Math-Net.Ru

Общероссийский математический портал

M. Ya. Kelbert, Yu. M. Sukhov, Branching diffusions on H^d with variable fission: The Hausdorff dimension of the limiting set, *Теория вероятн. и ее примен.*, 2006, том 51, выпуск 1, 241–255

DOI: <https://doi.org/10.4213/tvp155>

Использование Общероссийского математического портала Math-Net.Ru подразумевает, что вы прочитали и согласны с пользовательским соглашением

<http://www.mathnet.ru/rus/agreement>

Параметры загрузки:

IP: 3.93.64.190

26 апреля 2023 г., 13:05:57

© 2006 г.

KELBERT M.*, SUHOV Yu. M.**

**BRANCHING DIFFUSIONS ON $\mathbf{H}^d$ WITH VARIABLE FISSION:
THE HAUSDORFF DIMENSION OF THE LIMITING SET**

В статье результаты работ [14] и [11] о хаусдорфовой размерности предельного множества однородного гиперболического ветвящегося диффузионного процесса распространяются на случай непостоянного механизма деления. Точнее, мы рассматриваем неоднородный ветвящийся диффузионный процесс на пространстве Лобачевского $\mathbf{H}^d$ и предполагаем, что параметры процесса равномерно сходятся к их предельным значениям на абсолюте $\partial\mathbf{H}^d$. В этих предположениях устанавливается формула для хаусдорфовой размерности $h(\Lambda)$ предельного (случайного) множества $\Lambda \subseteq \partial\mathbf{H}^d$, которая согласуется с формулами, полученными в [14] и [11] для однородного случая. Метод основан на свойствах минимального решения уравнения Штурма–Лиувилля с потенциалом, принимающим два значения, и на элементах гармонического анализа на $\mathbf{H}^d$.

Ключевые слова и фразы: пространство Лобачевского, ветвящийся диффузионный процесс, предельное множество, хаусдорфова размерность, орисферическая проекция, эквидистантная проекция, уравнение Штурма–Лиувилля, минимальное положительное решение.

1. Introduction. Homogeneous branching processes on a d -dimensional Lobachevsky space $\mathbf{H}^d$ have been considered first by Lalley and Sellke [14] in the two-dimensional case (the branching diffusion on the Lobachevsky plane $\mathbf{H}^2$) and subsequently by Karpelevich, Pechersky, and Suhov [11] in a general case. Following [14], we call such process a homogeneous hyperbolic branching process (HHBP). A feature of such a process is that an «individual» Markov trajectory is attracted to the absolute but if the branching mechanism is «strong enough» the entropy vs. energy balance is in favour of entropy. As a result, «more branches are produced» than «driven to infinity». Consequently, the branching process population visits

* Mathematics Department, University of Wales Swansea, Singleton Park, Swansea, SA2 8PP, UK; Международный институт прогноза землетрясений и математической геофизики РАН, Варшавское ш., 79, 117556 Москва, Россия; e-mail: M. Kelbert@swansea.ac.uk

** Statistical Laboratory, DPMMS/CMS, University of Cambridge, Cambridge CB3 0WB, UK; Институт проблем передачи информации РАН, Б. Каретный пер., 19, 127994 Москва, Россия; e-mail: yms@statslab.cam.ac.uk

every compact at indefinitely large times. On the contrary, if the branching mechanism is «weak», the branching process population eventually stops visiting any compact. (Both properties take place with probability one.)

The main result of [14] and [11], for an HHBP on a Lobachevsky space, was as follows. (I) The Hausdorff dimension $h(\Lambda)$ of the limiting (random) set Λ on the absolute $\partial\mathbf{H}^d$, calculated in the so-called angular metric on $\partial\mathbf{H}^d$ (which, in the Klein model of $\mathbf{H}^d$ as the interior of a unit ball in $\mathbf{R}^d$, and with the centre at the origin, coincides with the Riemannian metric on the unit sphere in $\mathbf{R}^d$) is almost surely constant. (II) $h(\Lambda)$ changes with the fission parameters in a peculiar fashion. For instance, set the diffusion coefficient of the individual motion to be 1 and consider the value $h(\Lambda)$ as a function of $\lambda > 0$, the fission rate, and κ , the mean number of offspring in a single fission, assuming that $\kappa > 1$. In fact, it is the product $v = \lambda(\kappa - 1)$, called the fission potential (FP), that plays the crucial role here, and we write $h(\Lambda)$ as a function $f(v)$. Then for $0 \leq v < v_{\text{cr}}$, $f(v)$ monotonically increases with v from 0 to $(d-1)/2$, where d is the dimension of the space, and then jumps to the value $d-1$, the dimension of the absolute. This phenomenon is a consequence of some algebraic properties related to the hyperbolic geometry (the Harish-Chandra theorem; see [5] and [6, Theorem 10.1.15]). The value v_{cr} and the function $v \mapsto f(v)$ have a transparent meaning in terms of large deviations (see [11]).

More precisely, $h(\Lambda)$ as a function of v is

$$f(v) = \begin{cases} \frac{d-1 - ((d-1)^2 - 8v)^{1/2}}{2} & \text{if } 0 \leq v \leq \frac{(d-1)^2}{8}, \\ d-1 & \text{if } v > \frac{(d-1)^2}{8}. \end{cases} \quad (1.1)$$

with $v_{\text{cr}} = (d-1)^2/8$. As Λ is a closed continuum which, for $v > (d-1)^2/8$, is everywhere dense on $\partial\mathbf{H}^d$, it follows that $\Lambda = \partial\mathbf{H}^d$ for $v > (d-1)^2/8$.

It is interesting to assess the Hausdorff dimension of the limiting set for a nonhomogeneous hyperbolic branching diffusion (NHBD). A simple case here is where the individual motion is still homogeneous, but the parameters of the branching mechanism, i.e., the fission rate and the mean number of offspring, depends on $z \in \mathbf{H}^d$: $\lambda = \lambda(z)$ and $\kappa = \kappa(z)$. More precisely, let $p_k(z)$ be the probability that in the course of fission at point z , the number of offspring produced equals k , then $\kappa(z) = \sum_{k=1}^{\infty} kp_k(z)$. Define the *local fission potential* (LFP) by $v(z) = \lambda(z)(\kappa(z) - 1)$. We assume that, when point z tends to the absolute $\partial\mathbf{H}^d$ (say in the Euclidean metric, in the Klein model of $\mathbf{H}^d$), both $\lambda(z)$ and probability distribution $P(z) = \{p_k(z)\}$ approach from below their limits λ^+ and $P^+ = \{p_k^+\}$, with $\kappa(z)$, approaching from below the mean value $\kappa^+ = \sum_{k=1}^{\infty} kp_k^+$ and $v(z)$ approaching from below the limiting value say $v^+ = \lambda^+(\kappa^+ - 1)$. For the precise formulation, see Theorem 2.1 below. The result of this paper is that in this case the Hausdorff

dimension $h(\Lambda)$ of the limiting set $\Lambda \subset \partial\mathbf{H}^d$ equals $f(v^+)$, the value given by the right-hand side of (1.1), for a fixed fission mechanism.

2. Main results. As was mentioned, in the Klein model, the d -dimensional Lobachevsky space $\mathbf{H}^d$ is represented by the interior $\{x = (x_1, \dots, x_d): x_1^2 + \dots + x_d^2 < 1\}$ of the unit ball around the origin in $\mathbf{R}^d$, with a special Riemannian metric ρ which is called the Klein metric. The origin in $\mathbf{R}^d$ will be referred to as the centre of $\mathbf{H}^d$. The absolute $\partial\mathbf{H}^d$ in this model is the unit sphere in $\mathbf{R}^d$ around the origin. The union $\mathbf{H}^d \cup \partial\mathbf{H}^d$ endowed with the standard Euclidean metric on $\mathbf{R}^d$ will be denoted by $\bar{\mathbf{H}}^d$; it is simply the closed unit ball in $\mathbf{R}^d$. Next, in metric ρ , geodesics of $\mathbf{H}^d$ entering the absolute at a given point $w \in \partial\mathbf{H}^d$ are (open) directed chords with the endpoint at w ; the family of all such geodesics is denoted by Σ_w . Further, in metric ρ , (i) horoballs with the pole at w are d -dimensional ellipsoids lying inside $\mathbf{H}^d$ and touching $\partial\mathbf{H}^d$ at w , and (ii) horospheres (the boundary manifolds of horoballs) with the pole at w are ellipsoidal $(d-1)$ -dimensional surfaces touching $\partial\mathbf{H}^d$ at w and orthogonal to every geodesic $g \in \Sigma_w$ at the (unique) point of their intersection within $\mathbf{H}^d$ (by definition, point w does not lie in horoballs or horospheres). The family of horoballs with pole at w is denoted by $\mathcal{B}_w$, and the family of horospheres with pole at w by $\mathcal{S}_w$.

There is a number of possible metrics that can be introduced on $\partial\mathbf{H}^d$. The standard Riemannian metric on the unit sphere in $\mathbf{R}^d$ is one example; it is imbedded in the class of angular metrics, where the distance between points $w, w' \in \partial\mathbf{H}^d$ is taken to be the angle between geodesics $g_w \in \Sigma_w$ and $g_{w'} \in \Sigma_{w'}$ passing through a fixed point in $\mathbf{H}^d$. Another example is the stereographic projection metric, depending on a reference point $w^0 \in \partial\mathbf{H}^d$. It turns out that the Hausdorff dimension of the (random) set of interest in this paper is the same in each of these metrics; for definiteness, we refer in Theorem 2.1 to the standard Riemannian metric on $\partial\mathbf{H}^d$.

For detailed definitions, see, e.g., [11].

We will consider NHBD Θ on $\mathbf{H}^d$, starting from the centre of $\mathbf{H}^d$, where the individual motion is a «standard» homogeneous hyperbolic Brownian motion (HHBM), with diffusion coefficient 1. Next, we will assume that the fission rate $\lambda(z)$ and the probability distribution $P(z) = \{p_k(z), k \geq 1\}$ of the number of offspring in a fission at point $z \in \partial\mathbf{H}^d$, with mean $\kappa(z) \in (1, \infty)$, are measurable in $z \in \mathbf{H}^d$ and satisfy the following condition. There exists $\lambda^+ > 0$ and a probability distribution $P^+ = \{p_k^+, k \geq 1\}$, with mean $\kappa^+ \in (1, \infty)$ and FP $v^+ := \lambda^+(\kappa^+ - 1)$ such that

(a) $\lambda(z) \leq \lambda^+$ and $P(z) \preceq P^+$ (here and below $\preceq$ stands for stochastic ordering of probability distributions (on $\mathbf{Z}_+ = \{1, 2, \dots\}$)),

(b) for any $\varepsilon > 0$, there exists a point $w = w(\varepsilon)$ on the absolute $\partial\mathbf{H}^d$, a horoball $B(\varepsilon) \in \mathcal{B}_w$, a number $\lambda^- \in (\lambda^+ - \varepsilon, \lambda^+)$, and a probability

distribution $P^- = \{p_k^-, k \geq 1\} \preceq P^+$ with $\|P^+ - P^-\| < \varepsilon$, mean $\kappa^- \in (\kappa^+ - \varepsilon, \kappa^+)$, and FP $v^- = \lambda^-(\kappa^- - 1) \in (v^+ - \varepsilon, v^+)$ such that for all $z \notin B(\varepsilon)$,

$$\lambda^- \leq \lambda(z) \leq \lambda^+, \quad P^- \preceq P \preceq P^+, \quad \kappa^- \leq \kappa(z) \leq \kappa^+, \quad (2.1)$$

implying $v^- \leq v(z) \leq v^+$. Here $\|P^+ - P^-\| = \sum_{k=1}^{\infty} |p_k^+ - p_k^-|$ stands for the ℓ_1 -norm.

An example where this assumption holds is where offspring probabilities $p_k(z)$ do not depend on fission point $z \in \mathbf{H}^d$ (and hence coincide with probabilities p_k^+). Here $\kappa(z) = \kappa^+$ is constant, and the only varying parameter is fission rate $\lambda(z)$. The condition is that there exists a constant value $\lambda^+ > 0$ such that for any $\varepsilon > 0$ we can find $w = w(\varepsilon) \in \partial\mathbf{H}^d$ and a horoball $B(\varepsilon) \in \mathcal{B}_w$ such that $0 \leq \lambda^+ - \lambda(z) \leq \varepsilon$, for all $z \in \mathbf{H}^d \setminus B(\varepsilon)$. The limiting FP is $v^+ = \lambda^+(\kappa - 1)$.

The result of this paper is the following theorem.

Theorem 2.1. *Consider a NHBD Θ on $\mathbf{H}^d$, starting from the centre of $\mathbf{H}^d$, where the individual motion is a HHBP, and the fission rate $\lambda(z)$ and the probability distribution $P(z)$ of the number of offspring at point $z \in \mathbf{H}^d$ satisfy the above assumption. Define $\Lambda \subseteq \partial\mathbf{H}^d$ as the (random) set of accumulation points, for the NHBD Θ , in the Euclidean metric on $\overline{\mathbf{H}^d}$.*

(i) *Assume that limiting fission rate $v^+ \leq \frac{1}{8}(d-1)^2$. Then the Hausdorff dimension $h(\Lambda)$ ($= h_{\Theta}(\Lambda)$) is almost surely constant and equals*

$$\frac{1}{2} \left[d - 1 - \sqrt{(d-1)^2 - 8v^+} \right].$$

(ii) *Assume that $v^+ > \frac{1}{8}(d-1)^2$. Then with probability 1, Λ coincides with $\partial\mathbf{H}^d$.*

R e m a r k. Under assumption $v^+ > \frac{1}{8}(d-1)^2$, branching process Θ is recurrent, in the sense defined in [3] (i.e., with probability 1, for any neighborhood $\mathbf{O}$ of the origin, members of the population in process Θ will visit $\mathbf{O}$ at indefinitely large times). The latter fact implies that with probability 1, Λ coincides with $\partial\mathbf{H}^d$. However, for completeness, we will sketch a direct proof.

Statement (ii) of Theorem 2.1 can be deduced, following an argument developed originally in [14] and repeated in [11], from the following theorem.

Theorem 2.2. *Under the conditions of Theorem 2.1 (ii), there exists a point $w \in \partial\mathbf{H}^d$ and a horoball $B \in \mathcal{B}_w$ such that, with probability 1, for any compact $K \subset \mathbf{H}^d \setminus B$ there exists an infinite trajectory of process Θ remaining in K .*

Statement (i) of Theorem 2.1 follows from two assertions; the first assertion is again not very hard:

Theorem 2.3. *Under the conditions of Theorem 2.1 (i), the Hausdorff dimension $h(\Lambda)$ is almost surely constant and does not exceed $\frac{1}{2}[d-1-\sqrt{(d-1)^2-8v^+}]$.*

Analogues of Theorems 2.2 and 2.3 can be established for a large class of branching processes on $\mathbf{H}^d$. The most delicate fact, where we use detailed information about process Θ , is the following lower bound.

Theorem 2.4. *Under the conditions of Theorem 2.1 (i), the Hausdorff dimension $h(\Lambda)$ is almost surely constant and is greater than or equal to $\frac{1}{2}[d-1-\sqrt{(d-1)^2-8v^+}]$.*

The proof of Theorems 2.2–2.4 is based on an analysis of particular examples of the NHBD, where the LFP takes at most two values, v_1 outside and v_2 inside a horoball. In particular, the HHBD Θ^+ with a constant fission rate and the same probability distribution of the number of offspring:

$$\lambda(z) \equiv \lambda^+, \quad P(z) \equiv P^+, \quad (2.2)$$

will give a majorant for original NHBD Θ . On the other hand, consider the NHBD $\Theta^- (= \Theta_\varepsilon^-)$ with the fission rate

$$\lambda^-(z) = \begin{cases} 0, & z \in B(\varepsilon), \\ \lambda^-, & z \in \mathbf{H}^d \setminus B(\varepsilon), \end{cases} \quad (2.3)$$

the offspring number probabilities

$$p_k(z) = \begin{cases} \delta_{1k}, & z \in B(\varepsilon), \\ p_k^-, & z \in \mathbf{H}^d \setminus B(\varepsilon), \end{cases} \quad (2.4)$$

and the LFP

$$v^-(z) = \begin{cases} 0, & z \in B(\varepsilon), \\ \lambda^-(\kappa^- - 1), & z \in \mathbf{H}^d \setminus B(\varepsilon), \end{cases} \quad (2.5)$$

where point $w \in \partial\mathbf{H}^d$, horoball $B(\varepsilon) \in \mathcal{B}_w$, and values λ^- and p_k^- are as in Theorem 2.1, with $\kappa^- = \sum_{k=1}^{\infty} kp_k^-$. Process Θ^- will be a minorant for original NHBD Θ . More precisely, by performing a standard coupling of three branching processes, we obtain that almost surely the respective limiting sets

$$\Lambda^- \subseteq \Lambda \subseteq \Lambda^+, \quad (2.6)$$

implying the corresponding bounds for the Hausdorff dimensions:

$$h(\Lambda^-) \leq h(\Lambda) \leq h(\Lambda^+). \quad (2.7)$$

Note that, as process Θ^+ is an HHBP, $h(\Lambda^+)$ is computed by means of [14] and [11]; cf. (1.1). Therefore, it suffices to prove the following Theorems 2.2A and 2.3A concerning only process Θ^- .

Theorem 2.2A. *Under the conditions of Theorem 2.1 (ii), for any $\varepsilon > 0$ small enough, with probability one, for any compact $K \subset \mathbf{H}^d \setminus B(\varepsilon)$ there exists an infinite trajectory of process Θ^- remaining in K . Here $B(\varepsilon)$*

is the horoball from Theorem 2.1. This implies that with probability one, $h(\Lambda^-) = d - 1$. Thus, in process Θ^- , with probability one, $\Lambda^- = \partial\mathbf{H}^d$.

Theorem 2.3A. *Under the conditions of Theorem 2.1 (i), the Hausdorff dimension $h(\Lambda^-)$ in process Θ^- is almost surely constant and equals*

$$\frac{1}{2} \left[d - 1 - \sqrt{(d - 1)^2 - 8(v^+ - \varepsilon)} \right].$$

As goes back to [14], the study of $h(\Lambda^-)$ is based on the so-called horospheric projection. See also [11].

The horospheric projection arises when we fix a geodesic $g \in \Sigma_w$ and send a point z on a horosphere $S \in \mathcal{S}_w$ to the (unique) point of intersection $g \cap S$. We choose the orientation on g , where pole w corresponds to $+\infty$, and fix the origin on g as the point of intersection $g \cap S_w(0)$, where $S_w(0)$ is the (unique) horosphere from $\mathcal{S}_w$ passing through the centre of $\mathbf{H}^d$; this allows us to identify g with a real line $\mathbf{R}$. Then, given $z \in \mathbf{H}^d$, denote by $y = y(z)$ the point of $\mathbf{R}$ represented by the above intersection $g \cap S$, where $S (= S_w(z))$ is the (unique) horosphere from $\mathcal{S}_w$ passing through z . The horospheric projection is

$$\Pi_g^{\text{ho}}: \mathbf{H} \rightarrow \mathbf{R}; \quad z \mapsto y(z). \quad (2.8)$$

A key observation used in [14] and [11] was that the horospheric projection of an HHBD on $\mathbf{H}^d$ is a homogeneous branching diffusion (HBD) on $\mathbf{R}$, with drift $-(d - 1)/2$. In the current study an important role is played by the following lemma.

Lemma 2.1. *Take $w = w(\varepsilon) \in \partial\mathbf{H}^d$ and horoball $B(\varepsilon) \in \mathcal{B}_w$ as in the definition of NHBD Θ^- . Let $g \in \Sigma_w$ be a geodesic with the pole at w . The horospheric projection $\Pi_g^{\text{ho}}\Theta^-$ of process Θ^- is an NHBD on $\mathbf{R}$, with drift $-(d - 1)/2$ and the LFP*

$$v^-(y) = \begin{cases} 0, & y \geq r, \\ \lambda^-(\kappa^- - 1), & y < r, \end{cases} \quad (2.9)$$

where r is the coordinate of the point of intersection $g \cap B(\varepsilon)$.

The assertion of Lemma 2.1 follows directly from the construction. Properties of NHBDs on $\mathbf{R}$ are discussed in Section 3 below.

For reader's convenience, we illustrate our results in the case $d = 2$ (the Lobachevsky plane), in the Poincaré model, where $\mathbf{H}^2$ is represented by an open upper complex half-plane, with $z = x + iy$, $x \in \mathbf{R}$, $y > 0$. Here, the «centre» of $\mathbf{H}^2$ is point i , while the absolute $\partial\mathbf{H}^2$ is the extended real line $\bar{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$. Choosing $w = \infty$ identifies the geodesics $g \in \Sigma_w$ as vertical half-lines $\{z: \operatorname{Re} z = x, \operatorname{Im} z > 0\}$ labelled by $x \in \mathbf{R}$. In this case, the horospheres (horocycles) $S \in \mathcal{S}_w$ are represented by horizontal lines $\{z: \operatorname{Im} z = y\}$, and horoballs (horocircles) by half-planes $\{z: \operatorname{Im} z \geq y\}$, labelled by $y > 0$. The horocyclic projection is $z = x + yi \mapsto y$. The union

$\overline{\mathbf{H}}^d = \mathbf{H}^d \cup \partial\mathbf{H}^2$ is a standard compactification of closed upper complex half-plane. The HHBM $Z_t = (X_t, Y_t)$ has components X_t and Y_t giving the solution to

$$dX_t = Y_t dW_t^{(1)}, \quad dY_t = Y_t dW_t^{(2)}, \quad (2.10)$$

where $(W_t^{(1)}, W_t^{(2)})$ are standard independent Brownian motions on $\mathbf{R}$. The horocyclic projection of Z_t is process $\ln Y_t$ which is a standard Brownian motion on $\mathbf{R}$, with drift $-\frac{1}{2}$.

In this context, an example of an NHBD Θ , where the assumptions of Theorem 2.1 are fulfilled, is as follows.

- (1) The single-particle motion is Z_t .
- (2) Fission rate $\lambda(z)$ depends only on variable y , with $\lambda(y) \nearrow \lambda^+$ as $y \searrow 0$.
- (3) Offspring probabilities $p_k(z)$ are fixed, i.e., do not depend on $z \in \mathbf{H}^2$.

3. Branching diffusion on a line: Fission potential with two values. It is convenient to analyze a general model of NHBD on a line $\mathbf{R}$, starting from an arbitrary point $y \in \mathbf{R}$, where both the branching rate $\lambda(y)$ and probabilities $p_k(y)$, $k = 1, 2, \dots$, to produce k offsprings are variable. In fact, it is only the expected value $\kappa(y) = \sum_{k=1}^{\infty} k p_k(y)$ that matters in our presentation. Set

$$v(y) = \lambda(y)(\kappa(y) - 1)$$

and, given $L > 0$, consider an ordinary differential equation with a boundary condition:

$$\frac{1}{2} \frac{d^2 u}{dy^2}(y) - \frac{d-1}{2} \frac{du}{dy}(y) + v(y)u(y) = 0, \quad u(-L) = 1. \quad (3.1)$$

After the substitution $w e^{(d-1)y/2} = u$ we obtain the problem

$$\frac{1}{2} \frac{d^2 w}{dy^2}(y) + \left[v(y) - \frac{(d-1)^2}{8} \right] w(y) = 0, \quad w(-L) = e^{(d-1)L/2}. \quad (3.2)$$

Now assume that $v(y)$ takes two values:

$$v(y) = \begin{cases} v_2, & \text{for } y \geq r, \\ v_1, & \text{for } y < r. \end{cases} \quad (3.3)$$

When $0 \leq v_1, v_2 \leq (d-1)^2/8$, the minimal nonnegative solution w_L^+ to problem (3.2) takes the following form.

(a) For $y \leq r$

$$w_L^+(y) = e^{(d-1)L/2} \times \left\{ \left(\frac{(d-1)^2}{8} - v_1 \right)^{1/2} \operatorname{ch} \left[(r-y) \left(\frac{(d-1)^2}{4} - 2v_1 \right)^{1/2} \right] + \left(\frac{(d-1)^2}{8} - v_2 \right)^{1/2} \operatorname{sh} \left[(r-y) \left(\frac{(d-1)^2}{4} - 2v_1 \right)^{1/2} \right] \right\}$$

$$\times \left\{ \left(\frac{(d-1)^2}{8} - v_1 \right)^{1/2} \operatorname{ch} \left[(r+L) \left(\frac{(d-1)^2}{4} - 2v_1 \right)^{1/2} \right] + \left(\frac{(d-1)^2}{8} - v_2 \right)^{1/2} \operatorname{sh} \left[(r+L) \left(\frac{(d-1)^2}{4} - 2v_1 \right)^{1/2} \right] \right\}^{-1}. \quad (3.4)$$

(b) For $y > r$

$$w_L^+(y) = e^{(d-1)L/2} \times \left(\frac{(d-1)^2}{8} - v_1 \right)^{1/2} \exp \left\{ - (y-r) \left(\frac{(d-1)^2}{4} - 2v_2 \right)^{1/2} \right\} \times \left\{ \left(\frac{(d-1)^2}{8} - v_1 \right)^{1/2} \operatorname{ch} \left[(r+L) \left(\frac{(d-1)^2}{4} - 2v_1 \right)^{1/2} \right] + \left(\frac{(d-1)^2}{8} - v_2 \right)^{1/2} \operatorname{sh} \left[(r+L) \left(\frac{(d-1)^2}{4} - 2v_1 \right)^{1/2} \right] \right\}^{-1}. \quad (3.5)$$

The corresponding function u_L^+ obtained from w_L^+ by the above substitution gives the minimal nonnegative solution to (3.1).

As in [14] and [11], an important role in our analysis is played by an NHBD starting at 0 with LFP $v(y)$ as in (3.3) and an individual Brownian motion with the drift $-(d-1)/2$ absorbed at $-n$, $n \in \mathbf{Z}$. The latter means that a particle remains «frozen» after the time when it hits point $-n$. A convenient object will be $M_r(-n)$, the random number of particles stuck at $-n$ over the infinite time in the absorbed NHBD. The following lemma holds true.

Lemma 3.1. *For any $r \in \mathbf{R}$, the sequence $(M_r(n), n = 0, 1, 2, \dots)$ forms an age-dependent Galton–Watson process. Also, for any $r \in \mathbf{R}$, the sequence $(M_r(-n), n = 0, 1, 2, \dots)$ forms an age-dependent Galton–Watson process. The mean values*

$$\mu_r(\pm n) = \mathbf{E}M_r(\pm n)$$

of these processes satisfy the following properties.

(i) *Assume that $0 \leq v_1, v_2 \leq (d-1)^2/8$. The mean value $\mu_r(-n)$, $n = 0, 1, 2, \dots$, of process $(M_r(-n))$ equals*

$$e^{n(d-1)/2} \left\{ \left(\frac{(d-1)^2}{8} - v_1 \right)^{1/2} \operatorname{ch} \left[r \left(\frac{(d-1)^2}{4} - 2v_1 \right)^{1/2} \right] + \left(\frac{(d-1)^2}{8} - v_2 \right)^{1/2} \operatorname{sh} \left[r \left(\frac{(d-1)^2}{4} - 2v_1 \right)^{1/2} \right] \right\} \times \left\{ \left(\frac{(d-1)^2}{8} - v_1 \right)^{1/2} \operatorname{ch} \left[(r+n) \left(\frac{(d-1)^2}{4} - 2v_1 \right)^{1/2} \right] + \left(\frac{(d-1)^2}{8} - v_2 \right)^{1/2} \operatorname{sh} \left[(r+n) \left(\frac{(d-1)^2}{4} - 2v_1 \right)^{1/2} \right] \right\}^{-1} \quad (3.6)$$

in the case $r > 0$, and

$$e^{n(d-1)/2} \left(\frac{(d-1)^2}{4} - 2v_1 \right)^{1/2} \exp \left[-r \left(\frac{(d-1)^2}{4} - 2v_2 \right)^{1/2} \right] \\ \times \left\{ \left(\frac{(d-1)^2}{8} - v_1 \right)^{1/2} \operatorname{ch} \left[(r+n) \left(\frac{(d-1)^2}{4} - 2v_1 \right)^{1/2} \right] \right. \\ \left. + \left(\frac{(d-1)^2}{8} - v_2 \right)^{1/2} \operatorname{sh} \left[(r+n) \left(\frac{(d-1)^2}{4} - 2v_1 \right)^{1/2} \right] \right\}^{-1} \quad (3.7)$$

in the case $-n \leq r < 0$. In particular, for all $r \in \mathbf{R}$, $\mu_r(n) > 1$ for $n \geq -r$, i.e., for any $r \in \mathbf{R}$ process $(M_r(-n))$ is supercritical.

Further, the mean value $\mu_r(n)$, $n = 0, 1, 2, \dots$, of process $(M_r(n))$ is equal to

$$e^{-n(d-1)/2} \left(\frac{(d-1)^2}{4} - 2v_2 \right)^{1/2} \exp \left[-r \left(\frac{(d-1)^2}{4} - 2v_2 \right)^{1/2} \right] \\ \times \left\{ \left(\frac{(d-1)^2}{8} - v_1 \right)^{1/2} \operatorname{ch} \left[(r-n) \left(\frac{(d-1)^2}{4} - 2v_2 \right)^{1/2} \right] \right. \\ \left. + \left(\frac{(d-1)^2}{8} - v_1 \right)^{1/2} \operatorname{sh} \left[(r-n) \left(\frac{(d-1)^2}{4} - 2v_2 \right)^{1/2} \right] \right\}^{-1} \quad (3.8)$$

for $r > 0$, and

$$e^{-n(d-1)/2} \left\{ \left(\frac{(d-1)^2}{8} - v_2 \right)^{1/2} \operatorname{ch} \left[r \left(\frac{(d-1)^2}{4} - 2v_2 \right)^{1/2} \right] \right. \\ \left. - \left(\frac{(d-1)^2}{8} - v_1 \right)^{1/2} \operatorname{sh} \left[r \left(\frac{(d-1)^2}{4} - 2v_2 \right)^{1/2} \right] \right\} \\ \times \left\{ \left(\frac{(d-1)^2}{8} - v_2 \right)^{1/2} \operatorname{ch} \left[(r-n) \left(\frac{(d-1)^2}{4} - 2v_2 \right)^{1/2} \right] \right. \\ \left. - \left(\frac{(d-1)^2}{8} - v_1 \right)^{1/2} \operatorname{sh} \left[(r-n) \left(\frac{(d-1)^2}{4} - 2v_2 \right)^{1/2} \right] \right\}^{-1} \quad (3.9)$$

for $r < 0$. In particular, $\mu_r(n) < 1$ for all $n = 1, 2, \dots$, i.e., for any $r \in \mathbf{R}$, process $(M_r(n))$ is subcritical.

(ii) Now suppose that $v_1 > (d-1)^2/8$. Then $\mu_r(\pm n) = \infty$ for $|n|$ large enough (depending on the difference $v_1 - (d-1)^2/8$). Further, consider the problem

$$\frac{1}{2} \frac{d^2 w}{dy^2} + V(y) w(y) = 0, \quad y \in \mathbf{R}, \quad y \geq -n, \quad (3.10)$$

where, for some $r > 0$,

$$V(y) = \begin{cases} v_2 - \frac{(d-1)^2}{8} & \text{if } y \geq r, \\ v_1 - \frac{(d-1)^2}{8} & \text{if } y < r, \end{cases} \quad (3.11)$$

with the boundary condition

$$w(-n) = 1. \quad (3.12)$$

This problem has no solution which is positive on $[-n, \infty)$.

P r o o f. Consider a Brownian motion on $\mathbf{R}$, with potential $V(y)$, starting from an initial point y . By the Feynman-Kac formula,

$$w(y) = \mathbf{E}_y \exp \left[\int_0^{\tau_n} V(Y_s) ds \right], \quad y \in \mathbf{R}, \quad (3.13)$$

where the expectation is relative to the distribution of the Brownian motion (Y_t) on $\mathbf{R}$, starting at y , and $\tau_n = \inf\{t \geq 0: Y_t = -n\}$. Using equation (2.6.1) of [1], we obtain, from (3.13), relations (3.4), (3.5) up to an exponential factor $e^{-n(d-1)/2}$.

We now consider the analog of problem (3.10)–(3.12) for an NHBD on $\mathbf{R}$ with drift $-(d-1)/2$ and LFP $v(y)$ as in (3.3). Let $u(y) = M_r(-n; y)$ be the mean number of particles absorbed at $-n$ for a process starting at y , so that $u(0) = M_r(-n; 0) = M_r(-n)$. Then $u(y)$ gives the minimal nonnegative solution to problem (3.1), with $L = n$, see [3]. After the substitution $e^{(d-1)y/2}w = u$ we obtain for w equation (3.10) with

$$V(y) = v(y) - \frac{(d-1)^2}{8}, \quad (3.14)$$

and the boundary condition

$$w(-n) = e^{n(d-1)/2}. \quad (3.15)$$

The minimal nonnegative solution to this problem is given by (3.4), (3.5). Substituting $y = 0$ we obtain for $u_n^+(0)$ expressions (3.6), (3.7). This proves statement (i). Statement (ii) can be verified by direct inspection. Lemma 3.1 is proved.

The following lemma provides a key relation used in analysis of recurrence properties of an NHBD in Section 4 and the calculation of the Hausdorff dimension in Section 5.

Lemma 3.2. Assume that $v_1, v_2 \leq (d-1)^2/8$. Then, for the minimal nonnegative solution, u_L^+ , to (3.1), the value $u_L^+(0)$ has the following asymptotics:

$$\lim_{L \rightarrow \infty} \frac{1}{L} \ln u_L^+(0) = \frac{d-1}{2} - \sqrt{\frac{(d-1)^2}{4} - 2v_1}. \quad (3.16)$$

P r o o f. Calculate the leading term of asymptotic expansion of the function $u_L^+(0)$ as $L \rightarrow \infty$. It follows from (3.6) that

$$u_L^+(0) \sim C \exp \left[L \frac{d-1}{2} - L \sqrt{\frac{(d-1)^2}{4} - 2v_1} \right].$$

4. Branching diffusion on $\mathbf{H}^d$: Recurrence properties. This section summarizes «rough» properties of an NHBD on $\mathbf{H}^d$ which can be deduced from Lemma 3.2. From now on, we assume that all processes on $\mathbf{H}^d$ start from the centre and all processes on $\mathbf{R}$ from the origin. Given $\varepsilon > 0$, let $w \in \partial \mathbf{H}^d$ and the horoball $B(\varepsilon) \in \mathcal{B}_w$ be as in Theorem 2.1. Fix a geodesic $g \in \Sigma_w$, and consider the horoballs $B_n (= B_{n,g}) \in \mathcal{B}_w$ given by

$$B_n = \{z \in \mathbf{H}^d: \Pi_g^{\text{ho}}(z) \geq n\}, \quad n \in \mathbf{Z}. \quad (4.1)$$

Then, for n large enough, $B_n \supset B(\varepsilon)$.

Consider BDs $\Theta_n^{(\pm)}$ on $\mathbf{H}^d$, with the absorbing condition on horosphere $S_n = \partial B_n \in \mathcal{S}_w$, such that their horospheric projections $\Pi_g^{\text{ho}} \Theta_n^{(\pm)}$ are BDs on $\mathbf{R}$, with drift $-(d-1)/2$, absorbed at point $-n$, and with LFP $v(y)$ of the form

$$\begin{aligned} v(y) &= v_1, \quad \text{for the process } \Theta_n^{(+)}, \\ v(y) &= \begin{cases} v_2, & \text{for } y \geq r, \\ v_1, & \text{for } y < r, \end{cases} \quad \text{for the process } \Theta_n^{(-)}. \end{aligned} \quad (4.2)$$

Lemma 4.1. Assume that $0 \leq v_1, v_2 \leq (d-1)^2/8$. For all $N = 1, 2, \dots$, with probability one, for all integer $n \geq N$, the number of particles in B_N in NHBD $\Theta_n^{(\pm)}$ is eventually 0. Hence, with probability one, for every compact subset $K \subset \mathbf{H}^d$, there exists a time t_K such that for all $t \geq t_K$ the number of particles from process $\Theta_n^{(\pm)}$ located in K is 0.

P r o o f. We give the proof for the (more complicated) case of process $\Theta_n^{(-)}$. Fix $n \geq N$ and let u_n^+ be the minimal nonnegative solution to (3.1), with $v(y)$ given by (3.3). Choose $y_1 > 0$ such that $u_n^+(y) < 1$ for all $y > y_1$; this is possible because $u_n^+(y) \rightarrow 0$ as $y \rightarrow \infty$, as follows from (3.5). We will construct a subcritical Galton–Watson process, Γ_m , with discrete time $m = 1, 2, \dots$ and the mean number of offspring $\nu = u_n^+(y_1)$. Namely, consider the horospheric projection $\Pi_g^{\text{ho}} \Theta_n^{(-)}$ of process $\Theta_n^{(-)}$ to a geodesic $g \in \Sigma_w$. More precisely, start a branching diffusion on $\mathbf{R}$, with a single particle placed at 0, with drift $-(d-1)/2$, the absorbing condition at point $-n$ and LFP $v(y)$ from (3.3) (further specifications of the branching mechanism are actually not important). The number Γ_1 of offspring at time 1 in the Galton–Watson process equals the number of particles from the population of this branching diffusion which reach level y_1 . Further, consider the number of particles from the population of the branching diffusion reaching level $y_2 > y_1$; this will give Γ_2 . Value y_2 is chosen so that $u_{n+y_1}^+(y_2) = \nu$. Next,

we choose value $y_3 > y_2$, with $u_{n+y_2}^+(y_3) = \nu$. Then we identify Γ_3 as the number of particles in the branching diffusion reaching level y_3 , and so on.

Since process Γ_n is subcritical, it dies out with probability one. In terms of the branching diffusion on $\mathbf{R}$ it means that there will be no particles beyond a certain level y_T (where label $T = 1, 2, \dots$ is random). In other words, all particles in the above branching diffusion on $\mathbf{R}$ will be absorbed at point $-n$ before they reach level y_T .

Now observe that the transition probability, in the process of individual motion, from K to the domain above level S is positive. Assuming that there are particles in K for arbitrarily large times, we immediately run into contradiction. Therefore, the number of particles lying in the horoball B_N in original process $\Theta_n^{(-)}$ will be eventually zero. Lemma 4.1 is proved.

Lemma 4.2. *Assume that $v_1 > (d-1)^2/8$. Then with probability one there exists an infinite path $\{\mathcal{P}(t), t \geq 0\}$, of the NHBD $\Theta_n^{(\pm)}$ tree which remains forever in a compact subset K , i.e., the distance from point $\mathcal{P}(t)$ to the centre of $\mathbf{H}^d$ remains bounded as $t \rightarrow \infty$.*

Furthermore, with probability one, for each nonempty open subset $U \subset \mathbf{H}^d$, there are particles from the NHBP population in U at indefinitely large times.

P r o o f. Again consider the case of the process $\Theta_n^{(-)}$ only. Following an argument from [14, Section 3.3, Corollary 3], repeated in [11, Theorem 4.4], consider a modification, $\Theta_{n,k}^{(-)}$, of process $\Theta_n^{(-)}$ in which, in addition to horosphere S_{-n} , particles are frozen upon reaching the horosphere S_k . Since for process $(M_r(n))$, according to Lemma 3.1, $\mathbf{E}M_r(k) = \infty$, for $k \geq 1$ sufficiently large, the expected number of particles whose trajectories, in process $\Theta_{n,k}^{(-)}$, did not reach S_{-n} before freezing in S_k is strictly greater than 1. Thus, the branching process counting particles hitting S_k before S_{-n} is supercritical. As in [14], this property allows us to construct an infinite tree in a compact domain K with positive probability. However, if the process of producing an infinite tree from a given particle fails, one starts it again from the next available particle, and so on, before the success is achieved.

Lemmas 4.1 and 4.2 allows us to deduce Theorem 2.2A.

5. Limiting set on the absolute. In this section we prove Theorem 2.3A. From now on we assume that the value v^+ satisfies

$$0 \leq v^+ \leq \frac{(d-1)^2}{8}. \quad (5.1)$$

We will refer to (5.1) as the subcriticality condition. Recall that we deal with NHBD Θ^- , where the LFP is given by (2.5).

Recall that the (closed) limiting set $\Lambda^- \subseteq \partial\mathbf{H}^d$ is defined as the set of accumulation points, in the Euclidean metric on $\overline{\mathbf{H}}^d$, as $t \rightarrow \infty$, for (branching) trajectories of process Θ^- . As was said before, to calculate the Hausdorff dimension of Λ^- (when it is a proper subset of $\partial\mathbf{H}^d$), we have to specify

a metric on $\partial\mathbf{H}^d$. A natural metric is the angular metric φ , mentioned in Introduction. The angular metric depends on a choice of a «centre» (it is usually the origin, i.e., the centre of the unit ball). However, for the purpose of calculations another metric is more convenient: the so-called stereographic projection metric $\chi = \chi_{w^0}$, determined by a choice of a reference point (a pole) $w_0 \in \partial\mathbf{H}^d$. For details, see [11, Section 3.4].

As in [14] and [11], we establish an upper bound and a lower bound for $h(\Lambda^-)$; the former is proved in Theorem 5.1, and the latter in Theorem 5.2. Symbol $\mathbb{P}$ stands for the distribution of process Θ^- .

Lemma 5.1. *For any $w \in \partial\mathbf{H}^d$, we have*

$$\mathbb{P}\{w \in \Lambda^-\} = 0.$$

The proof of Lemma 5.1 follows from the fact that $\Lambda^- \subseteq \Lambda^+$ while equality $\mathbb{P}\{w \in \Lambda^-\} = 0$ holds as Θ^+ is an HHBD (see [11, Lemma 5.4]).

As in [11], Lemma 5.1 implies the following result.

Lemma 5.2. *With probability 1, the Hausdorff dimensions of Λ^- calculated in metrics φ and χ coincide.*

In the rest of the paper, when speaking about set Λ^- , we refer to metric χ . In particular, $D(\Lambda^-)$ in Lemma 5.5 below stands for the diameter of Λ^- in metric χ . Furthermore, we choose reference point w^0 (determining metric χ) coinciding with $w(\varepsilon)$, i.e., the point of the absolute figuring in Theorem 2.1. It is also convenient to fix a geodesic g ($= g_{w(\varepsilon)} \in \Sigma_{w(\varepsilon)}$) which is the diameter of the unit ball pointing at $w(\varepsilon)$.

Set $\mathbf{H}_0^d = \{x \in \mathbf{H}^d: x_d = 0\}$; Klein's metric ρ generates on manifold $\mathbf{H}_0^d$ the structure of $\mathbf{H}^{d-1}$, the Lobachevsky space in dimension $d-1$. The next lemma generalizes the bound of Lemma 5.3 of [11] to the case under consideration. Here we use the *equidistant projection*, relative to $\mathbf{H}_0^d$, on geodesic g :

$$x \in \mathbf{H}^d \mapsto \Pi^{\text{ed}}(x) \in \mathbf{R}.$$

See [11, Section 3.2]. Pictorially, the equidistant projection moves point x along the manifold $\mathbf{B}$ ($= \mathbf{B}_{x, \mathbf{H}_0^d} = \{\tilde{x} \in \mathbf{H}^d: \rho(\tilde{x}, \mathbf{H}_0^d) = \rho(x, \mathbf{H}_0^d)\}$); manifold $\mathbf{B}$ intersects g at a single point, x^* , and $\Pi^{\text{ed}}(x)$ is the image of x^* on the straight line. Denote by $\mathcal{A}_l = \{x \in \mathbf{H}^d: x_d > \text{th } l\}$, $l \geq 0$.

Lemma 5.3. *Set $s = \Pi^{\text{ed}}(x)$ and assume that $s < 0$ and $|s|$ is large enough. Then $\mathbb{P}\{\Theta^- \text{ does not reach } \mathcal{A}_0\} \geq 1 - ce^{sp_1}$, where $c > 0$ is a constant and*

$$p_1 = \frac{1}{2} \left[d - 1 + ((d-1)^2 - 8(v^+ - \varepsilon))^{1/2} \right]. \quad (5.2)$$

P r o o f. This lemma follows because the process Θ^- is majorized by HHBP Θ_1^- , where the FP is constant and is equal to $v^+ - \varepsilon$. In particular, $\mathbb{P}\{\Theta^- \text{ does not reach } \mathcal{A}_0\} \geq \mathbb{P}\{\Theta_1^- \text{ does not reach } \mathcal{A}_0\}$. But for Θ_1^- , the bound (5.2) was established in Lemma 5.3 of [11].

Similarly, we have the following analog of Lemma 5.4 of [11].

Lemma 5.4. *For $s > 0$ large enough, $\mathbb{P}\{\Theta^- \text{ does not reach } \mathcal{A}_s\} \geq 1 - ce^{-sp_1}$.*

In turn, Lemmas 5.3 and 5.4 lead to the following result.

Lemma 5.5. *For $t > 0$ large enough,*

$$\mathbb{P}\{D(\Lambda^-) > t\} \leq cD_0^{p_1}t^{-p_1}. \quad (5.3)$$

Here c is the constant from Lemmas 5.3 and 5.4, and D_0 is the constant from Lemma 3.6 in [11].

From Lemma 5.5, following [11], we deduce the theorem.

Theorem 5.1. *Under condition (5.1), the Hausdorff dimension of set Λ^- calculated in metric χ is $\leq \frac{1}{2}[d-1 - ((d-1)^2 - 8(v^+ - \varepsilon))^{1/2}]$ a.s.*

To prove the lower bound, we use Lemma 3.2. In the context of this section it takes the following form.

Lemma 5.6. *For the LFP (2.5),*

$$\lim_{L \rightarrow \infty} \frac{1}{L} \ln u_L^+(0) = \frac{d-1}{2} - \sqrt{\frac{(d-1)^2}{4} - 2(v^+ - \varepsilon)}. \quad (5.4)$$

The lower bound is established in following theorem.

Theorem 5.2. *Under condition (5.1), the Hausdorff dimension of set Λ^- calculated in metric χ is $\geq [d-1 - ((d-1)^2 - 8(v^+ - \varepsilon))^{1/2}]/2$ a.s.*

P r o o f o f T h e o r e m 5.2. The argument follows Section 7 of [14] (see also Section 5 of [11]). We consider the process $X_1, X_2, \dots$ in $\mathbf{H}^d$ analogous to that introduced in [14, Section 7.2]. Namely, fix $l > 0$ and impose an absorbing condition in the domain $\mathbf{B}_1 = \{x \in \mathbf{H}^d: \Pi_g^{\text{ho}}(x) < -l\}$ (the complement of a horoball). That is, particles from Θ^- are considered frozen, where they hit $\mathbf{B}_1$. Among the particles frozen in $\mathbf{B}_1$ we randomly choose one; its position is denoted by X_1 . Then revive the chosen particle and create an absorbing condition for its offspring in the domain $\mathbf{B}_2 = \{x \in \mathbf{H}^d: \Pi_g^{\text{ho}}(x) < -l + \Pi_g^{\text{ho}}(X_1)\}$. Among the offspring frozen in $\mathbf{B}_2$ choose randomly again one; its position is denoted by X_2 , etc. As in [14], it is possible to prove that a.s. there exists $\lim_{n \rightarrow \infty} X_n$ (in the topology of $\bar{\mathbf{H}}^d$) giving a random point $W \in \partial\mathbf{H}^d$.

The distribution of W has a bounded probability density with respect to the measure on $\partial\mathbf{H}^d$ generated by the Lebesgue measure on the linear manifold $\{x \in \mathbf{R}^{d+1}: x_d = 0, x_{d+1} = 1\}$ under stereographic projection Π^{st} . This follows from the fact that sequence $\{X_0 = x, X_1, X_2, \dots\}$ is distributed as the process of individual motion starting at x and observed at the times $T_1, T_2, \dots$, when it subsequently hits the domains $\{x \in \mathbf{H}^d: \Pi_g^{\text{ho}}(x) < -l\}$, $\{x \in \mathbf{H}^d: \Pi_g^{\text{ho}}(x) < -l + \Pi_g^{\text{ho}}(X_1)\}$, etc. Using this fact and Lemma 5.6, the proof of Theorem 5.2 is completed as in [14].

Acknowledgments. The first author thanks Statistical Laboratory, DPMMS, University of Cambridge, for the hospitality during regular visits in 2003–2005. The second author thanks Dublin Institute for Advanced Studies and Institut des Hautes Etudes Scientifiques, Bures-sur-Yvette, France, for the hospitality during visits in 2002–2004. The second author acknowledges support from the RDEES Programme, ESF, towards research visits in 2003.

REFERENCES

1. Бородин А. Н., Салминен П. Справочник по броуновскому движению. Факты и формулы. СПб.: Лань, 2000, 639 с.
2. Falconer K. J. Fractal Geometry. Mathematical Foundations and Applications. Chichester: Wiley, 1990, 288 p.
3. Grigor'yan A., Kelbert M. Recurrence and transience of branching diffusion processes on Riemannian manifolds. — Ann. Probab., 2003, v. 31, № 1, p. 244–284.
4. Гельфанд И. М., Граев М. И., Виленкин Н. Я. Интегральная геометрия и связанные с ней вопросы теории представлений. М.: Физматгиз, 1962, 656 с. (Обобщенные функции, вып. 5.)
5. Harish-Chandra. Spherical functions on a semisimple Lie group. — Amer. J. Math., 1958, v. 80, p. 241–310.
6. Хелгасон С. Дифференциальная геометрия и симметрические пространства. М.: Мир, 1964, 533 с.
7. Helgason S. Differential Geometry, Lie Groups, and Symmetric Spaces. New York–London: Academic Press, 1978, 628 p.
8. Ито К., Маккин Г. Диффузионные процессы и их траектории. М.: Мир, 1968, 394 с.
9. Карпелевич Ф. И. Геометрия геодезических и собственных функций оператора Бельтрами–Лапласа на симметрических пространствах. — Труды Моск. матем. об-ва, 1965, т. 44, с. 48–185.
10. Karpelevich F. I., Kelbert M. Ya., Suhov Yu. M. The boundedness of branching Markov processes. — The Dynkin Festschrift. Markov Processes and Their Applications. Ed. by M. Freidlin. Boston: Birkhäuser, 1994, p. 143–152. (Progr. Probab., v. 34.)
11. Karpelevich F. I., Pechersky E. A., Suhov Yu. M. A phase transition for hyperbolic branching processes. — Comm. Math. Phys., 1998, v. 195, № 3, p. 627–642.
12. Karpelevich F. I., Suhov Yu. M. A criterion of boundedness of discrete branching random walk. — Classical and Modern Branching Processes. Ed. by K. B. Athreya and P. Jagers. New York: Springer-Verlag, 1997, p. 141–155. (IMA Vol. Math. Appl., v. 84.)
13. Karpelevich F. I., Suhov Yu. M. Boundedness of one-dimensional branching Markov processes. — J. Appl. Math. Stochastic Anal., 1997, v. 10, № 4, p. 307–332.
14. Lalley S., Sellke T. Hyperbolic branching Brownian motion. — Probab. Theory Relat. Fields, 1997, v. 108, № 2, p. 171–192.

Поступила в редакцию
4.IX.2005