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and

$$e^{-m} \sum_{k=1}^{\infty} \frac{m^k}{k!} A^k f \quad \text{does not converge}$$

as $m \rightarrow \infty$ on the set of positive Lebesgue measure.

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ON ALMOST SURE BEHAVIOR OF STABLE SUBORDINATORS OVER RAPIDLY INCREASING SEQUENCES¹⁾

Пусть $(X(t), t \geq 0)$ с $X(0) = 0$ — α -устойчивый субординатор, $0 < \alpha < 1$, и пусть (t_k) — возрастающая последовательность такая,

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что $t_{k+1}/t_k \rightarrow \infty$ при $k \rightarrow \infty$. Введем последовательность (a_t) положительных неубывающих функций от t таких, что $a(t)/t \leq 1$, и определим $Y(t) = X(t + a(t)) - X(t)$ и $Z(t) = X(t) - X(t - a(t))$, $t > 0$. Для $(X(t_k))$, $(Y(t_k))$ и $(Z(t_k))$, соответствующим образом нормированных, получены результаты типа закона повторного логарифма.

Ключевые слова и фразы: закон повторного логарифма, подпоследовательности, устойчивые субординаторы, оценки почти наверное.

1. Introduction. Let $W(t)$, $t \geq 0$, denotes a standard Wiener process. If (t_k) is such that $\limsup_{k \rightarrow \infty} (t_{k+1}/t_k) < \infty$, then proceeding as in [1] one can show that $\limsup_{k \rightarrow \infty} (W(t_k)/\sqrt{2t_k \ln \ln t_k}) = 1$ a.s. But for some sequences (t_k) with $t_{k+1}/t_k \rightarrow \infty$ we have $\lim_{k \rightarrow \infty} (W(t_k)/\sqrt{2t_k \ln \ln t_k}) = 0$ a.s. In such cases the normalizing sequence $\sqrt{2t_k \ln \ln t_k}$ will not be precise enough to give a.s. bounds for $W(t_k)$. In general whenever $t_{k+1}/t_k \rightarrow \infty$, R. Schwabe and A. Gut [5] have pointed out that $\sqrt{2t_k \ln \ln t_k}$ is no longer the proper normalizing sequence and it has to be replaced by $\sqrt{2t_k \ln k}$. (These results have been obtained by the above authors for partial sums of independent and identically distributed random variables with finite variance.) This observation motivated us to examine whether similar things happen in the case of stable subordinators. The answer turns out to be affirmative as established in the next section.

We first present the following result from [6], on the behavior of the limit supremum of $(X(t_k)/t_k^{1/\alpha})^{1/\ln \ln t_k}$ and limit infimum of $(X(t_k)/(t_k^{1/\alpha}(\ln \ln t_k)^{(\alpha-1)/\alpha}))$ for sequences (t_k) , when (t_k) is at most geometrically increasing and when (t_k) is at least geometrically increasing. The case $t_{k+1}/t_k \rightarrow \infty$ comes under the class of at least geometrically increasing sequences.

Theorem A. Let $X(t)$ denote a stable subordinator with index α , $0 < \alpha < 1$. Define $\theta_\alpha = \alpha(1-\alpha)^{(1-\alpha)/\alpha}(\cos(\pi\alpha/2))^{-1/\alpha}$. If $\limsup_{k \rightarrow \infty} (t_{k+1}/t_k) < \infty$, then

$$\limsup_{k \rightarrow \infty} \left(\frac{X(t_k)}{t_k^{1/\alpha}} \right)^{1/\ln \ln t_k} = e^{1/\alpha} \quad \text{a.s.}$$

and

$$\liminf_{k \rightarrow \infty} \frac{X(t_k)}{t_k^{1/\alpha} (\ln \ln t_k)^{(\alpha-1)/\alpha}} = \theta_\alpha \quad \text{a.s.}$$

If $\liminf_{k \rightarrow \infty} t_{k+1}/t_k > 0$, then

$$\limsup_{k \rightarrow \infty} \left(\frac{X(t_k)}{t_k^{1/\alpha}} \right)^{1/\ln \ln t_k} = e^{\lambda/\alpha} \quad \text{a.s.,}$$

where $\lambda = \inf\{\varepsilon > 0: \sum_{k=k_0}^{\infty} (\ln t_k)^{-\varepsilon} < \infty\}$, and

$$\liminf_{k \rightarrow \infty} \frac{X(t_k)}{t_k^{1/\alpha} (\ln \ln t_k)^{(\alpha-1)/\alpha}} = \lambda \theta_\alpha \quad \text{a.s.,}$$

where $\lambda = \inf\{\varepsilon > 0: \sum_{k=k_0}^{\infty} (\ln t_k)^\beta < \infty, \beta = \varepsilon^{\alpha/(\alpha-1)}\}$.

Remark. When (t_{k+1}/t_k) is at most geometrically fast we notice that both the limit supremum and the limit infimum remain unchanged, whereas when (t_{k+1}/t_k) is more than geometrically fast the limit supremum and the limit infimum both change depending on the speed of (t_k) . The limit supremum result depends on the right (heavy) tail of the process whereas limit infimum result depends on the tail near zero, which

is exponentially fast. When $t_k = 2^{k^2}$ (k times) one can show that limit supremum becomes 1 (the limit infimum of the same function is always 1 for any (t_k)) and the limit infimum (or limit) becomes ∞ . In [6] it has been observed that over any $(t_k) \uparrow \infty$, $\liminf_{k \rightarrow \infty} (X(t_k)/t_k^{1/\alpha})^{1/\ln \ln t_k} = 1$ a.s. and $\limsup_{k \rightarrow \infty} X(t_k)/t_k^{1/\alpha} (\ln \ln t_k)^{(\alpha-1)/\alpha} = \infty$ a.s. In [6] it was also shown that for certain sequences (t_k) that are faster than geometric the iterated logarithm results can still be obtained by replacing $\langle \ln \ln \rangle$ by $\langle \ln \ln \ln \rangle$. By following [5], we now obtain a.s. results with $\ln k$ in place of $\ln \ln t_k$ whenever $t_{k+1}/t_k \rightarrow \infty$. We also obtain similar results for $(Y(t_k))$ and $(Z(t_k))$.

2. Main results.

Theorem 1. *Let (t_k) be such that $t_{k+1}/t_k \rightarrow \infty$ as $k \rightarrow \infty$. Then*

$$(A) \quad \limsup_{k \rightarrow \infty} \left(\frac{X(t_k)}{t_k^{1/\alpha}} \right)^{1/\ln k} = e^{1/\alpha} \quad a.s.$$

Moreover, if $t_{k+1}/t_k \geq k^{(1+\delta)}$, for some $\delta > 0$, then

$$(B) \quad \liminf_{k \rightarrow \infty} \frac{X(t_k)}{t_k^{1/\alpha} (\ln k)^{(\alpha-1)/\alpha}} = \theta_\alpha \quad a.s.$$

P r o o f. We first establish (A). Observe that $t^{-1/\alpha}X(t) = X(1)$, in distribution, where $X(1)$ is a positive stable random variable with index α . Hence for any given $\varepsilon > 0$, by the tail behavior of $X(1)$, we have

$$\mathbf{P}\{X(t_k) > t_k^{1/\alpha} k^{(1+\varepsilon)/\alpha}\} < Ck^{-(1+\varepsilon)}.$$

By the Borel–Cantelli lemma,

$$\mathbf{P}\{X(t_k) > t_k^{1/\alpha} k^{(1+\varepsilon)/\alpha} \text{ i.o.}\} = 0 \quad (1)$$

(here «i.o.» stands for «infinitely often»). Also, using the fact that $t_{k+1}/t_k \rightarrow \infty$ as $k \rightarrow \infty$ one can show that for large k , $(X(t_{k+1}) - X(t_k))/t_k^{1/\alpha}$ is distributionally equivalent to $X(1)$, and hence

$$\mathbf{P}\{X(t_k) - X(t_{k-1}) > t_k^{-1/\alpha} k^{(1-\varepsilon)/\alpha}\} \geq Ck^{-(1-\varepsilon)}.$$

Since $(X(t_k) - X(t_{k-1}))$ are mutually independent and $\sum_{k=1}^{\infty} k^{-(1-\varepsilon)} = \infty$, again by applying the Borel–Cantelli lemma one gets that

$$\mathbf{P}\{X(t_k) - X(t_{k-1}) > t_k^{-1/\alpha} k^{(1-\varepsilon)/\alpha} \text{ i.o.}\} = 1. \quad (2)$$

From the fact that $X(t)$ is increasing we have $X(t_k) > X(t_k) - X(t_{k-1})$. Hence (2) implies that

$$\mathbf{P}\{X(t_k) > t_k^{-1/\alpha} k^{(1-\varepsilon)/\alpha} \text{ i.o.}\} = 1. \quad (3)$$

Now (1) and (3) together establish (A) of the theorem.

For any $\varepsilon > 0$, we have

$$\begin{aligned} \mathbf{P}\{t_k^{-1/\alpha} X(t_k) < (1 - \varepsilon) \theta_\alpha (\ln k)^{(\alpha-1)/\alpha}\} &\leq C(\ln k)^{-(1-\alpha)/2\alpha} \exp\{-(1 + \varepsilon_1) \ln k\} \\ &= C(\ln k)^{-(1-\alpha)/\alpha} k^{-(1+\varepsilon)} \end{aligned}$$

for some $\varepsilon_1 > 0$. Hence by the Borel–Cantelli lemma one gets

$$\mathbf{P}\{t_k^{-1/\alpha} X(t_k) < (1 - \varepsilon) \theta_\alpha (\ln k)^{-(1-\alpha)/\alpha} \text{ i.o.}\} = 0. \quad (4)$$

Again recalling that for large k , $t_k^{-1/\alpha}(X(t_k) - X(t_{k-1}))$ has the same distribution as $X(1)$, one gets

$$\mathbf{P}\{t_k^{-1/\alpha}(X(t_k) - X(t_{k-1})) < (1 + \varepsilon) \theta_\alpha (\ln k)^{-(1-\alpha)/\alpha}\} \geq Ck^{-(1-\varepsilon)}.$$

By the Borel–Cantelli lemma, we have

$$\mathbf{P}\{t_k^{-1/\alpha}(X(t_k) - X(t_{k-1})) < (1 + \varepsilon) \theta_\alpha (\ln k)^{-(1-\alpha)/\alpha} \text{ i.o.}\} = 1. \quad (5)$$

Also,

$$\begin{aligned} \mathbf{P}\{t_k^{-1/\alpha} X(t_{k-1}) > \varepsilon \theta_\alpha (\ln k)^{-(1-\alpha)/\alpha}\} \\ \leq \mathbf{P}\{t_{k-1}^{-1/\alpha} X(t_{k-1}) > \varepsilon t_k^{1/\alpha} t_{k-1}^{-1/\alpha} \theta_\alpha (\ln k)^{-(1-\alpha)/\alpha}\} \leq Ck^{-(1+\delta)}, \end{aligned}$$

for some $\delta > 0$. By the Borel–Cantelli lemma one gets

$$\mathbf{P}\{t_k^{-1/\alpha} X(t_{k-1}) > \varepsilon \theta_\alpha (\ln k)^{-(1-\alpha)/\alpha} \text{ i.o.}\} = 0. \quad (6)$$

Relations (4), (5), and (6) together complete the proof of (B).

3. Boundary crossings associated with the law of the iterated logarithm.

Define, for any $\varepsilon > 0$,

$$U_k = \begin{cases} 1 & \text{if } \left(\frac{X(t_k)}{t_k^{1/\alpha}} \right)^{1/\ln k} > e^{(1+\varepsilon)/\alpha}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $N_\varepsilon = \sum_{k=1}^\infty U_k$. Note that N_ε is a proper random variable giving the number of boundary crossings of $(X(t_k)/t_k^{1/\alpha})^{1/\ln k}$. Then $\mathbf{E}N_\varepsilon = \sum_{k=1}^\infty \mathbf{P}\{U_k = 1\} \leq \sum_{k=1}^\infty k^{-(1+\varepsilon)} < \infty$. Hence the expected number of boundary crossings is finite. Similarly, if N_ε^* is the number of crossings of the lower boundary $(1 - \varepsilon)\theta_\alpha$ of the sequence $(X(t_k)/t_k^{1/\alpha}(\ln k)^{(\alpha-1)/\alpha})$, then one can show that $\mathbf{E}N_\varepsilon^* < \infty$.

We now discuss the behavior of $(Y(t_k))$ when (t_k) satisfies $t_{k+1}/t_k \rightarrow \infty$ as $k \rightarrow \infty$. First we present the following known result for $(Y(t))$ as $t \rightarrow \infty$.

Theorem B. Let $d(t) = \ln(t/a_t) + \ln \ln t$. Then

$$\mathbf{P}\left\{\limsup \left(\frac{Y(t)}{a_t^{1/\alpha}} \right)^{1/d(t)} = e^{1/\alpha}\right\} = 1 \quad \text{and} \quad \mathbf{P}\left\{\liminf \left(\frac{Y(t)}{a_t^{1/\alpha}} \right)^{1/d(t)} = 1\right\} = 1.$$

For proof see [6].

R e m a r k. Let $t_k = \rho^k$, $\rho > 1$. When $a_t = t^{1/2}$, we note that $d(t) \sim (\ln t)/2$ and $\lim_{k \rightarrow \infty} (Y(t_k)/a_{t_k}^{1/\alpha})^{1/d(t_k)} = 1$ a.s. But if $a_t = t/2$, then $d(t) \sim \ln \ln t$ and $\limsup_{k \rightarrow \infty} (Y(t_k)/a_{t_k}^{1/\alpha})^{1/d(t_k)} = e^{1/\alpha}$ a.s. Hence for a geometrically increasing (t_k) , the behavior of $(Y(t_k))$ changes with the form of a_t . However, for (t_k) such that $t_{k+1}/t_k \rightarrow \infty$ as $k \rightarrow \infty$ we have a unified result as presented below.

Theorem 2. Let (t_k) be such that $t_{k+1}/t_k \rightarrow \infty$ as $k \rightarrow \infty$. Then

$$\mathbf{P}\left\{\limsup \left(\frac{Y(t_k)}{a_{t_k}^{1/\alpha}} \right)^{1/\ln k} = e^{1/\alpha}\right\} = 1 \quad \text{and} \quad \mathbf{P}\left\{\liminf \left(\frac{Y(t_k)}{a_{t_k}^{1/\alpha}} \right)^{1/\ln k} = 1\right\} = 1.$$

Further, all points in $[1, e^{1/\alpha}]$ are a.s. limit points of $(Y(t_k)/a_{t_k}^{1/\alpha})^{1/\ln k}$.

P r o o f. From the fact that $a_t < t$, note that $t_{k+1}/t_k \rightarrow \infty$ as $k \rightarrow \infty$ implies that $t_{k+1}/(t_k + a_{t_k}) \rightarrow \infty$ as $k \rightarrow \infty$ and hence for k large $(Y(t_k))$ are mutually independent. From the fact that $Y(t)/a_t^{1/\alpha}$ is distributionally same as $X(1)$ the limit supremum and limit infimum results in the theorem follow by a straightforward application of the Borel–Cantelli lemma.

A point e^p , belonging to $[1, e^{1/\alpha}]$, is a limit point, if for any $\varepsilon > 0$, $\mathbf{P}\{k^{(p-\varepsilon)/\alpha} < Y(t_k)/a_{t_k}^{1/\alpha} < k^{(p+\varepsilon)/\alpha} \text{ i.o.}\} = 1$. From the fact that $\mathbf{P}\{k^{(p-\varepsilon)/\alpha} < Y(t_k)/a_{t_k}^{1/\alpha} < k^{(p+\varepsilon)/\alpha}\} \geq k^{-(p-\varepsilon)}$ and $\sum_{k=1}^\infty k^{-(p-\varepsilon)} = \infty$, by applying the Borel–Cantelli lemma we note that e^p is an a.s. limit point of $(Y(t_k)/a_{t_k}^{1/\alpha})^{1/\ln k}$. The proof of the theorem is complete.

Theorem 3. Let (t_k) be such that $t_{k+1}/t_k \rightarrow \infty$ as $k \rightarrow \infty$. Then

$$\mathbf{P}\left\{\limsup_{k \rightarrow \infty} \frac{Y(t_k)}{a_{t_k}^{1/\alpha}(\ln k)^{(\alpha-1)/\alpha}} = \infty\right\} = 1 \quad \text{and} \quad \mathbf{P}\left\{\liminf_{k \rightarrow \infty} \frac{Y(t_k)}{a_{t_k}^{1/\alpha}(\ln k)^{(\alpha-1)/\alpha}} = \theta_\alpha\right\} = 1.$$

The proof follows the lines of Theorem 2, but by considering the tail behavior near zero. The details are omitted.

Define

$$Q_k = \frac{1}{a_{t_k}^{1/\alpha}} \left((Y(t_k))^{1/\ln k}, (Z(t_k))^{1/\ln k} \right), \quad R_k = \frac{1}{a_{t_k}^{1/\alpha}(\ln k)^{(\alpha-1)/\alpha}} (Y(t_k), Z(t_k))$$

and introduce the sets $S_1 = \{(x, y): x \geq 0, y \geq 0, xy \leq e^{1/\alpha}\}$ and $S_2 = \{(x, y): x \geq 0, y \geq 0, xy \geq \theta_\alpha\}$. Then we have the following result.

Theorem 4. Let $t_{k+1}/t_k \rightarrow \infty$ as $k \rightarrow \infty$. Then set of all a.s. limit points of (Q_k) coincides with S_1 and that of (R_k) coincides with S_2 .

P r o o f. For k large, $(Y(t_k), Z(t_k))$ becomes a mutually independent sequence with independent components. The proof follows by proceeding the lines of [4]. The details are omitted.

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