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METRIC MULTIVARIATE POISSON APPROXIMATION OF THE GENERALIZED MULTINOMIAL DISTRIBUTION

Цель данной статьи — использовать многомерное В-разложение Шарлье для многомерной метрической пуассоновской аппроксимации обобщенного полиномиального распределения (см. [3] и [12]). Найдены границы для полной вариации и точечной метрики.

Ключевые слова и фразы: пуассоновская аппроксимация, обобщенное полиномиальное распределение, многомерное В-разложение Шарлье, полная вариация, точечная метрика.

1. Introduction. Let $X_1, \dots, X_n$ ($n \in \mathbf{N} = \{1, 2, \dots\}$) be independent random vectors of $\mathbf{R}^k$, $k \in \mathbf{N}$, where $X_j = (X_j(1), \dots, X_j(k))$, $\mathbf{P}\{X_j = e_r\} = p_{j,r} \in [0, 1]$, and $\mathbf{P}\{X_j = 0\} = 1 - \sum_{r=1}^k p_{j,r} \in [0, 1]$, for $j \in \{1, \dots, n\}$ and $r \in \{1, \dots, k\}$. Here, e_r is the vector in $\mathbf{R}^k$ with 1 at position $r \in \{1, \dots, k\}$ and 0 otherwise. For $l \in \mathbf{N}$ and $r \in \{1, \dots, k\}$, let $\lambda_l = (\lambda_l(1), \dots, \lambda_l(k))$, $\lambda_l(r) = \sum_{j=1}^n p_{j,r}^l$, $\lambda_l(r) > 0$, $\lambda = \lambda_1$, $\theta = (\theta(1), \dots, \theta(k))$, $\theta(r) = \lambda_2(r)/\lambda(r)$. Set $S_n = \sum_{j=1}^n X_j$. Always, let $0^0 = 1$.

In this paper, the distribution P^{S_n} of S_n is approximated by the multivariate Poisson distribution $\mathcal{P}(t)$, for $t = (t_1, \dots, t_k) \in (0, \infty)^k$, and related finite signed measures of higher order. Here

$$\mathcal{P}(t)(\{m\}) = \exp\left(-\sum_{r=1}^k t_r\right) \prod_{r=1}^k \frac{t_r^{m_r}}{m_r!} \quad \text{for } m = (m_1, \dots, m_k) \in \mathbf{Z}_+^k,$$

and $\mathbf{Z}_+ = \{0, 1, 2, \dots\}$. As measures of accuracy, the following metrics are considered:

$$d_\tau(P, Q) = \sup_{A \subseteq \mathbf{Z}_+^k} |P(A) - Q(A)| = \frac{1}{2} \sum_{m \in \mathbf{Z}_+^k} |P(\{m\}) - Q(\{m\})| \quad (\text{total variation}), \quad (1)$$

$$d_\pi(P, Q) = \sup_{m \in \mathbf{Z}_+^k} |P(\{m\}) - Q(\{m\})| \quad (\text{point metric}), \quad (2)$$

where P and Q are finite signed measures concentrated on $\mathbf{Z}_+^k$ satisfying $P(\mathbf{Z}_+^k) = Q(\mathbf{Z}_+^k)$.

Various authors treated this approximation problem. The most important papers, concerning the total variation, came from Barbour [3] and Deheuvels and Pfeifer [12], using the Stein–Chen method and the semigroup method originally developed by Le Cam [16], respectively.

Barbour [3] showed that

$$d_\tau(P^{S_n}, \mathcal{P}(\lambda)) \leq \sum_{j=1}^n \min \left\{ \left[\frac{1}{2} + \log^+ \left(2 \sum_{r=1}^k \lambda(r) \right) \right] \sum_{r=1}^k \frac{p_{j,r}^2}{\lambda(r)}, \left(\sum_{r=1}^k p_{j,r} \right)^2 \right\}, \quad (3)$$

where $\log^+(x) = \max\{0, \log(x)\}$ for $x \in (0, \infty)$. Note that in the case $k = 1$, Barbour and Hall [4] proved that

$$\frac{1}{32} \min\{\lambda_2(1), \theta(1)\} \leq d_\tau(P^{S_n}, \mathcal{P}(\lambda)) \leq (1 - e^{-\lambda(1)}) \theta(1).$$

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Deheuvels and Pfeifer [12] developed some asymptotic formulae, which are too involved to state them here. It should be mentioned that their upper bounds for total variation [see their Theorem 2.2] are only useful for bounded $\sum_{j=1}^n (\sum_{r=1}^k p_{j,r})^2$. In the case $k = 1$, they used the univariate Charlier B expansion due to Charlier [7] to prove (see [13]):

$$d_r(P^{S_n}, \mathcal{P}(\lambda)) = \frac{1}{2} \lambda_2(1) e^{-\lambda(1)} \times \left(\frac{\lambda(1)^{a-1}}{a!} (a - \lambda(1)) + \frac{\lambda(1)^{b-1}}{b!} (\lambda(1) - b) \right) + R, \quad (4)$$

where $a = \lfloor \lambda(1) + \frac{1}{2} + \sqrt{\lambda(1) + \frac{1}{4}} \rfloor$, $b = \lfloor \lambda(1) + \frac{1}{2} - \sqrt{\lambda(1) + \frac{1}{4}} \rfloor$ and $|R| \leq \sqrt{2} \theta(1)^{3/2} / (1 - \sqrt{2\theta(1)})$, for $\theta(1) < \frac{1}{2}$ ($\lfloor x \rfloor \in \mathbf{Z}$ being defined by $x - 1 < \lfloor x \rfloor \leq x$, $x \in \mathbf{R}$).

Other publications, concerning the multivariate problem, are: [1], [2], [5], [9], [17], [23]–[26].

The univariate setting (i.e., the case $k = 1$) was investigated by many authors (for instance, see [4], [5], [6], [8], [10], [11], [13]–[16], [18]–[22]).

Various metrics were considered in the univariate setting (for instance, the total variation, Kolmogorov metric, Fortet–Mourier metric, point metric, and l^p metric between distribution functions), while in the multivariate case, the total variation and the Kolmogorov metric (for this, see [23]) were treated. Here, the point metric has not been considered before.

The aim of this paper is to introduce the multivariate Charlier B expansion to the given problem. The text refers to the works of Shorgin [22], Deheuvels and Pfeifer [13], and Roos [19], [20] concerning the univariate Poisson approximation with the help of the univariate Charlier B expansion. In Section 2, the multivariate Charlier B expansion of P^{S_n} is presented. Formulae and an estimate for the corresponding Charlier coefficients are derived. Section 3 is devoted to the main results. Bounds for the distances between P^{S_n} and Poisson related signed measures in the total variation and point metric are given. As one of the most interesting results, it is proven that $d_r(P^{S_n}, \mathcal{P}(\lambda))$ is of order $O((\sum_{r=1}^k \sqrt{\min\{\theta(r), \lambda_2(r)\}})^2)$ (see Corollary 1).

Note that the Kolmogorov metric could also have been considered in this paper; but the resulting bounds would have been of the same order as for the total variation and are therefore omitted.

2. The multivariate Charlier B expansion. Let

$$\pi(m, x) = \frac{e^{-x} x^m}{m!} \quad \text{for } m \in \mathbf{Z}_+, x \in [0, \infty).$$

For $f: \mathbf{Z}_+ \rightarrow \mathbf{R}$, let $\Delta f: \mathbf{Z}_+ \rightarrow \mathbf{R}$ be defined by $(\Delta f)(m) = f(m-1) - f(m)$, $m \in \mathbf{N}$, and $(\Delta f)(0) = -f(0)$. Further, let $\Delta^0 f = f$ and $\Delta^l f = \Delta(\Delta^{l-1} f)$ for $l \in \mathbf{N}$. Finally, let $\Delta^l \pi(m, x) := \Delta^l(\pi(\cdot, x))(m)$ for $m, l \in \mathbf{Z}_+$, $x \in [0, \infty)$. It is easy to show that

$$\sum_{m=0}^{\infty} \Delta^l \pi(m, x) z^m = (z-1)^l \exp(x(z-1)), \quad (5)$$

for $l \in \mathbf{Z}_+$, $x \in [0, \infty)$, $z \in \mathbf{C}$, where $\mathbf{C}$ denotes the set of complex numbers. The following lemma gives the main argument of this paper.

Lemma 1. For $m = (m_1, \dots, m_k) \in \mathbf{Z}_+^k$, $t = (t_1, \dots, t_k) \in (0, \infty)^k$,

$$P\{S_n = m\} = \sum_{l_1=0}^{\infty} \cdots \sum_{l_k=0}^{\infty} a_l(t) \prod_{r=1}^k \Delta^{l_r} \pi(m_r, t_r), \quad (6)$$

where l stands for $(l_1, \dots, l_k)$ and the coefficients $a_l(t)$ are defined by

$$\sum_{l_1=0}^{\infty} \cdots \sum_{l_k=0}^{\infty} a_l(t) z_1^{l_1} \cdots z_k^{l_k} = \exp \left(- \sum_{r=1}^k t_r z_r \right) \prod_{j=1}^n \left(1 + \sum_{r=1}^k p_{j,r} z_r \right). \quad (7)$$

P r o o f. Let $\varphi_{S_n}: \mathbf{C}^k \rightarrow \mathbf{C}$ denote the probability generating function of S_n , defined by

$$\begin{aligned} \varphi_{S_n}(z) &= \sum_{m_1=0}^{\infty} \cdots \sum_{m_k=0}^{\infty} \mathbf{P} \{S_n = (m_1, \dots, m_k)\} z_1^{m_1} \cdots z_k^{m_k} \\ &= \prod_{j=1}^n \left(1 + \sum_{r=1}^k p_{j,r} (z_r - 1) \right), \end{aligned}$$

for $z = (z_1, \dots, z_k) \in \mathbf{C}^k$. By the help of (7), this yields, in case of $|z_r| < 1$ for $r \in \{1, \dots, k\}$,

$$\begin{aligned} \varphi_{S_n}(z) &= \sum_{l_1=0}^{\infty} \cdots \sum_{l_k=0}^{\infty} a_l(t) \left[\prod_{r=1}^k (z_r - 1)^{l_r} \right] \exp \left(\sum_{r=1}^k t_r (z_r - 1) \right) \\ &= \sum_{l_1=0}^{\infty} \cdots \sum_{l_k=0}^{\infty} a_l(t) \prod_{r=1}^k \left[\sum_{m_r=0}^{\infty} \Delta^{l_r} \pi(m_r, t_r) z_r^{m_r} \right] \\ &= \sum_{m_1=0}^{\infty} \cdots \sum_{m_k=0}^{\infty} \left[\sum_{l_1=0}^{\infty} \cdots \sum_{l_k=0}^{\infty} a_l(t) \prod_{r=1}^k \Delta^{l_r} \pi(m_r, t_r) \right] z_1^{m_1} \cdots z_k^{m_k}. \end{aligned}$$

Here, summations may be interchanged, because $|\Delta^l \pi(m, x)| \leq 2^l$ holds for all $l, m \in \mathbf{Z}_+$, $x \in [0, \infty)$, and therefore the iterated series converge absolutely. By comparing the power series, the assertion follows.

By similar considerations as above, one shows that the iterated series (6) converge absolutely, so that, here, the order of summation is not relevant. The series (6) is called Charlier B expansion of P^{S_n} . The coefficients $a_l(t)$ are called Charlier coefficients of P^{S_n} . In order to derive upper bounds for the total variation and point metric, an inequality for the Charlier coefficients of P^{S_n} is needed.

In what follows, we use the notation $s = \sum_{r=1}^k l_r$ for $(l_1, \dots, l_k) \in \mathbf{Z}_+^k$, and $\eta(r, t_r) = 2\lambda_2(r) + (\lambda(r) - t_r)^2$ for $r \in \{1, \dots, k\}$ and $(t_1, \dots, t_k) \in (0, \infty)^k$.

Lemma 2. Let $l = (l_1, \dots, l_k) \in \mathbf{Z}_+^k$, $s \geq 1$, $t = (t_1, \dots, t_k) \in (0, \infty)^k$. Further, let $I_0(x) = \sum_{j=0}^{\infty} (x/2)^{2j} / (j!)^2$ be the modified Bessel function of the first kind and order 0 and $\beta(x) = I_0(x)e^{-x^2/4}$, $x \in \mathbf{R}$. Then

$$|a_l(t)| \leq \prod_{r=1}^k \left[\frac{(\eta(r, t_r) s e)^{l_r/2}}{2^{l_r/2} l_r^{l_r}} \beta \left(\frac{\sqrt{2} |\lambda(r) - t_r| l_r}{\sqrt{s \eta(r, t_r)}} \right) \right]. \quad (8)$$

P r o o f. Let $R_1, \dots, R_k \in (0, \infty)$. By Cauchy's theorem,

$$\begin{aligned} a_l(t) &= \frac{1}{(2\pi)^k R_1^{l_1} \cdots R_k^{l_k}} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} h(R_1 e^{ix_1}, \dots, R_k e^{ix_k}) \\ &\quad \times \exp \left(-i \sum_{r=1}^k l_r x_r \right) dx_1 \cdots dx_k, \end{aligned} \quad (9)$$

where

$$h(z) = \varphi_{S_n}(z_1 + 1, \dots, z_k + 1) \exp\left(-\sum_{r=1}^k t_r z_r\right), \quad z = (z_1, \dots, z_k) \in \mathbf{C}^k.$$

Because $\cos x_1 \cos x_2 + \sin x_1 \sin x_2 = \cos(x_1 - x_2)$, $x_1, x_2 \in \mathbf{R}$, and $1 + x \leq e^x$, $x \in \mathbf{R}$,

$$\begin{aligned} |h(R_1 e^{ix_1}, \dots, R_k e^{ix_k})| &\leq \prod_{j=1}^n \left| 1 + \sum_{r=1}^k p_{j,r} R_r e^{ix_r} \right| \left| \exp\left(-\sum_{r=1}^k t_r R_r e^{ix_r}\right) \right| \\ &= \prod_{j=1}^n \left[1 + 2 \sum_{r=1}^k p_{j,r} R_r \cos(x_r) + \sum_{r_1=1}^k \sum_{r_2=1}^k p_{j,r_1} p_{j,r_2} R_{r_1} R_{r_2} \cos(x_{r_1} - x_{r_2}) \right]^{1/2} \\ &\quad \times \exp\left(-\sum_{r=1}^k t_r R_r \cos(x_r)\right) \\ &\leq \exp\left(\sum_{r=1}^k (\lambda(r) - t_r) R_r \cos(x_r) + \frac{1}{2} \sum_{j=1}^n \left[\sum_{r=1}^k p_{j,r} R_r \right]^2\right). \end{aligned}$$

Since $I_0(x) = (1/\pi) \int_0^\pi \exp(x \cos(y)) dy$, $x \in \mathbf{R}$, this leads to

$$\begin{aligned} |a_l(t)| &\leq \frac{1}{R_1^{l_1} \dots R_k^{l_k}} \exp\left(\frac{1}{2} \sum_{j=1}^n \left[\sum_{r=1}^k p_{j,r} R_r \right]^2 + \frac{1}{4} \sum_{r=1}^k (\lambda(r) - t_r)^2 R_r^2\right) \\ &\quad \times \prod_{r=1}^k \beta((\lambda(r) - t_r) R_r). \end{aligned}$$

Using Cauchy's inequality, the following estimate is valid:

$$\sum_{j=1}^n \left[\sum_{r=1}^k p_{j,r} R_r \right]^2 = \sum_{r_1=1}^k \sum_{r_2=1}^k \left(\sum_{j=1}^n p_{j,r_1} p_{j,r_2} \right) R_{r_1} R_{r_2} \leq \left(\sum_{r=1}^k \sqrt{\lambda_2(r)} R_r \right)^2.$$

Hence, $|a_l(t)| \leq g(R_1, \dots, R_k) \prod_{r=1}^k \beta((\lambda(r) - t_r) R_r)$, where $g: (0, \infty)^k \rightarrow \mathbf{R}$, defined by

$$g(R_1, \dots, R_k) = \frac{1}{R_1^{l_1} \dots R_k^{l_k}} \exp\left(\frac{1}{2} \left[\sum_{r=1}^k \sqrt{\frac{\eta(r, t_r)}{2}} R_r \right]^2\right),$$

attains its minimum for $\tilde{R} = (\tilde{R}_1, \dots, \tilde{R}_k)$, $\tilde{R}_r = \sqrt{2}l_r / \sqrt{s\eta(r, t_r)}$, $r \in \{1, \dots, k\}$. Substituting $\tilde{R}$ for $(R_1, \dots, R_k)$, the desired result is shown.

Note that $0 < \beta(x) = \beta(|x|) \leq 1$, $x \in \mathbf{R}$. The next lemma gives a recursive formula for the Charlier coefficients of P^{S_n} . Lemmata 2 and 3 are multivariate generalizations of results of [19], [20] (for $k = 1$ and arbitrary t) and [22] (for $k = 1$ and $t = \lambda$).

Lemma 3. Let $b \in \{1, \dots, k\}$, $l = (l_1, \dots, l_k) \in \mathbf{Z}_+^k$, $l_b \geq 1$, $t = (t_1, \dots, t_k) \in (0, \infty)^k$. Then

$$\begin{aligned} a_l(t) &= -\frac{t_b}{l_b} a_{l-e_b}(t) - \frac{1}{l_b} \sum_{j \in A} \frac{l_b - j_b}{s - J} \binom{s - J}{l_1 - j_1, \dots, l_k - j_k} (-1)^{s+J} \\ &\quad \times a_j(t) \left[\sum_{i=1}^n \prod_{r=1}^k p_{i,r}^{l_r - j_r} \right], \end{aligned} \quad (10)$$

where A denotes the set of all $j = (j_1, \dots, j_k) \in \mathbf{Z}_+^k$ such that $0 \leq j_r \leq l_r$ for $r \in \{1, \dots, k\} \setminus \{b\}$ and $0 \leq j_b \leq l_b - 1$, and further J is the sum $\sum_{r=1}^k j_r$.

P r o o f. It suffices to prove the assertion for $b = 1$. Let

$$z = (z_1, \dots, z_k) \in (-1, 1)^k, \quad g(z) = \exp(f(z)),$$

$$f(z) = - \sum_{r=1}^k t_r z_r + \sum_{j=1}^n \ln \left(1 + \sum_{r=1}^k p_{j,r} z_r \right).$$

Then

$$\begin{aligned} \frac{\partial^{l_1+\dots+l_k}}{\partial z_1^{l_1} \dots \partial z_k^{l_k}} g(z) &= \sum_{j_1=0}^{l_1-1} \binom{l_1-1}{j_1} \frac{\partial^{l_2+\dots+l_k}}{\partial z_2^{l_2} \dots \partial z_k^{l_k}} \left[\frac{\partial^{j_1}}{\partial z_1^{j_1}} g(z) \frac{\partial^{l_1-j_1}}{\partial z_1^{l_1-j_1}} f(z) \right] \\ &= \sum_{j_1=0}^{l_1-1} \sum_{j_2=0}^{l_2} \dots \sum_{j_k=0}^{l_k} \binom{l_1-1}{j_1} \binom{l_2}{j_2} \dots \binom{l_k}{j_k} \\ &\quad \times \frac{\partial^{j_1+\dots+j_k}}{\partial z_1^{j_1} \dots \partial z_k^{j_k}} g(z) \frac{\partial^{(l_1-j_1)+\dots+(l_k-j_k)}}{\partial z_1^{l_1-j_1} \dots \partial z_k^{l_k-j_k}} f(z). \end{aligned}$$

Hence,

$$\begin{aligned} a_l(t) &= \frac{1}{\prod_{r=1}^k (l_r!)} \frac{\partial^{l_1+\dots+l_k}}{\partial z_1^{l_1} \dots \partial z_k^{l_k}} g(z) \Big|_{z=0} \\ &= \frac{1}{l_1} \sum_{j_1=0}^{l_1-1} \sum_{j_2=0}^{l_2} \dots \sum_{j_k=0}^{l_k} \frac{a_j(t)}{(l_1-1-j_1)!(l_2-j_2)! \dots (l_k-j_k)!} \\ &\quad \times \frac{\partial^{(l_1-j_1)+\dots+(l_k-j_k)}}{\partial z_1^{l_1-j_1} \dots \partial z_k^{l_k-j_k}} f(z) \Big|_{z=0}. \end{aligned}$$

Since

$$\frac{\partial^{l_1+\dots+l_k}}{\partial z_1^{l_1} \dots \partial z_k^{l_k}} f(z) \Big|_{z=0} = \begin{cases} \lambda(c) - t_c & \text{if } (l_1, \dots, l_k) = e_c, \quad c \in \{1, \dots, k\}, \\ \sum_{j=1}^n \left(\prod_{r=1}^k p_{j,r}^{l_r} \right) (s-1)! (-1)^{s-1} & \text{otherwise,} \end{cases}$$

for $(l_1, \dots, l_k) \in \mathbf{Z}_+^k$, $s \geq 1$, the assertion follows immediately.

It is clear that $a_0(t) = 1$ and, as the preceding lemma shows, for $r, v \in \{1, \dots, k\}$, $r \neq v$, $t = (t_1, \dots, t_k) \in (0, \infty)^k$,

$$a_{e_r}(t) = \lambda(r) - t_r, \quad (11)$$

$$a_{2e_r}(t) = \frac{1}{2} ((\lambda(r) - t_r)^2 - \lambda_2(r)), \quad (12)$$

$$a_{e_r+e_v}(t) = (\lambda(r) - t_r) (\lambda(v) - t_v) - \sum_{j=1}^n p_{j,r} p_{j,v}. \quad (13)$$

3. Main results. For the first main result of this paper, the next technical lemma is necessary.

Lemma 4. Let $k \in \mathbf{N}$, $l_1, \dots, l_k \in \mathbf{Z}_+$. Then

$$\frac{s^s}{\prod_{r=1}^k l_r^{l_r}} \leq \left(\frac{s!}{\prod_{r=1}^k (l_r!)} \right)^2 = \binom{s}{l_1, \dots, l_k}^2 \quad (14)$$

P r o o f. It suffices to assume that $l_1, \dots, l_k \in \mathbb{N}$, $k \in \{2, 3, \dots\}$. Then

$$\frac{s^s}{\prod_{r=1}^k l_r^{l_r}} = \prod_{r=1}^{k-1} \frac{u_{r+1}^{u_{r+1}}}{u_r^{u_r} l_{r+1}^{l_{r+1}}},$$

where $u_r = \sum_{m=1}^r l_m$ for $r \in \{1, \dots, k\}$. For $a \in \mathbb{N}$, $b \in \{1, \dots, a-1\}$,

$$\frac{a^a}{b^b(a-b)^{a-b}} = \left(\frac{a}{b}\right)^b \left(\frac{a}{a-b}\right)^{a-b} \leq \left[\prod_{m=1}^b \frac{a-m+1}{b-m+1}\right] \left[\prod_{m=1}^{a-b} \frac{a-m+1}{a-b-m+1}\right] = \left(\frac{a}{b}\right)^2.$$

The proof is easily completed.

For $f: \mathbb{Z}_+ \rightarrow \mathbb{R}$, let

$$\|f\|_\infty = \sup_{m \in \mathbb{Z}_+} |f(m)| \quad \text{and} \quad \|f\|_1 = \sum_{m=0}^{\infty} |f(m)|.$$

Theorem 1. Let

$$t = (t_1, \dots, t_k) \in (0, \infty)^k \quad \text{and} \quad \gamma(r, t_r) = \eta(r, t_r) \min \left\{ \frac{1}{2t_r}, e \right\}$$

for $r \in \{1, \dots, k\}$. If $\sum_{r=1}^k \sqrt{2\gamma(r, t_r)} < 1$ then

$$d_r(P^{S_n}, \mathcal{P}(t)) \leq \sum_{r=1}^k |\lambda(r) - t_r| \min \left\{ (2t_r e)^{-1/2}, 1 \right\} + \frac{(\sum_{r=1}^k \sqrt{\gamma(r, t_r)})^2}{1 - \sum_{r=1}^k \sqrt{2\gamma(r, t_r)}}. \quad (15)$$

If $Q(u, t)$ denotes the finite signed measure concentrated on $\mathbb{Z}_+^k$ with counting density

$$Q(u, t)(\{m\}) = \sum_{s=0}^u \sum_{l \in A_s} \left[a_l(t) \prod_{r=1}^k \Delta^{l_r} \pi(m_r, t_r) \right], \quad (16)$$

for $m = (m_1, \dots, m_k) \in \mathbb{Z}_+^k$, where $u \in \mathbb{N}$ and $A_s = \{(l_1, \dots, l_k) \in \mathbb{Z}_+^k \mid \sum_{r=1}^k l_r = s\}$ for $s \in \mathbb{Z}_+$, then, in the case $\sum_{r=1}^k \sqrt{2\gamma(r, t_r)} < 1$,

$$d_r(P^{S_n}, Q(u, t)) \leq 2^{(u-1)/2} \frac{(\sum_{r=1}^k \sqrt{\gamma(r, t_r)})^{u+1}}{1 - \sum_{r=1}^k \sqrt{2\gamma(r, t_r)}}. \quad (17)$$

P r o o f. Let

$$T = \sum_{r=1}^k |\lambda(r) - t_r| \min \left\{ (2t_r e)^{-1/2}, 1 \right\}.$$

By the use of

$$\left\| \Delta^b \pi(\cdot, x) \right\|_1 \leq \min \left\{ \left[\frac{2b}{xe} \right]^{b/2}, 2^b \right\}, \quad x \in (0, \infty), \quad b \in \mathbb{Z}_+$$

(see [13]) in addition to (6), (8), (11), and (14), the inequality (15) is shown as follows. Assume that

$$\sum_{r=1}^k \sqrt{2\gamma(r, t_r)} < 1,$$

then

$$\begin{aligned}
 d_\tau(P^{S_n}, \mathcal{P}(t)) &\leq \frac{1}{2} \sum_{m \in \mathbf{Z}_+^k} \left| \sum_{l \in \mathbf{Z}_+^k \setminus A_0} a_l(t) \prod_{r=1}^k \Delta^{l_r} \pi(m_r, t_r) \right| \\
 &\leq \frac{1}{2} \sum_{l \in \mathbf{Z}_+^k \setminus A_0} |a_l(t)| \prod_{r=1}^k \|\Delta^{l_r} \pi(\cdot, t_r)\|_1 \\
 &\leq T + \frac{1}{2} \sum_{s=2}^\infty \sum_{l \in A_s} \prod_{r=1}^k \left[\frac{(\eta(r, t_r) s e)^{l_r/2}}{2^{l_r/2} l_r^{l_r}} \min \left\{ \left(\frac{2l_r}{t_r e} \right)^{l_r/2}, 2^{l_r} \right\} \right] \\
 &\leq T + \frac{1}{2} \sum_{s=2}^\infty \sum_{l \in A_s} \binom{s}{l_1, \dots, l_k} \prod_{r=1}^k (2\gamma(r, t_r))^{l_r/2} \\
 &= T + \frac{(\sum_{r=1}^k \sqrt{\gamma(r, t_r)})^2}{1 - \sum_{r=1}^k \sqrt{2\gamma(r, t_r)}}.
 \end{aligned}$$

The rest of the assertion is proven analogously.

It is easy to verify that $Q(1, \lambda) = \mathcal{P}(\lambda)$ and, for $m = (m_1, \dots, m_k) \in \mathbf{Z}_+^k$, $t = (t_1, \dots, t_k) \in (0, \infty)^k$,

$$Q(1, t)(\{m\}) = \left[\prod_{r=1}^k \pi(m_r, t_r) \right] \left[1 + \sum_{r=1}^k \frac{1}{t_r} (\lambda(r) - t_r)(m_r - t_r) \right], \quad (18)$$

$$\begin{aligned}
 Q(2, t)(\{m\}) &= \left[\prod_{r=1}^k \pi(m_r, t_r) \right] \left[1 + \sum_{r=1}^k \frac{1}{t_r} (\lambda(r) - t_r)(m_r - t_r) \right. \\
 &\quad \left. + \frac{1}{2} \left(\sum_{r=1}^k \frac{1}{t_r} (\lambda(r) - t_r)(m_r - t_r) \right)^2 - \frac{1}{2} \sum_{j=1}^n \left(\sum_{r=1}^k \frac{p_{j,r}(m_r - t_r)}{t_r} \right)^2 \right. \\
 &\quad \left. - \frac{1}{2} \sum_{r=1}^k \frac{m_r}{t_r^2} [(\lambda(r) - t_r)^2 - \lambda_2(r)] \right]. \quad (19)
 \end{aligned}$$

As a consequence of Theorem 1,

$$d_\tau(P^{S_n}, \mathcal{P}(\lambda)) \leq \frac{(\sum_{r=1}^k \sqrt{\gamma(r)})^2}{1 - \sum_{r=1}^k \sqrt{2\gamma(r)}} \quad \text{if } \sum_{r=1}^k \sqrt{2\gamma(r)} < 1, \quad (20)$$

where $\gamma(r) = \gamma(r, \lambda(r))$, $r \in \{1, \dots, k\}$. The following corollary shows that the singularity in (20) can be removed.

Corollary 1. *Let*

$$\delta(r) = \min \{ \theta(r), \lambda_2(r) \}$$

for $r \in \{1, \dots, k\}$. *Then*

$$\frac{1}{32} \max_{1 \leq r \leq k} \delta(r) \leq d_\tau(P^{S_n}, \mathcal{P}(\lambda)) \leq \frac{1}{2 - \sqrt{3}} \left(\sum_{r=1}^k \sqrt{\gamma(r)} \right)^2 \leq \frac{2ke}{2 - \sqrt{3}} \sum_{r=1}^k \delta(r). \quad (21)$$

P r o o f. The first inequality is shown as follows. For $r \in \{1, \dots, k\}$,

$$\begin{aligned}
 d_\tau(P^{S_n}, \mathcal{P}(\lambda)) &\geq \sup_{A \subseteq \mathbf{Z}_+} \left| \mathbf{P}\{S_n \in \mathbf{Z}_+^{r-1} \times A \times \mathbf{Z}_+^{k-r}\} - \mathcal{P}(\lambda)(\mathbf{Z}_+^{r-1} \times A \times \mathbf{Z}_+^{k-r}) \right| \\
 &= d_\tau(\mathcal{PB}(n; p_{1,r}, \dots, p_{n,r}), \mathcal{P}(\lambda(r))),
 \end{aligned}$$

where $\mathcal{PB}(n; p_{1,r}, \dots, p_{n,r})$ is the Poisson binomial distribution with parameters $n \in \mathbb{N}$, $p_{1,r}, \dots, p_{n,r} \in [0, 1]$, i.e., the distribution of the sum of n independent random variables $Y_1, \dots, Y_n$ with $\mathbf{P}\{Y_j = 1\} = 1 - \mathbf{P}\{Y_j = 0\} = p_{j,r}$, $j \in \{1, \dots, n\}$. Hence, using a lower bound for the latter term obtained by Barbour and Hall [4],

$$d_\tau(P^{S_n}, \mathcal{P}(\lambda)) \geq \frac{1}{32} \max_{1 \leq r \leq k} \delta(r).$$

For the second inequality, it suffices to assume that $x := \sum_{r=1}^k \sqrt{2\gamma(r)} < 1$. But in this case, $d_\tau(P^{S_n}, \mathcal{P}(\lambda)) \leq \min\{1, f(x)\} \leq x^2/(2(2 - \sqrt{3}))$, where $f(y) = y^2/(2(1 - y))$, $y \in [0, 1)$. By application of Cauchy's inequality, the third inequality is shown.

By Corollary 1 and

$$\frac{1}{k} \left(\sum_{r=1}^k \theta(r) \right)^2 \leq \sum_{r=1}^k \theta(r)^2 \leq \sum_{r=1}^k \delta(r) \leq \sum_{r=1}^k \theta(r),$$

it follows that, for fixed or bounded k , $d_\tau(P^{S_n}, \mathcal{P}(\lambda))$ tends to zero if and only if $\sum_{r=1}^k \theta(r)$ tends to zero.

In the general case, one cannot remove the factor k from the right hand side of (21): An inequality of type

$$d_\tau(P^{S_n}, \mathcal{P}(\lambda)) \leq M k^\alpha \sum_{r=1}^k \delta(r)$$

with absolute constants $M \in (0, \infty)$ and $\alpha \in [0, 1)$ cannot hold. In order to verify this assertion, let, for example, $p_{j,r} = 1/(kn)$ for all $j \in \{1, \dots, n\}$, $r \in \{1, \dots, k\}$. Using an identity by Deheuvels and Pfeifer [12, Lemma 5.1] in addition to an asymptotic result by Deheuvels and Pfeifer [11], $d_\tau(P^{S_n}, \mathcal{P}(\lambda)) = d_\tau(\mathcal{B}(n, 1/n), \mathcal{P}(1)) \sim 3/(4en)$, $n \rightarrow \infty$. Under these assumptions, Corollary 1 leads to $d_\tau(P^{S_n}, \mathcal{P}(\lambda)) \leq 2e/((2 - \sqrt{3})n)$.

Note that Corollary 1 shows that $d_\tau(P^{S_n}, \mathcal{P}(\lambda)) \leq c \sum_{r=1}^k \theta(r)$, with $c \leq (2ke)/(2 - \sqrt{3})$, whereas Barbour's result (3) implies

$$c \leq \frac{1}{2} + \log^+ \left(2 \sum_{r=1}^k \lambda(r) \right).$$

The next result concerns the point metric, for which the following lemma is needed.

Lemma 5. *The following inequalities are valid:*

$$\|\Delta^l \pi(\cdot, x)\|_\infty \leq \min \left\{ c \left[\frac{l}{xe} \right]^{(l+1)/2}, 2^l \right\}, \quad l \in \mathbb{N}, \quad x \in (0, \infty), \quad (22)$$

where $c = (\sqrt{e}/2)(1 + \sqrt{\pi/2})$,

$$\|\Delta^0 \pi(\cdot, x)\|_\infty \leq \min \left\{ (2xe)^{-1/2}, 1 \right\}, \quad x \in (0, \infty). \quad (23)$$

P r o o f. First, note that

$$\|\Delta^l \pi(\cdot, x)\|_\infty \leq \|\Delta^l \pi(\cdot, x)\|_1 \leq 2^l, \quad \text{for } l \in \mathbb{Z}_+, \quad x \in [0, \infty).$$

The rest of (22) and (23) are results of Shorgin [22] and Deheuvels and Pfeifer [13], respectively.

Note that a result by Roos [19, Satz 6.31] (being published in a subsequent paper) asserts that (22) remains valid if c is replaced by $\frac{\sqrt{e}}{2}(1 + \sqrt{\pi/8})$.

Theorem 2. *Let $l = (t_1, \dots, t_k) \in (0, \infty)^k$. Let c be a constant satisfying inequality (22) and*

$$c_0 = (2e)^{-1/2} \max\{c, 1\}.$$

Finally, let

$$H = \sum_{r=1}^k |\lambda(r) - t_r| \min \left\{ \frac{c}{t_r e}, 2 \right\} \prod_{\substack{v=1 \\ v \neq r}}^k \min \left\{ (2t_v e)^{-1/2}, 1 \right\}.$$

If $\sum_{r=1}^k \sqrt{2\gamma(r, t_r)} < 1$ then

$$d_{\pi} \left(P^{S_n}, \mathcal{P}(t) \right) \leq H + 2c_0^k \left[\prod_{r=1}^k \sqrt{\frac{2\gamma(r, t_r)}{\eta(r, t_r)}} \right] \frac{(\sum_{r=1}^k \sqrt{\gamma(r, t_r)})^2}{1 - \sum_{r=1}^k \sqrt{2\gamma(r, t_r)}}. \quad (24)$$

If $Q(u, t)$ denotes the finite signed measure with counting density (16), for $u \in \mathbb{N}$, then, under assumption of $\sum_{r=1}^k \sqrt{2\gamma(r, t_r)} < 1$,

$$d_{\pi} \left(P^{S_n}, Q(u, t) \right) \leq 2^{(u+1)/2} c_0^k \left[\prod_{r=1}^k \sqrt{\frac{2\gamma(r, t_r)}{\eta(r, t_r)}} \right] \frac{(\sum_{r=1}^k \sqrt{\gamma(r, t_r)})^{u+1}}{1 - \sum_{r=1}^k \sqrt{2\gamma(r, t_r)}}. \quad (25)$$

P r o o f. Let A_s be defined as in Theorem 1 and assume that

$$\sum_{r=1}^k \sqrt{2\gamma(r, t_r)} < 1.$$

Using (6), (11), (22), (23), (8), (14) and $b \leq 2^{b-1}$, $b \in \mathbb{N}$, the assertion follows as in the proof of Theorem 1:

$$\begin{aligned} d_{\pi} \left(P^{S_n}, \mathcal{P}(t) \right) &\leq H + \sum_{s=2}^{\infty} \sum_{l \in A_s} \prod_{r=1}^k \left[\frac{(\eta(r, t_r) s e)^{l_r/2}}{2^{l_r/2} l_r^{l_r}} \|\Delta^{l_r} \pi(\cdot, t_r)\|_{\infty} \right] \\ &\leq H + \sum_{s=2}^{\infty} \sum_{l \in A_s} \prod_{r=1}^k \left[\frac{s^{l_r/2}}{l_r^{l_r/2}} c_0 \sqrt{\frac{2\gamma(r, t_r)}{\eta(r, t_r)}} (2\gamma(r, t_r))^{l_r/2} \right] \\ &\leq H + 2c_0^k \left[\prod_{r=1}^k \sqrt{\frac{2\gamma(r, t_r)}{\eta(r, t_r)}} \right] \frac{(\sum_{r=1}^k \sqrt{\gamma(r, t_r)})^2}{1 - \sum_{r=1}^k \sqrt{2\gamma(r, t_r)}}. \end{aligned}$$

The rest of the assertion is proven analogously.

Theorem 2 yields

$$d_{\pi} \left(P^{S_n}, \mathcal{P}(\lambda) \right) \leq 2c_0^k \left[\prod_{r=1}^k \sqrt{\frac{\gamma(r)}{\lambda_2(r)}} \right] \frac{(\sum_{r=1}^k \sqrt{\gamma(r)})^2}{1 - \sum_{r=1}^k \sqrt{2\gamma(r)}} \quad \text{if } \sum_{r=1}^k \sqrt{2\gamma(r)} < 1, \quad (26)$$

where c_0 is defined as in Theorem 2. One can assume that $c_0 < 1$.

REFERENCES

1. Ahmad I. A. On the Poisson approximation of multinomial probabilities. — Statist. Probab. Lett., 1985, v. 3, p. 55–56.
2. Аренбаев Н. К. Асимптотическое поведение полиномиального распределения. — Теория вероятн. и ее примен., 1976, т. XXI, в. 4, с. 826–831.
3. Barbour A. D. Stein's method and Poisson process convergence. — J. Appl. Probab., 1988, v. 25 A, p. 175–184.
4. Barbour A. D., Hall P. On the rate of Poisson convergence. — Math. Proc. Cambridge Philos. Soc., 1984, v. 95, p. 473–480.

5. Barbour A. D., Holst L., Janson S. Poisson Approximation. — Oxford: Clarendon Press, 1992, 277 p.
6. Боровков К. А. Уточнение пуассоновской аппроксимации. — Теория вероятн. и ее примен., 1988, т. XXXIII, в. 2, с. 364–368.
7. Charlier C. V. L. Die zweite Form des Fehlergesetzes. — Ark. Mat. Astr. Fys., 1905, v. 2, 15, p. 1–8.
8. Chen L. H. Y. Poisson approximation for dependent trials. — Ann. Probab., 1975, v. 3, p. 534–545.
9. Chen L. H. Y., Roos M. Compound Poisson approximation for unbounded functions on a group, with application to large deviations. — Probab. Theory Relat. Fields, 1995, v. 103, p. 515–528.
10. Deheuvels P., Pfeifer D. A semigroup approach to Poisson approximation. — Ann. Probab., 1986, v. 14, p. 663–676.
11. Deheuvels P., Pfeifer D. Operator semigroups and Poisson convergence in selected metrics. — Semigroup Forum, 1986, v. 34, p. 203–224. Errata. Semigroup Forum, 1987, v. 35, p. 251.
12. Deheuvels P., Pfeifer D. Poisson approximations of multinomial distributions and point processes. — J. Multivariate Anal., 1988, v. 25, p. 65–89.
13. Deheuvels P., Pfeifer D. On a relationship between Uspensky's theorem and Poisson approximations. — Ann. Inst. Statist. Math., 1988, v. 40, p. 671–681.
14. Kerstan J. Verallgemeinerung eines Satzes von Prochorow und Le Cam. — Z. Wahrscheinlichkeitstheor. verw. Geb., 1964, v. 2, p. 173–179.
15. Крючонис Й. Уточнение аппроксимации обобщенного биномиального распределения свертками пуассоновских мер. — Литов. матем. сб., 1986, т. 26, с. 53–69.
16. Le Cam L. An approximation theorem for the Poisson binomial distribution. — Pacific J. Math., 1960, v. 10, p. 1181–1197.
17. McDonald D. R. On the Poisson approximation to the multinomial distribution. — Canad. J. Statist., 1980, v. 8, p. 115–118.
18. Пресман Э. Л. Аппроксимация по вариации распределения суммы независимых бернуллиевских величин законом Пуассона. — Теория вероятн. и ее примен., 1985, т. XXX, в. 2, с. 391–396.
19. Roos B. Metrische Poisson-Approximation. Ph. D. thesis, Fachbereich Mathematik, Universität Oldenburg, 1996.
20. Roos B. Asymptotics and sharp bounds in the Poisson approximation to the Poisson binomial distribution. Preprint № 96-14. Hamburg: Institut für Mathematische Stochastik, Universität Hamburg, 1996.
21. Serfling R. J. A general Poisson approximation theorem. — Ann. Probab., 1975, v. 3, p. 726–731.
22. Шоргин С. Я. Аппроксимация обобщенного биномиального распределения. — Теория вероятн. и ее примен., 1977, т. XXII, в. 4, с. 867–871.
23. Sintès Blanc A. On the Poisson approximation for some multinomial distributions. — Statist. Probab. Lett., 1991, v. 11, p. 1–6.
24. Wang Y. H. Coupling methods in approximation. — Canad. J. Statist., 1986, v. 14, p. 69–74.
25. Wang Y. H. A multivariate extension of Poisson's theorem. — Canad. J. Statist., 1989, v. 17, p. 241–245.
26. Witte H.-J. On the optimality of multivariate Poisson approximation. — Stochastic Process. Appl., 1993, v. 44, p. 75–88.

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