость почти на верное.

1. Introduction. The main purpose of the paper is to prove a kind of individual ergodic theorem for unbounded operators in $\mathbb{L}_2$ over a probability space. Let us start with a few remarks concerning bounded operators. It is well known that, in general, the strong law of large numbers does not hold for the orbits of a unitary operator in $\mathbb{L}_2$. The asymptotic behaviour of the ergodic averages of a normal contraction operator A in $\mathbb{L}_2(\Omega, \mathcal{F}, \mu)$ heavily depends on the spectrum of A close to the value one. More precisely, let $A = \int_{\{|z| \leq 1\}} z E(dz)$ be the spectral representation of A , and let $S_n = n^{-1} \sum_{k=0}^n A^k$. Gaposhkin proved [6], [7] that, for $\xi \in \mathbb{L}_2$,

$$S_n \xi \rightarrow E\{1\} \xi \quad \mu\text{-a.e.} \quad (1)$$

if and only if

$$E\{z: 0 < |1 - z| < 2^{-n}\} \xi \rightarrow 0 \quad \mu\text{-a.e.} \quad (2)$$

A similar result for noncontractive bounded normal operators can be obtained [9] by passing from the classical arithmetic (Cesàro) means to the Borel methods of summability.

Let us recall the definition of these methods.

For $\alpha > 0$ and a sequence $x = (\xi_n)$ of numbers (or vectors), let us put

$$B_\alpha(t, x) = \alpha e^{-t} \sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)} \xi_n, \quad t > 0. \quad (3)$$

The function $B_\alpha(t, x)$ is called the B_α -transform of the sequence $x = (\xi_n)$. If $\lim_{t \rightarrow \infty} B_\alpha(t, x) = \xi$, then we say that (ξ_n) is summable (limitable) to ξ by the method B_α , and write $\xi_n \rightarrow \xi(B_\alpha)$ or $B_\alpha\text{-}\lim \xi_n = \xi$.

The final results in the paper concern «discrete» Borel methods, i.e., $B_\alpha(t, x)$ in (3) is taken for $t = 1, 2, \dots$ only, though in the proofs we often consider the transform $B_\alpha(t, x)$ with continuous parameter (for example by taking derivative of B_α with respect to t). Also we take $\alpha = 2^{-\nu}$, $\nu = 1, 2, \dots$, only.

The role of linear summability methods in probability theory has been examined by many authors. Let us mention here the papers [2], [3] related to this subject in the context of strong law of large numbers.

Borel methods of summability have already been intensively investigated by Mittag-Leffler as a powerful method in the theory of analytic continuations [4], [8]. Let us mention here the famous representation given by Mittag-Leffler

$$\int_0^\infty e^{-t} \sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)} z^n dt = \frac{1}{1 - z}$$

valid in the domain G_α containing the origin and bounded by the curve $r = (\cos(\theta/\alpha))^{-\alpha}$, $|\theta| < \pi\alpha/2$; $0 < \alpha < 1$. Taking α close to zero, one gets the summability of the series $\sum_{k=0}^\infty z^k$ in nearly the whole Mittag-Leffler's star for $z \rightarrow 1/(1 - z)$. More precisely, $\bigcup_{\alpha > 0} G_\alpha = \mathbb{C} \setminus [1, \infty)$.

It is worth noting here that the sequence $\{z^n\}$ is B_α -summable to zero for $z \in G_\alpha$, cf. [11].

The results just formulated suggest that the Borel summability methods may be efficient for unbounded normal operators with the spectral representation

$$A = \int_{\Delta} z E(dz),$$

where Δ is close to G_α .

Before formulating the main results let us adopt one more definition.

Let H be an unbounded operator acting in a Banach space X . For $\alpha > 0$ and $\beta > 0$, a vector $x \in X$ is said to be (α, β) -analytic for H if the function

$$z \rightarrow \sum_{n \geq 0} \frac{\|H^n x\|^\beta}{\Gamma(n\alpha + 1)} z^n$$

is entire analytic. The $(1, 1)$ -analyticity coincides with the analyticity in the usual sense, cf. [5], [10]. The notion of (α, β) -analyticity seems to deserve a more careful analysis but in this paper we shall only use the $(\alpha, 2)$ -analyticity in the context of Borel summability methods. In the proofs presented in the paper we follow some ideas of Włodarski [11] and Gaposhkin [6], [7], and classical procedures well known in the theory of orthogonal series which go back to Plancherel. Some preliminary results can be found in [9]. We give the short proofs of them for the sake of completeness.

2. Main results. Our main goal is to prove some pointwise convergence theorems for the Borel means of the orbits of a normal unbounded operator A in $\mathbb{L}_2$ over a probability space $(\Omega, \mathcal{F}, \mu)$. The conditions implying the almost everywhere convergence of the Borel transform associated to the operator A are described in terms of the spectral measure E of A .

Let us formulate the following theorems.

Theorem 2.1. *Let*

$$A = \int_{\Delta} z E(dz) \quad (4)$$

be the spectral representation of a normal (unbounded, in general) operator in $\mathbb{L}_2(\Omega, \mathcal{F}, \mu)$. For $\alpha = 2^{-k}$ and $0 < d < 1 - \cos(\pi/2^{k+1})$, let us define a set $D_{\alpha,d} \subset \mathbb{C}$ by putting

$$D_{\alpha,d} = \{z: \operatorname{Re} z \leq 1 - d\} \cup \{z: |z| \leq 1\} \cup \{z: \operatorname{Re} z^{2^k} \leq 1 - d\}. \quad (5)$$

Assume that, for the operator (4), we have the inclusion $\Delta \subset D_{\alpha,d}$. Let $\xi \in \mathbb{L}_2$ be $(\alpha, 2)$ -analytic for A and such that

$$\int_{\Delta \cap \{|z| < 1\}} \frac{\nu_{\xi}(dz)}{|1 - z|^2} < \infty, \quad (6)$$

where $\nu_{\xi}(dz) = \|E(dz)\xi\|^2$.

Then

$$B_{\alpha}\text{-}\lim A^n \xi = E\{1\} \xi, \quad \mu\text{-a.e.} \quad (7)$$

We stress that B_{α} in (7) denotes the «discrete» Borel method.

Let us have a look at the picture of $D_{\alpha,d}$ defined by (5). Denoting by D the domain

$$D = \left\{ z = r e^{i\theta}: |\theta| < \frac{\pi}{2^{k+1}}, r > \left(\frac{1-d}{\cos 2^k \theta} \right)^{2^{-k}} \right\},$$

we take all the images of D_s under the rotation $z \rightarrow z e^{i\beta_s}$, where $\beta_s = (\pi/2^{k-1})s$; $s = 0, \pm 1, \dots, \pm 2^{k-2}$. To obtain $D_{\alpha,d}$ we remove from the plane the set $\bigcup_s D_s \cap \{z: \operatorname{Re} z > 1 - d\}$ adding (a part of) the disc $\{z: |z| \leq 1\}$.

The condition (6) is very restrictive. It can be dropped in the case when near the value 1 the spectrum Δ of A is situated in the angle

$$Q(\alpha_0) = \left\{ 1 + r e^{i\theta}: r \geq 0, |\pi - \theta| \leq \alpha_0 < \frac{\pi}{2} \right\}, \quad (8)$$

for some α_0 . Indeed, we have the following result which is in the spirit of the Gaposhkin theorem. It means that for some vectors in $\mathbb{L}_2$, the almost everywhere convergence of the Borel transform associated to the operator A is characterized just by the Gaposhkin condition (2).

Theorem 2.2. *For $\alpha = 2^{-k}$ and $0 < d < 1 - \cos(\pi/2^{k+1})$, let $D_{\alpha,d}$ be defined by (5). Let A be a normal (unbounded, in general) operator with the spectral representation (4). Assume that $\Delta \subset D_{\alpha,d}$ and, additionally, that we have*

$$\Delta \cap \{z: |z| \leq 1\} \subset Q(\alpha_0), \quad (9)$$

$Q(\alpha_0)$ being the angle defined by (8).

Let ξ be a vector in $\mathbb{L}_2$ which is $(\alpha, 2)$ -analytic for A .

Then the conditions

$$B_{\alpha}\text{-}\lim A^n \xi = E\{1\} \xi \quad \mu\text{-a.e.} \quad (10)$$

and

$$E\{z: 0 < |1 - z| < 2^{-n}\} \xi \rightarrow 0 \quad \mu\text{-a.e.} \quad (11)$$

are equivalent.

As an immediate consequence of the above theorem we get the following result.

Corollary 2.1. *Let*

$$A = \int_{-\infty}^1 z E(dz) \quad (12)$$

be the spectral representation of a self-adjoint operator A in $\mathbb{L}_2(\Omega, \mathcal{F}, \mu)$. Let ξ be a vector in $\mathbb{L}_2$ which is analytic (i.e., $(1, 1)$ -analytic) for A .

Then the conditions

$$B_1\text{-}\lim_{n \rightarrow \infty} A^n \xi = \lim_{m \rightarrow \infty} e^{-m} \sum_{n=0}^{\infty} \frac{m^n}{n!} A^n \xi = E\{1\} \xi \quad \mu\text{-a.e.} \quad (13)$$

and

$$E\{\lambda: 1 - 2^{-n} < \lambda < 1\} \xi \rightarrow 0 \quad \mu\text{-a.e.} \quad (14)$$

are equivalent.

3. Auxiliary results. Before starting the proof of Theorem 2.1, let us begin with several lemmas. Let us consider the operator A given by (4). The following lemma is basic for all further considerations. In the sequel it will be constantly used without any reference.

Throughout, as a rule, the constant C is different in different places.

Lemma 3.1. *If ξ is $(\alpha, 2)$ -analytic for A , then for $x = (A^k \xi)$ the Borel transform $B_\alpha(t, x)$ exists and, putting $\mu_\xi(dz) = E(dz) \xi$, we have*

$$B_\alpha(t, x) = \int_{\Delta} B_\alpha(t, \zeta) \mu_\xi(dz), \quad \zeta = (z^k), \quad (15)$$

that is,

$$\alpha e^{-t} \sum_{n \geq 0} \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)} A^n \xi = \int_{\Delta} \alpha e^{-t} \sum_{n \geq 0} \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)} z^n \mu_\xi(dz).$$

P r o o f. By the $(\alpha, 2)$ -analyticity of ξ , we have

$$\sum_{n \geq 0} \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)} \|A^n \xi\| \leq C \sum_{n \geq 0} \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)} \|A^n \xi\|^2 < \infty,$$

so the series

$$\sum_{n \geq 0} \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)} A^n \xi$$

is convergent in $\mathbb{L}_2$, and the transform $B_\alpha(t, x)$ is well defined. To prove (15) let us remark that the function

$$g(z) = \sum_{n \geq 0} \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)} |z|^{2n} \quad \text{is } \nu_\xi\text{-integrable,} \quad (16)$$

where $\nu_\xi(dz) = \|E(dz) \xi\|^2$. This is clear since

$$\infty > \sum_{n \geq 0} \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)} \|A^n \xi\|^2 = \sum_{n \geq 0} \int_{\Delta} |z|^{2n} \nu_\xi(dz) \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)} = \int_{\Delta} g(z) \nu_\xi(dz).$$

In particular, we have

$$\int_{\Delta} |z|^k \nu_\xi(dz) < \infty, \quad k = 0, 1, 2, \dots \quad (17)$$

Moreover, there exists an $L > 0$ such that

$$\gamma_N(z) = \left(\sum_{n=0}^N \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)} |z|^n \right)^2 \leq g(z), \quad (18)$$

for $|z| > L$ and $N = 1, 2, \dots$.

Indeed, putting

$$a_{nk} = \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)} |z|^n \frac{t^{k\alpha}}{\Gamma(k\alpha + 1)} |z|^k,$$

we have

$$\gamma_N(z) = \sum_{n=0}^N \sum_{k=0}^N a_{nk} \leq \sum_{n=0}^{2N} \sum_{k=0}^n a_{nk} = \sum_{n=0}^{2N} \sum_{k=0}^n a_{k,n-k},$$

so

$$\gamma_N(z) \leq \sum_{n=0}^{2N} |z|^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \sum_{k=0}^n \frac{\Gamma(n\alpha+1)}{\Gamma(k\alpha+1) \Gamma((n-k)\alpha+1)}.$$

Using the Stirling formula, after rather standard calculations, we get

$$\sum_{k=0}^n \frac{\Gamma(n\alpha+1)}{\Gamma(k\alpha+1) \Gamma((n-k)\alpha+1)} \leq L^n,$$

for L large enough.

Thus, for $|z| > L$,

$$\gamma_N(z) \leq \sum_{n=0}^{2N} |z|^n L^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \leq g(z).$$

Let us write

$$\sum_{n \geq 0} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} A^n \xi = \sum_{n \geq 0} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \left[\int_{\Delta \cap \{|z| \leq L\}} z^n \mu_\xi(dz) + \int_{\Delta \cap \{|z| > L\}} z^n \mu_\xi(dz) \right].$$

Clearly, to prove (15) it is enough to show

$$\sum_{n \geq 0} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \int_{\Delta \cap \{|z| > L\}} z^n \mu_\xi(dz) = \int_{\Delta \cap \{|z| > L\}} \sum_{n \geq 0} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} z^n \mu_\xi(dz).$$

To do this, let us notice that

$$\sum_{n \geq 0} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \int_{\Delta \cap \{|z| > L\}} z^n \mu_\xi(dz) = \lim_{N \rightarrow \infty} \int_{\Delta} \left[\sum_{n=0}^N \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} z^n \right] \mu_\xi(dz),$$

so, since the measure μ_ξ is orthogonally scattered, it suffices to prove that

$$\int_{\Delta \cap \{|z| > L\}} \left| \sum_{n=0}^N \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} z^n - \sum_{n \geq 0} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} z^n \right|^2 \nu_\xi(dz) \rightarrow 0.$$

But this easily follows from (18) and ν_ξ -integrability of $g(z)$ defined in (16), by the Lebesgue dominated convergence theorem. Lemma 3.1 is proved.

Lemma 3.2. For $k = 1, 2, \dots$, the following equality holds:

$$\sum_{n=0}^{\infty} \frac{t^{n2^{-k}}}{\Gamma(n2^{-k}+1)} z^n = \exp(tz^{2^k}) \left[1 + \sum_{\nu=1}^{2^k-1} \alpha_\nu^{(k)}(z) \int_{[0, z^{2^k} t]} \frac{e^{-v} v^{\nu 2^{-k}-1}}{\Gamma(\nu 2^{-k})} dv \right], \quad (19)$$

where $\alpha_\nu^{(k)}(z) = z^\nu / [z^{2^k \nu}]^{1/2^k} = e^{i\theta_\nu^{(k)}(z)}$.

The functions $\alpha_\nu^{(k)}(z)$ are determined by fixing the rational power $w \rightarrow w^{1/2^k}$ as taking its values in the angle $\{z = re^{i\theta} : r \geq 0, -\pi/2^{k-1} < \theta \leq \pi/2^{k-1}\}$. In particular, $\alpha_1^{(1)}(z) = -1$ for $\operatorname{Re} z < 0$, and $\alpha_\nu^{(k)}(1) = 1$ for $1 \leq \nu \leq 2^k - 1$, $k = 1, 2, \dots$.

P r o o f. For $k = 1$, this lemma has been proved by Włodarski [11]. We follow the way indicated in [11]. Let us note that the function

$$f(t) = \sum_{n \geq 0} \frac{t^{n2^{-k}}}{\Gamma(n2^{-k}+1)} z^n$$

satisfies the differential equation $f'(t) = g(t) + z^{2^k} f(t)$ with

$$g(t) = \sum_{\nu=1}^{2^k-1} \frac{t^{\nu 2^{-k}-1}}{\Gamma(\nu 2^{-k})} z^\nu.$$

This is easily checked. Consequently, we have

$$f(t) = \exp(z^{2^k} t) \left[1 + \int_0^t \exp(-z^{2^k} t) \sum_{\nu=1}^{2^k-1} \frac{t^{\nu 2^{-k}-1}}{\Gamma(\nu 2^{-k})} z^\nu dt \right].$$

The substitution $z^{2^k} t = v$ leads to formula (19).

For the later convenience let us put

$$\varphi_t(z) = e^{-t(1-z^{2^k})}, \quad \beta_t(z) = 1 + \sum_{\nu=1}^{2^k-1} \alpha_\nu^{(k)}(z) \int_{[0, z^{2^k} t]} \frac{e^{-v} v^{\nu 2^{-k}-1}}{\Gamma(\nu 2^{-k})} dv. \quad (20)$$

Then Lemma 3.2 says that we have the following representation of the Borel transform:

$$B_{2^{-k}}(t, \zeta) = 2^{-k} \varphi_t(z) \beta_t(z), \quad \zeta = (z^\nu). \quad (21)$$

Lemma 3.3. *If z is a complex number with $0 < \operatorname{Arg} z < \pi/2$, $\frac{1}{2} \leq |z| \leq 1$, and $\alpha > -1$, then*

$$\left| \int_{[0, zt]} e^{-u} u^\alpha du - \int_{[0, |z|t]} e^{-u} u^\alpha du \right| \leq C e^{-\gamma_0 t} \quad (22)$$

with some $0 < \gamma_0 < \frac{1}{2}$ and $C > 0$ (we can assume $\gamma_0 < \frac{1}{2}$ for a future convenience).

In particular,

$$\int_{[0, zt]} e^{-u} u^\alpha du \longrightarrow \Gamma(\alpha + 1) \quad \text{as } t \rightarrow \infty. \quad (23)$$

We omit a standard proof.

Lemma 3.4. *Let $B = \{|z| \leq 1\} \cap \{|1 - z| < \sigma\}$, where σ is taken small enough to have $\alpha_\nu^{(k)}(z) = 1$, for $\nu = 1, 2, \dots, 2^k - 1$, and $|z|^{2^k} \geq \frac{1}{2}$. Then, for $\zeta = (z^n)$ with $z \in B$ and $\mathbf{1} = (1, 1, \dots)$, the estimation*

$$|B_{2^{-k}}(t, \zeta - \mathbf{1})| \leq C[|1 - z|t + e^{-\gamma t}] \quad (24)$$

holds for some $C > 0$ and $0 < \gamma < \frac{1}{2}$.

P r o o f. For $z \in B$, by Lemma 3.2 applied both to ζ and $\mathbf{1}$, we obtain

$$\begin{aligned} 2^{-k} B_{2^{-k}}(t, \zeta - \mathbf{1}) &= [e^{-t(1-z^{2^k})} - 1] \left[1 + \sum_{\nu=1}^{2^k-1} \frac{1}{\Gamma(\nu 2^{-k})} \int_{[0, z^{2^k} t]} e^{-u} u^{\nu 2^{-k}-1} du \right] \\ &+ \sum_{\nu=1}^{2^k-1} \frac{1}{\Gamma(\nu 2^{-k})} \left[\int_{[0, z^{2^k} t]} e^{-u} u^{\nu 2^{-k}-1} du - \int_{[0, t]} e^{-u} u^{\nu 2^{-k}-1} du \right]. \end{aligned} \quad (25)$$

We have the estimation

$$\left| e^{t(z^{2^k}-1)} - 1 \right| \leq |z^{2^k} - 1| \int_0^t e^{u(\operatorname{Re} z^{2^k}-1)} du \leq C|1 - z|t. \quad (26)$$

Consequently, since $\int_{|z|^{2^k} t} e^{-u} du \leq \int_{t/2}^\infty e^{-u} du \leq C e^{-t/2}$ for $t > 2$ we get (24) by Lemma 3.3, (25), and (26).

Lemma 3.5. *The following estimations hold for the Borel transform $B_{2^{-k}}(t, \zeta)$, $\zeta = (z^k)$.*

(a) *If $\operatorname{Re} z^{2^k} \leq 1 - d$, then*

$$|B_{2^{-k}}(t, \zeta)| \leq C e^{-dt/2} [1 + \max(1, |z|^{2^k})]. \quad (27)$$

(b) *Let $G = \{z = re^{i\theta} : r \geq 0, |\theta| < \pi/2^{k+1}\} \cap \{z : \operatorname{Re} z^{2^k} > 1 - d\} \cap \{z : |z| \leq 1\}$. Then, for $z \in G$*

$$|B_{2^{-k}}(t, \zeta)| \leq C e^{-t|1-z|^{2/2}}. \quad (28)$$

(c) *If $\operatorname{Re} z \leq 1 - d$, then*

$$|B_{2^{-k}}(t, \zeta)| \leq C 2^k e^{-d2^{-k}t}. \quad (29)$$

P r o o f. (a) Let us remark that, for $\operatorname{Re} z < 1$, $t > 1$, and $\beta > -1$, we have that

$$\left| e^{t(z-1)} \int_{[0,zt]} u^\beta e^{-u} du \right| \leq |zt|^{\beta+1} \max(e^{-t}, e^{-t(1-\operatorname{Re} z)}) \quad (30)$$

(a standard proof can be omitted).

Consequently, if $\operatorname{Re} z^{2^k} \leq 1 - d$, we have by Lemma 3.2 and (30) that

$$\begin{aligned} |B_{2^{-k}}(t, \zeta)| &\leq C e^{-td} + \sum_{\nu=1}^{2^k-1} \left| e^{t(z^{2^k}-1)} \int_{[0, z^{2^k} t]} e^{-v} v^{\nu 2^{-k}-1} dv \right| \\ &\leq C e^{-td} + C \left[\sum_{\nu=1}^{2^k-1} |z|^\nu t^{\nu 2^{-k}} \right] \max(e^{-t}, e^{-t(1-\operatorname{Re} z^{2^k})}) \\ &\leq C e^{-td} \left[1 + \sum_{\nu=1}^{2^k-1} |z|^\nu t^{\nu 2^{-k}} \right] \leq C e^{-dt/2} [1 + \max(1, |z|^{2^k})]. \end{aligned}$$

(b) For $z = r e^{i\theta} \in G$, we have $|\theta| < \pi/2^{k+1}$. Thus $\operatorname{Re} z^{2^k} = r^{2^k} \cos 2^k \theta \leq r \cos \theta = \operatorname{Re} z$. Moreover, if $|z| \leq 1$, then $1 - \operatorname{Re} z \geq \frac{1}{2} |1 - z|^2$, and consequently, we get

$$1 - \operatorname{Re} z^{2^k} \geq 1 - \operatorname{Re} z \geq \frac{1}{2} |1 - z|^2.$$

Since $|\alpha_\nu^{(k)}(z)| = 1$, we get (28) by Lemma 3.2 and (23).

(c) We have $B_1(t, \zeta) = e^{-t(1-z)}$. Let us take the following transformation:

$$W(f)(t) = \frac{e^{-t}}{2\sqrt{\pi t}} \int_0^\infty \exp\left(-\frac{u^2}{4t} + u\right) f(u) du$$

(defined, say, for continuous functions $f: (0, \infty) \rightarrow \mathbf{R}$; cf. [11, p. 140]).

As shown in [11], the following formula holds:

$$W(B_{2^{-k}}(\cdot, \zeta)(t)) = B_{2^{-(k+1)}}(\zeta, t), \quad t > 0. \quad (31)$$

Clearly $|W(f)(t)| \leq W(|f|)(t)$, so $|W^k(f)(t)| \leq W^k(|f|)(t)$. Then, for $f(u) = e^{-u(1-z)}$ with $\operatorname{Re} z \leq 1 - d$, we get by (31) that $|B_{2^{-k}}(t, \zeta)| = |W^k(B_1(u, \zeta))(t)| \leq W^k(e^{-du})(t)$.

Let us note that, for $t > 1$,

$$\begin{aligned} 0 \leq W(e^{-du})(t) &\leq \frac{e^{-t}}{2\sqrt{\pi t}} \int_0^{t/2} e^u du + \frac{e^{-t}}{2\sqrt{\pi t}} \int_{t/2}^\infty \exp\left(-\frac{u^2}{4t} + u\right) e^{-du} du \\ &\leq e^{-t/2} + W(1)(t) e^{-dt/2} \leq 2e^{-dt/2}. \end{aligned}$$

Consequently, $W^k(e^{-du}) \leq 2^k e^{-d2^{-k}t}$, and we get (29).

4. Proof of Theorem 2.1. In the proof we follow a general idea of Gaposhkin [6]. Instead of Cesàro means the Borel transforms and their suitable estimations are considered (cf. [9]).

Let us set

$$Z_k = \{z \in \Delta: 2^{-(k+1)} \leq |1 - z| < 2^{-k}\}, \quad k = 0, 1, 2, \dots$$

Then, by (6), we have that $\sum_k \nu_\xi(Z_k) 2^{2k} < \infty$, which implies the μ -a.e. convergence of the orthogonal series $\sum_k E(Z_k) \xi$, and we get the Gaposhkin condition $E\{z \in \Delta: 0 < |1 - z| < 2^{-n}\} \xi \rightarrow 0$ μ -a.e. (the above remark is due to the referee).

Put, for $x = (A^\nu \xi)_{\nu=0}^\infty$,

$$\delta_n = B_{2^{-k}}(2^n, x) - E\{z \in \Delta: |1 - z| < 2^{-n}\} \xi; \quad (32)$$

we shall show that

$$\sum \|\delta_n\|^2 < \infty. \quad (33)$$

Let us write

$$\begin{aligned} \sum_{n=1}^{\infty} \|\delta_n\|^2 &= \sum_{n=1}^{\infty} \int_{\Delta \cap \{0 < |1-z| \leq 2^{-n}\}} |B_{2^{-k}}(2^n, \zeta) - 1|^2 d\nu_{\xi}(z) \\ &\quad + \int_{\Delta \cap \{|1-z| \geq 2^{-n}\}} |B_{2^{-k}}(2^n, \zeta)|^2 d\nu_{\xi}(z). \end{aligned}$$

By Lemma 3.2, we have

$$\begin{aligned} 0 < 1 - B_{2^{-k}}(t, 1) &= 1 - 2^{-k} \left[1 + \sum_{\nu=1}^{2^k-1} \frac{1}{\Gamma(\nu 2^{-k})} \int_0^t e^{-u} u^{\nu 2^{-k}-1} du \right] \\ &= 2^{-k} \sum_{\nu=1}^{2^k-1} \frac{1}{\Gamma(\nu 2^{-k})} \int_t^{\infty} e^{-u} u^{\nu 2^{-k}-1} du \leq C \int_t^{\infty} e^{-u} du \leq C e^{-t}. \end{aligned}$$

Consequently, by Lemma 3.4 (with a fixed σ), assuming that n is large enough, say $n \geq n_0$, to have $|1-z| < 2^{-n} < \sigma$ we can write

$$\begin{aligned} |B_{2^{-k}}(2^n, \zeta) - 1|^2 &\leq 2 \left[|B_{2^{-k}}(2^n, \zeta - 1)|^2 + |B_{2^{-k}}(2^n, 1) - 1|^2 \right] \\ &\leq C[4^n |1-z|^2 + e^{-\gamma 2^n}], \end{aligned}$$

By Lemma 3.5, we have that

$$\begin{aligned} \sum_{n=1}^{\infty} \|\delta_n\|^2 &\leq C \sum_{n=1}^{\infty} \int_{\Delta \cap \{z: |1-z| < 2^{-n}\}} [4^n |1-z|^2 + e^{-\gamma 2^n}] \nu_{\xi}(dz) \\ &\quad + C \sum_{n=1}^{\infty} \int_{\{z: \operatorname{Re} z 2^k \leq 1-d\} \cap \Delta} e^{-d 2^{n-1}} [1 + \max(1, |z|^{2^k})] \nu_{\xi}(dz) \\ &\quad + C \sum_{n=1}^{\infty} \int_{\{z: \operatorname{Re} z \leq 1-d\}} 2^k e^{-d 2^{n-k}} \nu_{\xi}(dz) \\ &\quad + C \sum_{n=1}^{\infty} \int_{\Delta \cap \{z: |1-z| > 2^{-n}\} \cap G} e^{-2^{n-1} |1-z|^2} \nu_{\xi}(dz), \end{aligned}$$

where G is defined in Lemma 3.5 (b).

Finally, taking into account (17), we get

$$\begin{aligned} \sum_{n=1}^{\infty} \|\delta_n\|^2 &\leq C \left[\sum_{n=1}^{\infty} \int_{\Delta \cap \{z: |1-z| < 2^{-n}\}} 4^n |1-z|^2 \nu_{\xi}(dz) \right. \\ &\quad \left. + \sum_{n=1}^{\infty} \int_{\Delta \cap \{z: |1-z| > 2^{-n}\} \cap G} e^{-2^{n-1} |1-z|^2} \nu_{\xi}(dz) + 1 \right]. \end{aligned}$$

Let $m(z)$ be a positive integer such that $|1-z| 2^{m(z)} \leq 1$ and $|1-z| 2^{m(z)+1} > 1$. Put

$$g(z) = |1-z|^2 \sum_{n=1}^{m(z)} 4^n + \sum_{n=m(z)+1}^{\infty} e^{-2^{n-1} |1-z|^2}.$$

To get (33) it is enough to check that the function g is integrable with respect to $\nu_{\xi}(dz)$. The first sum is bounded. The integrability of the second one follows easily from condition (6).

Having (33), to prove our theorem it suffices to show that, for $x = (A^k \xi)$,

$$\sigma_n = \max_{1 \leq m \leq 2^n} |B_{2^{-k}}(2^n + m, x) - B_{2^{-k}}(2^n, x)| \longrightarrow 0 \quad \mu\text{-a.e.} \quad (34)$$

We shall prove more, i.e.,

$$\sum_{n=1}^{\infty} \|\sigma_n\|^2 < \infty. \quad (35)$$

To this end let us put shortly $L_n(z) = B_{2^{-k}}(n, \zeta)$, where $\zeta = (z^\nu)$, and $\sigma_{n,m} = B_{2^{-k}}(2^n + m, x) - B_{2^{-k}}(2^n, x)$.

Writing m in the form $m = \sum_{r=1}^n \varepsilon_r 2^{n-r}$ with ε_r equal to 0 or 1, and applying the dyadic expansion method well known in the theory of orthogonal series [1], we obtain the following representation:

$$\sigma_{n,m} = \sum_{r=1}^n \varepsilon_r H_{r,n}^{(j_r)},$$

where $H_{r,n}^{(j)} = \int_{\Delta} K_{r,n}^{(j)}(z) E(dz) \xi$ with $K_{r,n}^{(j)}(z) = L_{2^n+j2^{n-r}}(z) - L_{2^n+(j-1)2^{n-r}}$, for $r = 1, \dots, n$, $j = 1, \dots, 2^r$.

Thus, we can write

$$\sigma_n = \max_{1 \leq m \leq 2^n} |\sigma_{n,m}| \leq \max_{(j_1, \dots, j_n)} \sum_{r=1}^n |H_{r,n}^{(j_r)}|,$$

where the maximum is taken over all vectors $(j_1, \dots, j_n)$ with different entries in $(1, \dots, 2^r)$.

Consequently, by the Schwarz inequality, we get

$$|\sigma_n|^2 \leq \max_{(j_1, \dots, j_n)} \sum_{r=1}^n |H_{r,n}^{(j_r)}|^2 r^2 \sum_{r=1}^n r^{-2} \leq 2 \sum_{r=1}^n r^2 \sum_{j=1}^{2^r} |H_{r,n}^{(j)}|^2,$$

so, finally,

$$\sum_{n=1}^{\infty} \|\sigma_n\|^2 \leq 2 \sum_{n=1}^{\infty} \sum_{r=1}^n r^2 \sum_{j=1}^{2^r} \|H_{r,n}^{(j)}\|^2.$$

One can easily show that by Lemma 3.5 and (17) to get (35) it suffices to prove that

$$\sum_{n=1}^{\infty} \sum_{r=1}^n r^2 \sum_{j=1}^{2^r} \int_G |K_{r,n}^{(j)}(z)|^2 d\nu_{\xi}(z) < \infty, \quad (36)$$

where $G = \{z = re^{i\theta} : |\theta| < \pi/2^{k+1}, r \geq 0\} \cap \{z : \operatorname{Re} z^{2^k} > 1 - d\} \cap \{z : |z| \leq 1\}$.

Putting $m_1 = 2^n + (j-1)2^{n-r}$, $m_2 = 2^n + j2^{n-r}2^r$, where $r = 1, \dots, n$, $j = 1, \dots, 2^r$, we write shortly $K_{r,n}^{(j)} = L_{m_2} - L_{m_1}$. By Lemma 3.2, we have $L_t(z) = 2^{-k} \varphi_t(z) \beta_t(z)$, where φ_t and β_t are given by (20).

In all further considerations, $z \in G$. For $t_1 < t_2$, we have the following estimations:

$$|\varphi_{t_2}(z) - \varphi_{t_1}(z)| \leq C|1-z|(t_2 - t_1), \quad |\varphi_{t_2}(z) - \varphi_{t_1}(z)| \leq \frac{C}{|1-z|} \frac{t_2 - t_1}{t_1}. \quad (37)$$

Indeed,

$$|\varphi_{t_2}(z) - \varphi_{t_1}(z)| \leq |1 - z^{2^k}| \int_{t_1}^{t_2} e^{-s(1 - \operatorname{Re} z^{2^k})} ds \leq C|1-z|(t_2 - t_1) e^{-t_1(1 - \operatorname{Re} z^{2^k})}, \quad (38)$$

and (36) follows. We get (37) from the last inequality noting that $1 - \operatorname{Re} z^{2^k} \geq \frac{1}{2}|1-z|^2$ (compare the proof of Lemma 3.5 (b)).

Lemma 3.3 leads us easily to the estimation, for $z \in G$

$$\left| \int_{[0, z^{2^k} t_2]} e^{-u} u^{\nu 2^{-k}-1} du - \int_{[0, z^{2^k} t_1]} e^{-u} u^{\nu 2^{-k}-1} du \right| \leq C e^{-\gamma t_1},$$

with some $0 < \gamma < \frac{1}{2}$.

Consequently,

$$|\beta_{t_2}(z) - \beta_{t_1}(z)| \leq C e^{-\gamma t_1} \quad \text{for } z \in G. \quad (39)$$

Let us write

$$\begin{aligned} \int_G |K_{r,n}^{(j)}|^2 d\nu_{\xi} &= \int_{G \cap \{|1-z| \leq 2^{-n}\}} |L_{m_2} - L_{m_1}|^2 d\nu_{\xi} \\ &+ \int_{G \cap \{2^{-n} < |1-z| \leq 2^{-(n-r)}\}} |L_{m_2} - L_{m_1}|^2 d\nu_{\xi} \\ &+ \int_{G \cap \{2^{-(n-r)} < |1-z| \leq \frac{1}{2}\}} |L_{m_2} - L_{m_1}|^2 d\nu_{\xi} = S_1 + S_2 + S_3. \end{aligned}$$

Let us put $\mathbb{Z}_m = \{z: 2^{-(m+1)} < |1-z| \leq 2^{-m}\}$ and $\tilde{\nu}_\xi(dz) = |1-z|^{-2} \nu_\xi(dz)$.

Since β_t and φ_t are bounded on G we can write $|L_{t_2} - L_{t_1}| \leq C[|\varphi_{t_2} - \varphi_{t_1}| + |\beta_{t_2} - \beta_{t_1}|]$. Using (37) and (39), we get

$$S_1 \leq C 2^{2n-2r} \sum_{k=n}^{\infty} 2^{-2k} \nu_\xi(\mathbb{Z}_k) + C e^{-\gamma 2^n}, \quad (40)$$

$$S_2 \leq C 2^{-2r} \sum_{k=n-r}^n \tilde{\nu}_\xi(\mathbb{Z}_k) + C e^{-\gamma 2^n}. \quad (41)$$

Since, for $z \in G$, $|L_{t_2} - L_{t_1}| \leq |L_{t_1}| + |L_{t_2}| \leq C e^{-t_1(1-\operatorname{Re} z 2^{2^k})} \leq C/(t_1|1-z|^2)$, we have

$$S_3 \leq C 2^{-2n} \sum_{k=1}^{n-r} 2^{2k} \tilde{\nu}_\xi(\mathbb{Z}_k). \quad (42)$$

Clearly, to prove (36) it suffices to show that

$$\sum_{n=1}^{\infty} \sum_{r=1}^n r^2 2^r (S_1 + S_2 + S_3) < \infty.$$

It can be done by standard calculations. For example, for S_3 ,

$$\begin{aligned} \sum_{n=1}^{\infty} 2^{-2n} \sum_{r=1}^n r^2 2^r \sum_{k=1}^{n-r} 2^{2k} \tilde{\nu}_\xi(\mathbb{Z}_k) &\leq \sum_{n=1}^{\infty} 2^{-2n} \sum_{k=1}^n 2^{2k} \tilde{\nu}_\xi(\mathbb{Z}_k) \sum_{r=1}^{n-k} r^2 2^r \\ &\leq C \sum_{n=1}^{\infty} 2^{-2n} \sum_{k=1}^n 2^{2k} \tilde{\nu}_\xi(\mathbb{Z}_k) (n-k)^3 2^{n-k} \leq C \sum_{k=1}^{\infty} 2^k \tilde{\nu}_\xi(\mathbb{Z}_k) \sum_{n=k}^{\infty} (n-k)^3 2^{-n} \\ &\leq C \sum_{k=1}^{\infty} 2^k \tilde{\nu}_\xi(\mathbb{Z}_k) \sum_{s=1}^{\infty} s^3 2^{-s} 2^{-k} \leq C \sum_{k=1}^{\infty} \tilde{\nu}_\xi(\mathbb{Z}_k) < \infty. \end{aligned}$$

We leave to the reader the estimations for S_1 and S_2 . They can also be found in [6]. The proof is completed.

5. Proof of Theorem 2.2. Let us remark that, for $z \in Q(\alpha_0) \cap \{z: |z| \leq 1\}$, we have

$$\frac{1 - \operatorname{Re} z}{|1 - z|} \geq \cos \alpha_0 = b_0 > 0. \quad (43)$$

In connection with this we reformulate Lemma 3.5 (b) as follows.

Lemma 5.1. Let $G_0 = G \cap Q(\alpha_0)$, G being defined in Lemma 3.5 (b), that is,

$$G_0 = \left\{ z = r e^{i\theta} : r \leq 0, |\theta| < \frac{\pi}{2^{k+1}} \right\} \cap \{\operatorname{Re} z 2^{2^k} > 1 - d\} \cap \{z: |z| \leq 1\} \cap Q(\alpha_0). \quad (44)$$

For $z \in G_0$ we have the estimation

$$|B_{2^{-k}}(t, \zeta)| \leq C e^{-b_0 t |1-z|}, \quad (45)$$

where $b_0 = \cos \alpha_0$ (see (43)).

P r o o f. Indeed, by (43) we have $1 - \operatorname{Re} z \geq b_0 |1 - z|$ and the rest of the argument is the same as for Lemma 3.5 (b).

Now the proof of Theorem 2.2 can be obtained by a fairly easy modification of that of Theorem 2.1. Roughly speaking, we can just follow the previous proof but we have to be careful with the estimations of $B_\alpha(t, \zeta)$ on the set G_0 , related to (43) and (45). In the sequel we shall use all the auxiliary results collected in Section 3 with only one exception: Lemma 5.1 will be used instead of Lemma 3.5 (b). Throughout, we shall keep the notation used in the proof of Theorem 2.1. In particular, we put $x = (A^k \xi)$ and set

$$\delta_n = B_{2^{-k}}(2^n, x) - E\{z \in \Delta: |1-z| < 2^{-n}\} \xi.$$

We get (33) in the same way as before but using the estimation (45) on the set $\Delta \cap \{z: |1-z| > 2^{-n}\} \cap G_0$, G_0 being defined by (44). In particular, letting $m(z)$ be a positive integer such that

$$|1-z| 2^{m(z)} \leq 1 \quad \text{and} \quad |1-z| 2^{m(z)+1} > 1,$$

and putting

$$g(z) = |1 - z|^2 \sum_{n=1}^{m(z)} 4^n + \sum_{n=m(z)+1}^{\infty} e^{-b_0 2^n |1-z|},$$

we get

$$\sum_{n=m(z)+1}^{\infty} e^{-b_0 2^n |1-z|} \leq \sum_{n=m(z)+1}^{\infty} \frac{1}{b_0 2^n |1-z|} \leq \frac{1}{b_0 2^{m(z)+1} |1-z|} \sum_{k=0}^{\infty} 2^{-k} \leq \frac{2}{b_0}$$

since $1/(|1-z|2^{m(z)+1}) \leq 1$, by definition of $m(z)$.

Now, as before, we have to show (35). Keeping consequently the previous notation and repeating the argument, we can see that it is enough to prove (instead of (36)) the estimation

$$\sum_{n=1}^{\infty} \sum_{r=1}^n r^2 \sum_{j=1}^{2^r} \int_{G_0} |K_{r,n}^{(j)}(z)|^2 \nu_{\xi}(dz) < \infty, \quad (46)$$

where G_0 is defined by (44).

Let us note that for $z \in G_0$ and $t_1 < t_2$ we have (instead of (37))

$$\begin{aligned} |\varphi_{t_2}(z) - \varphi_{t_1}(z)| &\leq C|1-z|(t_2 - t_1), \\ |\varphi_{t_2}(z) - \varphi_{t_1}(z)| &\leq C \frac{t_2 - t_1}{t_1} \end{aligned} \quad (47)$$

(by (38) and Lemma 5.1).

The estimation (39) for β_t with some $0 < \gamma < \frac{1}{2}$ holds for $z \in G \supset G_0$.

We are in a position to estimate

$$\begin{aligned} \int_{G_0} |K_{r,n}^{(j)}|^2 d\nu_{\xi} &= \int_{G_0 \cap \{|1-z| \leq 2^{-n}\}} |L_{m_2} - L_{m_1}|^2 d\nu_{\xi} \\ &+ \int_{G_0 \cap \{2^{-n} < |1-z| \leq 2^{-(n-r)}\}} |L_{m_2} - L_{m_1}|^2 d\nu_{\xi} \\ &+ \int_{G_0 \cap \{2^{-(n-r)} < |1-z| \leq \frac{1}{2}\}} |L_{m_2} - L_{m_1}|^2 d\nu_{\xi} = S_1 + S_2 + S_3. \end{aligned}$$

Note, that β_t and φ_t are bounded on G_0 , so

$$|L_{t_2} - L_{t_1}| \leq C[|\varphi_{t_2} - \varphi_{t_1}| + |\beta_{t_2} - \beta_{t_1}|].$$

Let us put

$$Z_m = \{z: 2^{-(m+1)} < |1-z| \leq 2^{-m}\}.$$

Using (47) and (35) we get

$$\begin{aligned} S_1 &\leq C 2^{2n-2r} \sum_{k=n}^{\infty} 2^{-2k} \nu_{\xi}(Z_k) + C e^{-\gamma 2^n}, \\ S_2 &\leq C 2^{-2r} \sum_{k=n-r}^n \nu_{\xi}(Z_k) + C e^{-\gamma 2^n}, \quad S_3 \leq C 2^{-2n} \sum_{k=1}^{n-r} 2^{2k} \nu_{\xi}(Z_k), \end{aligned}$$

since $|L_{t_2} - L_{t_1}| \leq C e^{-t_1(1-\operatorname{Re} z^{2^k})} \leq C/(t_1|1-z|)$.

Finally, it remains to prove that

$$\sum_{n=1}^{\infty} \sum_{r=1}^n r^2 2^r (S_1 + S_2 + S_3) < \infty,$$

which can be done by standard calculations.

6. Comments. We use the notation of the previous sections without any reference.

6.1. In Theorems 2.1 and 2.2, for an unbounded operator A it is not enough to assume that $\xi \in \bigcap_{n=1}^{\infty} D(A^n)$. The condition of analyticity of ξ cannot be dropped because it guarantees the existence of the Borel transform associated to $(A^n\xi)$. Let us consider the following natural example. Take the self-adjoint operator $A = -\frac{d^2}{dx^2}$ in a Hilbert space $L_2(0, 2\pi)$ with a usual domain $D(A)$. Clearly, $\Delta \subset (-\infty, 0)$. For A , there are functions $f \in C^\infty(0, 2\pi) = \bigcap_{n=1}^{\infty} D(A^n)$ such that

$$\liminf_{n \rightarrow \infty} \frac{\|A^n f\|}{n!} = \liminf_{n \rightarrow \infty} \left\| \frac{f^{(2n)}}{n!} \right\|_{L_2(0, 2\pi)} \geq a(f) > 0$$

(cf. [12]), so the Borel transform of $(A^n f)$ in Corollary 2.1 can not even be defined for such f .

6.2. Borel methods are very efficient for some rapidly divergent sequences, like $\{z^k\}$ with large $|z|$ (see also [8]). That is why they have been applied in Theorems 2.1 and 2.2 for unbounded normal operators. It is worth noting here that sometimes the Borel methods are *less efficient* than the arithmetic means. This is the case in a very natural context of the strong law of large numbers. Namely, for a sequence of independent identically distributed random variables (X_k) , the limit $B_1\text{-}\lim_k X_k = \mathbf{E}X_1$ exists almost everywhere if and only if the *second moment* $\mathbf{E}|X_1|^2$ is finite (cf. [2], [3]). A very restrictive assumption (6) in Theorem 2.1 is connected with a kind of inefficiency of Borel summability methods for normal operators with spectrum close to the arc of the circle $\{|z| = 1\}$ situated near $z = 1$. We leave aside the question of comparison of Borel and Cesàro means for (even contractive) normal operators. In our opinion, the full solution of this problem is difficult and deserves some investigations. We confine ourselves to the following example showing that the method used in the proof of Theorem 2.1 cannot be applied for contractive normal operators when the condition (6) is not satisfied. Let us take a normal operator of the form

$$A = \sum_{k=1}^{\infty} e^{i\theta_k} \widehat{\psi}_k,$$

where $0 < \theta_k < \pi/2$ with $\theta_k \downarrow 0$, and (ψ_k) is a trigonometric system, say, in $\mathbb{L}_2(0, 2\pi)$ and $\widehat{\psi}_k$ denotes the projection $f \rightarrow (f, \psi_k) \psi_k$, $f \in \mathbb{L}_2$. Then, clearly, for every $f \in \mathbb{L}_2$, $(C, 1)\text{-}\lim A^n f = 0$ almost everywhere (since the Gaposhkin condition is satisfied). At the same time, for $f \in \mathbb{L}_2$ we can fix (θ_n) in such a way that $\sum_n \|\delta_n\|^2 = \infty$, with (δ_n) defined in (32). Indeed, let us fix a function $f \in \mathbb{L}_2$, with $a_k = (f, \psi_k) \neq 0$. Moreover, let $m(k)$ and $M(k)$ be such integers that, putting $z_k = e^{i\theta_k}$, we have

$$\begin{aligned} 2^n |1 - z_k| \leq 1 & \quad \text{for } n \leq m(k), & 2^n |1 - z_k| > 1 & \quad \text{for } n > m(k), \\ 2^n |1 - z_k|^2 \leq 1 & \quad \text{for } n \leq M(k), & 2^n |1 - z_k|^2 > 1 & \quad \text{for } n > M(k). \end{aligned}$$

Let us fix $z_k = e^{i\theta_k}$ in such a way that, for $k = 1, 2, \dots$, we have

$$\dots \leq m(k) < M(k) \leq m(k+1) < M(k+1) \leq \dots$$

and $(M(k) - m(k))|a_k|^2 \geq 1$. Then, for (δ_n) appearing in formula (30), we get

$$\sum_n \|\delta_n\|^2 \geq \sum_{k=1}^{\infty} \sum_{m(k) < n \leq M(k)} |a_k|^2 e^{-2^n |1 - z_k|^2} \geq \sum_{k=1}^{\infty} (M(k) - m(k)) e^{-1} |a_k|^2 = +\infty.$$

We conclude this section with the following conjecture.

Conjecture. There exist a normal operator in $\mathbb{L}_2(0, 1)$ of the form

$$A = \sum_{k=1}^{\infty} e^{i\theta_k} \widehat{\psi}_k,$$

where (ψ_k) is an orthonormal system, $0 < \theta_k < \pi/2$, $\theta_k \downarrow 0$, and a function $f \in \mathbb{L}_2(0, 1)$ such that

$$\frac{1}{n} \sum_{k=1}^n A^k f \rightarrow 0 \quad \text{a.e.}$$

and

$$e^{-m} \sum_{k=1}^{\infty} \frac{m^k}{k!} A^k f \quad \text{does not converge}$$

as $m \rightarrow \infty$ on the set of positive Lebesgue measure.

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ON ALMOST SURE BEHAVIOR OF STABLE SUBORDINATORS OVER RAPIDLY INCREASING SEQUENCES¹⁾

Пусть $(X(t), t \geq 0)$ с $X(0) = 0$ — α -устойчивый субординатор, $0 < \alpha < 1$, и пусть (t_k) — возрастающая последовательность такая,

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