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ON THE CONTINUITY OF WEAK SOLUTIONS OF BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS¹⁾

В настоящей статье обсуждается понятие слабого решения общего обратного стохастического дифференциального уравнения (ОСДУ), введенное авторами и А. Рашкану в [2]. Изучена взаимосвязь между непрерывностью решений, потраекторной единственностью, единственностью по распределению и существованием потраекторно единственного сильного решения. Основной результат утверждает, что если все слабые решения ОСДУ непрерывны, то решение является потраекторно единственным. Следует отметить, что этот результат специфичен для ОСДУ и, разумеется, не имеет аналога в случае (прямых) стохастических дифференциальных уравнений. Как следствие, если существует

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некоторое слабое решение и все решения непрерывны, то существует потраекторно единственное решение и это решение является сильным. Более того, если управляющий (driving) процесс есть непрерывный локальный мартингал, допускающий предсказуемое представление, то верно и обратное. Другими словами, существование разрывных решений у ОСДУ — это естественное явление в случае, когда нет потраекторной единственности или, в частности, единственности по распределению. Примеры разрывных решений некоторого ОСДУ уже были даны в [2]. Это и явилось побудительным мотивом написания настоящей статьи, цель которой — изучить общую ситуацию.

Ключевые слова и фразы: обратное стохастическое дифференциальное уравнение, слабые решения, сильные решения, единственность по распределению, потраекторная единственность, непрерывность решений, разрывность решений.

1. Introduction. Weak solutions of forward stochastic differential equations (SDE) has been studied since a long time and proved to be useful not only in theory, but also in applications.

In a general setting, backward stochastic differential equations (BSDEs) were introduced by É. Pardoux and S. G. Peng [11] in 1990. Since that time, an extensive and profound theory of BSDEs was built and found important applications in such fields as financial mathematics, optimal control and stochastic games, partial differential equations and homogenization, see, e.g., [8].

According to this theory, solutions of BSDEs are always adapted to the filtration generated by the driving process. Usually, the underlying driving process is a Brownian motion, a diffusion process or a diffusion process with jumps. In most cases, the filtration generated by the driving process is that of a Brownian motion or, more generally, of a Brownian motion and a Poissonian random measure. In analogy to SDEs, it should be useful to call solutions of BSDEs adapted to the filtration generated by the driving process *strong* solutions.

One of the first attempts to introduce the notion of a weak solution for BSDEs was given only recently by the authors and A. Răşcanu [2] in a quite general set-up, not requiring the adaptedness with respect to the filtration generated by the driving process. As a necessary step, an example was provided in [3] showing that there exist BSDEs with a weak solution unique in law, but without any strong solution.

Certain examples of, what they call, weak solutions of a forward-backward SDE (FBSDE) were discussed by F. Antonelli and J. Ma [1]. Actually, there were considered examples in which the driving process is a weak solution of a certain (forward) SDE. Also a recent preprint by F. Delarue and G. Guatteri [7] on the same subject should be mentioned. However, using the decoupling method, in their paper the solution (Y, Z) of the backward equation is constructed as a functional of the solution X of the (decoupled) forward equation and hence (Y, Z) can be considered as a strong solution with new driving process X .

For a better understanding, the present paper is focused on «pure» BSDEs rather than FBSDEs to exclude the mixture of different effects caused by the forward equation on the one side and the backward equation on the other side.

In this paper, we are going to continue the discussion of weak solutions of a general BSDE in the sense of [2]. The starting point was an example of a simple BSDE with driving Brownian motion given by the authors in [4] (cf. Example 3.1 in the present paper) which possesses *discontinuous* solutions. It is obvious that these solutions must be weak solutions which are not strong. On the other side, the BSDE considered there also possesses strong solutions. This is, however, only possible if the uniqueness in law of the solution is violated. At first glance, this seemed to be a rather pathological example. But in a next step it became apparent that for quite general one-dimensional BSDEs with continuous driver the situation is similar (cf. [4, Example 6.2]): Whenever uniqueness in law fails, there exist discontinuous solutions and vice versa.

The purpose of the present paper is to systematically investigate the relationship between continuity of solutions and uniqueness in law for general BSDEs in the sense of [2]. It turns out that there is also a close connection to pathwise uniqueness and existence of strong solutions. The main result asserts that, if all weak solutions of a BSDE are continuous, then the solution is pathwise unique. As a consequence, if a weak solution exists and all solutions are continuous, then there exists a pathwise unique solution and this solution is strong. Moreover, if the driving process is a continuous local martingale satisfying the previsible representation property, then the converse is also true. In other words, the existence of discontinuous solutions to a BSDE is a natural phenomenon: It arises whenever pathwise uniqueness or, in particular, uniqueness in law is not satisfied, even if the driving process has best properties.

The paper is organized as follows.

In Section 2, some basic definitions and notation concerning the concept of BSDEs and their weak and strong solutions are introduced. Following [2], the existence theorem for weak solutions and the connection between pathwise uniqueness, uniqueness in law, and existence of strong solutions are stated for later use.

Section 3 starts from a detailed presentation of an example borrowed from [4] to get a feeling what could be expected in the general situation. Thereafter the main result on the relationship between continuity of solutions and pathwise uniqueness is stated and discussed. In particular, the special case is considered that the driving process is a multidimensional continuous local martingale which posses the previsible representation property.

The last Section 4 contains the proof of the main theorem.

2. Weak and strong solutions of BSDEs. This section is devoted to a brief recall of some basic definitions and notation related to strong and weak solutions for BSDEs (cf. [2]).

Some basic definitions. Let $T > 0$ be a real number (the terminal time). For any natural number $k \geq 1$, by $D(\mathbf{R}^k) := D([0, T]; \mathbf{R}^k)$ we denote the Skorokhod space of càdlàg (right continuous with left-hand limits) functions x on $[0, T]$ taking values in $\mathbf{R}^k$ such that

$$x(T-) := \lim_{t \uparrow T} x(t) = x(T),$$

and, by convention, $x(0-) = 0$. Let ζ be the coordinate mapping on $D(\mathbf{R}^k)$ defined by

$$\zeta_t(x) = x(t), \quad t \in [0, T], \quad x \in D(\mathbf{R}^k).$$

The following σ -algebras of subsets of $D(\mathbf{R}^k)$ are introduced:

$$\mathcal{D}_t^s := \mathcal{D}_t^s(\mathbf{R}^k) := \sigma(\{\zeta_u : u \in [s, t]\}), \quad 0 \leq s \leq t \leq T, \quad \mathcal{D} := \mathcal{D}(\mathbf{R}^k) := \mathcal{D}_T^0(\mathbf{R}^k).$$

We will always assume that there are given a measurable $\mathbf{R}^p$ -valued functional H on the space $(D(\mathbf{R}^d), \mathcal{D}(\mathbf{R}^d))$ and a measurable $\mathbf{R}^p$ -valued functional f on the product space

$$([0, T] \times D(\mathbf{R}^d) \times D(\mathbf{R}^p), \mathcal{B}([0, T]) \otimes \mathcal{D}(\mathbf{R}^d) \otimes \mathcal{D}(\mathbf{R}^p)),$$

where $\mathcal{B}([0, T])$ is the σ -algebra of Borel subsets of $[0, T]$.

Let μ be a probability measure on $(D(\mathbf{R}^d), \mathcal{D}(\mathbf{R}^d))$. Suppose that $(\Omega, \mathcal{F}, \mathbf{P})$ is a complete probability space, $\mathbf{F} = (\mathcal{F}_t)_{t \in [0, T]}$ a filtration satisfying the usual conditions, and X, Y are càdlàg processes defined on $(\Omega, \mathcal{F}, \mathbf{P})$ with values in $\mathbf{R}^d$ and $\mathbf{R}^p$, respectively.

By $\mathbf{F}^X = (\mathcal{F}_t^X)_{t \in [0, T]}$ (respectively, $\mathbf{F}^{X, Y} = (\mathcal{F}_t^{X, Y})_{t \in [0, T]}$) we denote the smallest of all filtrations $\mathbf{G}$ satisfying the usual conditions and being such that X (respectively, (X, Y)) is $\mathbf{G}$ -adapted.

Definition 2.1. (i) The six-tuple $(\Omega, \mathcal{F}, \mathbf{F}, \mathbf{P}, X, Y)$ is called a *weak solution* of BSDE (H, f, μ) if the following conditions are satisfied.

(W.i) The distribution of X on $D(\mathbf{R}^d)$ is μ .

(W.ii) X is $\mathbf{F}$ -adapted.

(W.iii) It holds

$$\mathbf{E}[|H(X)|] < +\infty, \quad \mathbf{E}\left[\int_0^T |f(s, X, Y)| ds\right] < +\infty.$$

(W.iv) Every $\mathbf{F}^X$ -martingale is also an $\mathbf{F}$ -martingale.

(W.v) The equation

$$Y_t = \mathbf{E}\left[H(X) + \int_t^T f(s, X, Y) ds \middle| \mathcal{F}_t\right], \quad t \in [0, T], \quad (1)$$

is fulfilled.

(ii) The process Y on the filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbf{P})$ is called a *strong solution of BSDE* (H, f, X) if $(\Omega, \mathcal{F}, \mathbf{F}, \mathbf{P}, X, Y)$ is a weak solution of BSDE (H, f, μ) and if Y is $\mathbf{F}^X$ -adapted.

In other words, a process Y on the filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbf{P})$ is a strong solution of BSDE (H, f, X) if and only if $(\Omega, \mathcal{F}, \mathbf{F}^X, \mathbf{P}, X, Y)$ is a weak solution of BSDE (H, f, μ) .

Note that $(\Omega, \mathcal{F}, \mathbf{F}^{X,Y}, \mathbf{P}, X, Y)$ is a weak solution of BSDE (H, f, μ) whenever $(\Omega, \mathcal{F}, \mathbf{F}, \mathbf{P}, X, Y)$ is one.

The triplet (H, f, μ) (respectively, (H, f, X)) is called the *data set* of BSDE (H, f, μ) (respectively, BSDE (H, f, X)), $H(X)$ the *terminal value*, f the *driver*, and X the *driving process*.

Usually the driving process X is a d -dimensional Brownian motion. For a forward-backward SDE, X is to be chosen as a solution of the given (decoupled) SDE.

A detailed discussion of the notion of a weak solution of BSDE (H, f, μ) can be found in [2]. For the sake of brevity, *weak solutions* are often simply called *solutions*.

It should be emphasized that condition (W.iv) of the above definition is very important. This condition means that the filtration $\mathbf{F}$ shall not be too large in comparison with $\mathbf{F}^X$. More precisely, (W.iv) is easily shown to be equivalent to the following condition:

(W.iv') Given $\mathcal{F}_t^X$ the σ -fields $\mathcal{F}_t$ and $\mathcal{F}_T^X$ are conditionally independent, for every $t \in [0, T]$.

Roughly speaking this means that the filtration $\mathbf{F}$ consists, additionally to $\mathbf{F}^X$, only of randomizations (i.e., independent experiments).

Here is a useful characterization of weak solutions of BSDE (H, f, μ) . Of course, an analogous characterization can also be stated for strong solutions.

Proposition 2.1. *Suppose that $(\Omega, \mathcal{F}, \mathbf{F}, \mathbf{P}, X, Y)$ satisfies conditions (W.i)–(W.iv) of Definition 2.1. Then $(\Omega, \mathcal{F}, \mathbf{F}, \mathbf{P}, X, Y)$ is a weak solution of BSDE (H, f, μ) if and only if Y satisfies the following equation:*

$$Y_t = H(X) + \int_t^T \{f(\cdot, X, Y)\}_s^{\mathbf{F}} ds - (M_T - M_t), \quad t \in [0, T], \quad (2)$$

where M is an $\mathbf{F}$ -martingale such that $M_0 = 0$. Here $\{f(\cdot, X, Y)\}_s^{\mathbf{F}}$ denotes the optional projection of the process $f(\cdot, X, Y)$ with respect to the filtration $\mathbf{F}$ (cf. [6]).

The proof of Proposition 2.1 is straightforward and therefore omitted.

Existence of weak solutions. Next the existence theorem for weak solutions of BSDE (H, f, μ) , proved in [2], is stated for later use.

The hypotheses used in this theorem include continuity properties of the driver in the Meyer–Zheng topology, which will now be introduced. For any natural number $k \geq 1$, the Meyer–Zheng topology on $D(\mathbf{R}^k)$ is induced by the following metric ρ on $D(\mathbf{R}^k)$:

$$\rho(x, y) = \int_0^T (|x(s) - y(s)| \wedge 1) ds, \quad x, y \in D(\mathbf{R}^k)$$

(cf. [10]). Obviously, the convergence in this metric is just the convergence in Lebesgue measure. The metric space $(D(\mathbf{R}^k), \rho)$ is separable, but not complete. However, $(D(\mathbf{R}^k), \rho)$ is a Luzin space: For every embedding in a compact metric space K , the

space $D(\mathbf{R}^k)$ is a Borel set in K ; cf. [5] (IV.40–46) and [10] for details. Moreover, the Meyer–Zheng topology is consistent with the given measurable structure on $D(\mathbf{R}^k)$: Let $\mathcal{B}(D(\mathbf{R}^k))$ be the σ -algebra of Borel subsets of $D(\mathbf{R}^k)$ in the Meyer–Zheng topology. Then $\mathcal{B}(D(\mathbf{R}^k)) = \mathcal{D}(\mathbf{R}^k)$.

Let (H, f, μ) be a data set described at the beginning of this section, with terminal functional H , driver f , and distribution μ of the driving process. It will always be assumed that H is integrable with respect to the probability measure μ on $(D(\mathbf{R}^d), \mathcal{D}(\mathbf{R}^d))$:

$$\int_{D(\mathbf{R}^d)} |H(x)| \mu(dx) < +\infty. \quad (3)$$

Here is the announced existence result. For its proof see [2].

Theorem 2.1. *Suppose that the following conditions are satisfied.*

(E.i) *f is bounded on $[0, T] \times D(\mathbf{R}^d) \times D(\mathbf{R}^p)$.*

(E.ii) *For every $(t, x) \in [0, T] \times D(\mathbf{R}^d)$, the mapping $f(t, x, \cdot)$ is $\mathcal{D}_T^t(\mathbf{R}^p)$ -measurable.*

(E.iii) *The mapping F on $D(\mathbf{R}^{d+p})$ defined by*

$$F(x, y) = f(\cdot, x, y), \quad (x, y) \in D(\mathbf{R}^{d+p}),$$

with values in $L^1([0, T]; \mathbf{R}^p)$ is continuous in the sense of Meyer–Zheng.

Then BSDE (H, f, μ) possesses a weak solution.

Let us give a few comments concerning the continuity condition (E.iii) of Theorem 2.1. It is equivalent to the following statement: If $z_n, z \in D(\mathbf{R}^{d+p})$ are such that

$$\lim_{n \rightarrow +\infty} \int_0^T (|z_n(s) - z(s)| \wedge 1) ds = 0, \quad \text{then} \quad \lim_{n \rightarrow +\infty} \int_0^T |f(s, z_n) - f(s, z)| ds = 0.$$

This is obviously satisfied if, ds -a.e. on $[0, T]$, the functional $f(s, \cdot, \cdot)$ on $D(\mathbf{R}^{d+p})$ is continuous with respect to the Meyer–Zheng topology. In particular, let φ be a measurable and bounded function on $[0, T] \times \mathbf{R}^d \times \mathbf{R}^p$ such that, for almost every $s \in [0, T]$, $\varphi(s, \cdot, \cdot)$ is continuous. We set

$$f(t, x, y) = \varphi(t, x(t), y(t)), \quad (t, x, y) \in [0, T] \times D(\mathbf{R}^{d+p}).$$

Then f satisfies the conditions (E.i)–(E.iii) of the theorem.

Pathwise uniqueness and uniqueness in law. In this last part of the present section, the notions of pathwise uniqueness and uniqueness in law of weak solutions of BSDE (H, f, μ) are introduced and a counterpart of the theorem of Yamada–Watanabe [13] for BSDEs is stated.

Definition 2.2. (i) *Pathwise uniqueness* is said to hold for BSDE (H, f, μ) , if whenever the six-tuples $(\Omega, \mathcal{F}, \mathbf{F}, \mathbf{P}, X, Y^i)$, $i = 1, 2$, are weak solutions of BSDE (H, f, μ) with the same filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbf{P})$ and the same driving process X , then $Y^1 = Y^2$ $\mathbf{P}$ -a.s.

(ii) *Joint uniqueness in law* is said to hold for BSDE (H, f, μ) , if whenever the six-tuples $(\Omega^i, \mathcal{F}^i, \mathbf{F}^i, \mathbf{P}^i, X^i, Y^i)$, $i = 1, 2$, are weak solutions of BSDE (H, f, μ) , then $\mathbf{P}_{(X^1, Y^1)}^1 = \mathbf{P}_{(X^2, Y^2)}^2$, i.e., the laws of (X^i, Y^i) with respect to $\mathbf{P}^i$ coincide.

(iii) *Uniqueness in law* is said to hold for BSDE (H, f, μ) , if whenever the six-tuples $(\Omega^i, \mathcal{F}^i, \mathbf{F}^i, \mathbf{P}^i, X^i, Y^i)$, $i = 1, 2$, are weak solutions of BSDE (H, f, μ) , then $\mathbf{P}_{Y^1}^1 = \mathbf{P}_{Y^2}^2$, i.e., the laws of Y^i with respect to $\mathbf{P}^i$ coincide.

Obviously, joint uniqueness in law implies uniqueness in law. The following statements are versions of the Yamada–Watanabe theorem and are borrowed from [2].

Proposition 2.2. *Pathwise uniqueness implies joint uniqueness in law (and, hence, uniqueness in law).*

Proposition 2.3. *If for BSDE (H, f, μ) there is a weak solution and pathwise uniqueness is satisfied, then there also exists a strong solution. Moreover, in this case, for every weak solution $(\Omega, \mathcal{F}, \mathbf{F}, \mathbf{P}, X, Y)$ of BSDE (H, f, μ) , the process Y is also a strong solution on $(\Omega, \mathcal{F}, \mathbf{P})$ of BSDE (H, f, X) .*

3. Continuity and pathwise uniqueness of solutions. To begin with, let us discuss a very simple example in detail for a better understanding of the situation.

Example 3.1. Let $d = p = 1$, $T = 1$, and $H = 0$. The driver f is defined by

$$f(t, x, y) = \varphi(y(t)) = 2\sqrt{y^+(t)} \wedge 1, \quad (t, y, x) \in [0, 1] \times D(\mathbf{R}^2),$$

where $a^+ = \max\{0, a\}$, $a \in \mathbf{R}$. Let W be a Brownian motion on $[0, 1]$ defined on $(\Omega, \mathcal{F}, \mathbf{P})$. Without loss of generality, it is assumed that there exists a $C \in \mathcal{F}$ such that C is independent of W and $0 < \mathbf{P}(C) < 1$.

1°. The continuous deterministic process $Y^{\max}$ defined by

$$Y_t^{\max} = (1 - t)^2, \quad t \in [0, 1],$$

is obviously a strong solution of BSDE $(0, f, W)$ on $(\Omega, \mathcal{F}, \mathbf{P})$.

2°. Another strong solution is $Y^{\min} \equiv 0$.

3°. The solutions $Y^{\max}$ and $Y^{\min} \equiv 0$ are different in law, the first one is the maximal and the second one the minimal solution.

4°. For the continuous random process $Y^0 = 1_C Y^{\max}$, it follows that the six-tuple $(\Omega, \mathcal{F}, \mathbf{F}^{W, Y^0}, \mathbf{P}, W, Y^0)$ is a weak solution of BSDE $(0, f, \mu)$, where μ is the Wiener measure. Its martingale part is zero. Notice that Y^0 is not adapted to $\mathbf{F}^W$ and hence is not a strong solution. For the filtration $\mathbf{F}^{W, Y^0}$ it holds $\mathcal{F}_t^{W, Y^0} = \sigma(\mathcal{F}_t^W \cup \{C\})$ for all $t \in [0, 1]$. The distribution Q^0 of Y^0 on $D(\mathbf{R})$ is just the mixture $Q^0 = \lambda Q^{\max} + (1 - \lambda) Q^{\min}$, where $Q^{\max}$ and $Q^{\min}$ denote the distribution of the maximal and minimal solution, respectively, and $\lambda = \mathbf{P}(C)$.

5°. For any u from $(0, 1)$, the process Y^u is introduced by

$$Y_t^u = \begin{cases} (\sqrt{\lambda}(1 - u) + u - t)^2, & t \in [0, u), \\ Y_t^0, & t \in [u, 1]. \end{cases}$$

It can easily be verified that the six-tuple $(\Omega, \mathcal{F}, \mathbf{F}^{W, Y^u}, \mathbf{P}, X, Y^u)$ is again a weak solution of BSDE $(0, f, \mu)$. This solution behaves like Y^0 on $[u, 1]$, and on $[0, u)$ it coincides with a unique strong solution of BSDE (H, f, W) with terminal time u and with terminal value $H = \mathbf{E}Y_u^0$. The process Y^u is deterministic on $[0, u)$ but random on $[u, 1]$. Hence it must have a jump in u , thus Y^u is not continuous and not quasi-left-continuous. A unique jump of Y^u and its martingale part M^u at u can easily be computed:

$$\Delta Y_u^u = \Delta M_u^u = (1 - u)^2(1_C - \lambda).$$

The filtration $\mathbf{F}^{W, Y^u}$ has the form

$$\mathcal{F}_t^{W, Y^u} = \begin{cases} \mathcal{F}_t^W, & t \in [0, u), \\ \sigma(\mathcal{F}_t^W \cup \{C\}), & t \in [u, 1]. \end{cases}$$

The martingale part M^u is given by

$$M_t^u = \mathbf{E}[(1 - u)^2 1_C | \mathcal{F}_t^{W, Y^u}] = \begin{cases} 0, & t \in [0, u), \\ (1 - u)^2 1_C - (1 - u)^2 \lambda, & t \in [u, 1]. \end{cases}$$

This completes the example.

One may wonder whether this example describes a natural phenomenon for general BSDEs. The aim of the present paper is to give an affirmative answer to this question.

In the remainder of the paper, it will always be assumed that (H, f, μ) is a data set as described in Section 2, with terminal functional H , driver f and distribution μ of the driving process, such that H is μ -integrable and conditions (E.i)–(E.iii) of Theorem 2.1 are satisfied.

The following theorem is the main result of this note.

Theorem 3.1. Suppose that all solutions of BSDE (H, f, μ) are continuous. Then pathwise uniqueness holds for BSDE (H, f, μ) .

The proof is postponed to Section 4.

Theorem 3.1 has important consequences.

Corollary 3.1. *Suppose that all solutions of BSDE (H, f, μ) are continuous. Then there exists a pathwise unique solution of BSDE (H, f, μ) , and this solution is a strong solution.*

P r o o f. According to Theorem 2.1 there is a solution of BSDE (H, f, μ) . By Theorem 3.1, the solution is pathwise unique. Finally, Proposition 2.3 yields that the solution is strong.

Corollary 3.2. (i) *If pathwise uniqueness does not hold for BSDE (H, f, μ) , then there exist discontinuous solutions of BSDE (H, f, μ) .*

(ii) *If uniqueness in law does not hold for BSDE (H, f, μ) , then there exist discontinuous solutions of BSDE (H, f, μ) .*

P r o o f. The first assertion is immediate from Theorem 3.1. Using Proposition 2.2, it follows that pathwise uniqueness does not hold if uniqueness in law does not hold. Hence, (ii) follows from (i).

R e m a r k 3.1. (i) It will be seen later from the proofs that under the conditions of Corollary 3.2 there even exists an uncountable family of discontinuous solutions.

(ii) In general, the driving process X being an arbitrary càdlàg process, for a weak solution $(\Omega, \mathcal{F}, \mathbf{F}, \mathbf{P}, X, Y)$ of BSDE (H, f, μ) , it may not be surprising if Y is discontinuous. But if we assume that X is continuous or that X is even a Brownian motion, the question is more delicate and Corollary 3.2 gives an unexpected answer.

In the remaining part of this section, let us concentrate on a special case which deserves particular interest: Suppose that μ is the probability law of a d -dimensional continuous local martingale satisfying the previsible representation property (cf. [12]).

Theorem 3.2. *Under the assumptions made above the following conditions on BSDE (H, f, μ) are equivalent:*

- (i) *Every solution is continuous;*
- (ii) *pathwise uniqueness holds;*
- (iii) *there exists a pathwise unique solution and this solution is strong;*
- (iv) *every solution is a strong solution.*

P r o o f. The implication (i) $\Rightarrow$ (ii) is a part of Theorem 3.1, while (ii) $\Rightarrow$ (iii) follows from Theorem 2.1 on the existence of solutions and Proposition 2.3 (cf. proof of Corollary 3.1). The implication (iii) $\Rightarrow$ (iv) is a consequence of Proposition 2.3. For proving the implication (iv) $\Rightarrow$ (i), it is sufficient to verify that every strong solution is continuous. To this end, let Y be a strong solution of BSDE (H, f, X) on the filtered probability space $(\Omega, \mathcal{F}, \mathbf{P}, \mathbf{F})$. By Proposition 2.1, applied to the filtration $\mathbf{F} = \mathbf{F}^X$, the process Y satisfies

$$Y_t = H(X) + \int_t^T \{f(\cdot, X, Y)\}_s^{\mathbf{F}^X} ds - (M_T - M_t), \quad t \in [0, T],$$

where M is an $\mathbf{F}^X$ -martingale such that $M_0 = 0$ and $\{f(\cdot, X, Y)\}_s^{\mathbf{F}^X}$ denotes the optional projection of the process $f(\cdot, X, Y)$ with respect to the filtration $\mathbf{F}^X$. Because of the predictable representation property of $(X, \mathbf{F}^X)$, there exists an $\mathbf{F}^X$ -predictable $\mathbf{R}^{p \times d}$ -valued process Z such that

$$M_t = \int_0^t Z_s dX_s, \quad t \in [0, T].$$

Hence, M is continuous. But this implies that Y is continuous, completing the proof of the theorem.

4. Proof of Theorem 3.1. Let us recall that (H, f, μ) is a data set (cf. beginning of Section 2) with terminal functional H , driver f , and distribution μ of the driving process such that H is μ -integrable and conditions (E.i)–(E.iii) of Theorem 2.1 are satisfied.

Lemma 4.1. *Let be given a solution $(\Omega, \mathcal{F}, \mathbf{F}, \mathbf{P}, X, Y)$ of BSDE (H, f, μ) and a filtration $\mathbf{G}$ satisfying the usual conditions and $\mathbf{F} \subseteq \mathbf{G}$. Suppose that every $\mathbf{F}$ -martingale is also a $\mathbf{G}$ -martingale. Then $(\Omega, \mathcal{F}, \mathbf{G}, \mathbf{P}, X, Y)$ is also a solution of BSDE (H, f, μ) .*

P r o o f. To verify conditions (W.i)–(W.v) of Definition 2.1 for $(\Omega, \mathcal{F}, \mathbf{G}, \mathbf{P}, X, Y)$, it suffices to consider (W.iv)–(W.v). Regarding (W.iv), let $(M, \mathbf{F}^X)$ be a martingale. Since $(\Omega, \mathcal{F}, \mathbf{F}, \mathbf{P}, X, Y)$ is a solution of BSDE (H, f, μ) , the process $(M, \mathbf{F})$ is also a martingale. Now the assumption of the lemma yields that $(M, \mathbf{G})$ is a martingale, too. Hence (W.iv) is valid. Finally, the assumption of the lemma is equivalent to $\mathbf{E}[L|\mathcal{F}_t] = \mathbf{E}[L|\mathcal{G}_t]$ for every integrable $\mathcal{F}_T$ -measurable random variable L and for all $t \in [0, T]$. This yields

$$\mathbf{E}\left[H(X) + \int_t^T f(s, X, Y) ds \mid \mathcal{F}_t\right] = \mathbf{E}\left[H(X) + \int_t^T f(s, X, Y) ds \mid \mathcal{G}_t\right],$$

and hence (W.v) is also valid. Lemma 4.1 is proved.

Let $\tau \in (0, T)$ and let the driver $\tilde{f}_\tau$ be defined by

$$\tilde{f}_\tau(t, x, y) = \mathbf{1}_{[\tau, T]}(t) f(t, x, y), \quad (t, x, y) \in [0, T] \times D(\mathbf{R}^{d+p}).$$

Lemma 4.2. *Suppose that for BSDE (H, f, μ) pathwise uniqueness does not hold. Then there exist $\tau \in (0, T)$ and a solution $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{G}}, \tilde{\mathbf{P}}, \tilde{X}, \tilde{Y})$ of BSDE $(H, \tilde{f}_\tau, \mu)$ such that*

$$\tilde{\mathbf{P}}\{\tilde{Y}_{\tau-} \neq \tilde{Y}_\tau\} > 0. \quad (4)$$

P r o o f. Together with Theorem 2.1, the assumption of the lemma implies that there are solutions $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{F}}, \tilde{\mathbf{P}}, \tilde{X}, \tilde{Y}^i)$, $i = 1, 2$, of BSDE (H, f, μ) , with the same filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{F}}, \tilde{\mathbf{P}})$ and the same driving process $\tilde{X}$, such that $\tilde{Y}^1$ and $\tilde{Y}^2$ are not $\mathbf{P}$ -a.s. pathwise equal. Hence there exists $\tau \in (0, T)$ with the property

$$\tilde{\mathbf{P}}\{\tilde{Y}_\tau^1 \neq \tilde{Y}_\tau^2\} > 0.$$

Since $\tilde{Y}^1, \tilde{Y}^2$ have at most countably many fixed discontinuities, τ can be chosen such that $\tilde{Y}_{\tau-}^i = \tilde{Y}_\tau^i$ $\mathbf{P}$ -a.s., $i = 1, 2$, holds. As a consequence, $\tilde{Y}_\tau^i$ is $\tilde{\mathcal{F}}_{\tau-}$ -measurable, $i = 1, 2$. Enlarging the probability space, if necessary, it can be assumed that there is $B \in \tilde{\mathcal{F}}$ which is independent of $\tilde{\mathcal{F}}_T$ and such that $0 < \lambda := \tilde{\mathbf{P}}(B) < 1$ holds. Now let us define $\tilde{\mathbf{G}}$ as the filtration

$$\tilde{\mathcal{G}}_t = \begin{cases} \tilde{\mathcal{F}}_t, & t \in [0, \tau), \\ \sigma(\tilde{\mathcal{F}}_t \cup \{B\}), & t \in [\tau, T], \end{cases}$$

and $\tilde{Y}$ as the process

$$\tilde{Y}_t = \begin{cases} \mathbf{1}_B \tilde{Y}_t^1 + \mathbf{1}_{B^c} \tilde{Y}_t^2, & t \in [\tau, T], \\ \mathbf{E}[\tilde{Y}_\tau | \mathcal{G}_t], & t \in [0, \tau). \end{cases}$$

It can easily be seen that $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{G}}, \tilde{\mathbf{P}}, \tilde{X}, \tilde{Y})$ is a solution of BSDE $(H, \tilde{f}_\tau, \mu)$. The set B being independent of $\tilde{\mathcal{G}}_{\tau-} = \tilde{\mathcal{F}}_{\tau-}$, it follows that

$$\tilde{Y}_{\tau-} = \mathbf{E}[\tilde{Y}_\tau | \tilde{\mathcal{G}}_{\tau-}] = \lambda \tilde{Y}_\tau^1 + (1 - \lambda) \tilde{Y}_\tau^2.$$

But the right-hand side is distinct from $\tilde{Y}_\tau$ on the set $\{\tilde{Y}_\tau^1 \neq \tilde{Y}_\tau^2\}$, proving the claim.

In the next step, a solution of BSDE (H, f, μ) on $[0, \tau]$ with terminal value $\tilde{Y}_{\tau-}$ is constructed. Glueing together this solution with the solution obtained in Lemma 4.2 restricted to $[\tau, T]$, will yield a solution of BSDE (H, f, μ) on $[0, T]$ which is *discontinuous* at τ .

With this aim, let $\hat{\mu} = \tilde{\mathbf{P}}_{(\tilde{X}, \tilde{Y})}$ be the distribution of $(\tilde{X}, \tilde{Y})$ on the space $(D(\mathbf{R}^{d+p}), \mathcal{D}(\mathbf{R}^{d+p}))$, where $(\tilde{X}, \tilde{Y})$ is the solution constructed in Lemma 4.2. Furthermore, a terminal functional $\hat{H}$ and a driver $\hat{f}_\tau$ are defined as follows:

$$\hat{H}(x, y) = y(\tau-), \quad (x, y) \in D(\mathbf{R}^{d+p}),$$

$$\hat{f}_\tau(t, (x, y), \hat{y}) = \begin{cases} 0, & t \in [\tau, T], \\ f(t, x, \hat{y}(y)_\tau), & t \in [0, \tau), \end{cases}$$

where the notation

$$\widehat{y}(y)_\tau(t) = \begin{cases} y(t), & t \in [\tau, T], \\ \widehat{y}(t), & t \in [0, \tau], \end{cases}$$

is used.

Lemma 4.3. Suppose that for BSDE (H, f, μ) pathwise uniqueness does not hold. Then there exists a solution $(\Omega, \mathcal{F}, \mathbf{F}, \mathbf{P}, (X, Y), \widehat{Y})$ of BSDE $(\widehat{H}, \widehat{f}_\tau, \widehat{\mu})$, with driving process (X, Y) having $\widehat{\mu}$ as its distribution.

P r o o f. To verify that $\widehat{f}_\tau$ is measurable on the product space $[0, T] \times D(\mathbf{R}^{d+p}) \times D(\mathbf{R}^p)$, it is sufficient to show that the mapping $(t, (x, y), \widehat{y}) \mapsto (t, x, \widehat{y}(y)_\tau)$ is measurable from this space into $[0, T] \times D(\mathbf{R}^d) \times D(\mathbf{R}^p)$. The proof is straight-forward and therefore omitted. The condition (E.i) for $\widehat{f}_\tau$ is obvious. Establishing (E.ii) for $\widehat{f}_\tau$, reduces to verify that, for every fixed $y \in D(\mathbf{R}^p)$, the mapping $\widehat{y} \rightarrow \widehat{y}(y)_\tau$ from $(D_T^t(\mathbf{R}^p), \mathcal{D}_T^t(\mathbf{R}^p))$ into itself is measurable, which is immediately seen. Condition (E.iii) for $\widehat{f}_\tau$ follows from the continuity of the mapping $((x, y), \widehat{y}) \mapsto (x, \widehat{y}(y)_\tau)$ in the Meyer-Zheng topology. Finally, $\widehat{H}$ is obviously a measurable and $\widehat{\mu}$ -integrable functional on $D(\mathbf{R}^{d+p})$. Hence all conditions of Theorem 2.1 are satisfied. Consequently, there exists a solution of BSDE $(\widehat{H}, \widehat{f}_\tau, \widehat{\mu})$, proving the assertion of the lemma.

After these preparations, it is now easy to complete the proof of Theorem 3.1. The statement of Theorem 3.1 amounts to saying that, if pathwise uniqueness does not hold for BSDE (H, f, μ) , then there exists a *discontinuous* solution of BSDE (H, f, μ) . To prove this implication, suppose that pathwise uniqueness is not satisfied for BSDE (H, f, μ) . Using Lemmas 4.2 and 4.3, there can be found a solution $(\Omega, \mathcal{F}, \mathbf{F}, \mathbf{P}, (X, Y), \widehat{Y})$ of BSDE $(\widehat{H}, \widehat{f}_\tau, \widehat{\mu})$, with driving process (X, Y) having $\widehat{\mu}$ as its distribution. The distribution $\mathbf{P}_{(X, Y)}$ of (X, Y) being the same as the distribution $\widehat{\mathbf{P}}_{(\widetilde{X}, \widetilde{Y})}$ of $(\widetilde{X}, \widetilde{Y})$, where $(\widetilde{X}, \widetilde{Y})$ is the solution pair of Lemma 4.2, implies that $(\Omega, \mathcal{F}, \mathbf{F}^{X, Y}, \mathbf{P}, X, Y)$ is a solution of BSDE $(H, \widehat{f}_\tau, \mu)$, and this solution has the property $\mathbf{P}\{Y_{\tau-} \neq Y_\tau\} > 0$. By property of the solution $(\Omega, \mathcal{F}, \mathbf{F}, \mathbf{P}, (X, Y), \widehat{Y})$ of BSDE $(\widehat{H}, \widehat{f}_\tau, \widehat{\mu})$ that every $\mathbf{F}^{X, Y}$ -martingale is also an $\mathbf{F}$ -martingale, it follows from Lemma 4.1 that $(\Omega, \mathcal{F}, \mathbf{F}, \mathbf{P}, X, Y)$ is also a solution of BSDE $(H, \widehat{f}_\tau, \mu)$. Now let us define

$$\overline{Y}_t = \begin{cases} Y_t, & t \in [\tau, T], \\ \widehat{Y}_t, & t \in [0, \tau]. \end{cases}$$

Then a straightforward verification of properties (W.i)–(W.v) shows that the six-tuple $(\Omega, \mathcal{F}, \mathbf{F}, \mathbf{P}, X, \overline{Y})$ is a solution of BSDE (H, f, μ) . By construction, it holds $\mathbf{P}\{\overline{Y}_{\tau-} \neq \overline{Y}_\tau\} > 0$. This finishes the proof of Theorem 3.1.

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A LEMMA ON STOCHASTIC MAJORIZATION AND PROPERTIES OF THE STUDENT DISTRIBUTION¹

Доказана общая лемма о стохастической мажоризации, позволяющая получить нижнюю и верхнюю границы для функции распределения Стьюдента. Обсуждается связь с оцениванием среднего значения нормального распределения посредством доверительных интервалов.

Ключевые слова и фразы: доверительные интервалы, нормальное распределение.

1. Introduction. Let T_n be a random variable having the Student distribution with n degrees of freedom,

$$\mathbf{P}\{T_n \leq x\} = S_n(x) = \int_{-\infty}^x s_n(z) dz,$$

where

$$s_n(z) = \frac{1}{\sqrt{n\pi}} \frac{\Gamma((n+1)/2)}{\Gamma(n/2)} \left(1 + \frac{z^2}{n}\right)^{-(n+1)/2}, \quad (1)$$

and let Z be a standard normal random variable,

$$\mathbf{P}\{Z \leq x\} = \Phi(x) = \int_{-\infty}^x \varphi(z) dz.$$

In terminology of stochastic majorization (see [3]), $|T_n|$ is stochastically larger than $|Z|$ which, in view of $\mathbf{P}\{T_n \leq 0\} = \mathbf{P}\{Z \leq 0\} = \frac{1}{2}$, is equivalent to

$$\mathbf{P}\{T_n > x\} > \mathbf{P}\{Z > x\}. \quad (2)$$

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