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S. Aspandiiarov, R. Iasnogorodski, General criteria of integrability of functions of passage-times for non-negative stochastic processes and their applications, *Теория вероятн. и ее примен.*, 1998, том 43, выпуск 3, 509–539

DOI: <https://doi.org/10.4213/tvp1557>

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26 апреля 2023 г., 14:38:25

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GENERAL CRITERIA OF INTEGRABILITY OF
FUNCTIONS OF PASSAGE-TIMES FOR
NON-NEGATIVE STOCHASTIC PROCESSES
AND THEIR APPLICATIONS

В этой статье мы изучаем вопрос об интегрируемости функций от момента первого попадания в компактные множества и функций от момента первого возвращения для случайных процессов с дискретным параметром. Мы рассматриваем сначала класс процессов с отрицательным сносом, принимающих значения в $\mathbf{R}_+$, и доказываем для них общие достаточные условия интегрируемости функций этих случайных моментов. Условия формулируются в «мартингальном духе», впервые предложенном Фостером, и обобщают соответствующие результаты, полученные ранее. Во второй части статьи мы обращаемся к тому же вопросу для отраженных случайных блужданий с нулевым сносом внутри области. Применяя результаты первой части, мы получаем условия интегрируемости некоторых функций от момента первого попадания и момента первого возвращения для отраженных случайных блужданий. Полученные оценки дают довольно тонкие результаты для первых упомянутых случайных моментов и дополняют соответствующие результаты в [1]. Наконец, мы выводим границы для скорости сходимости переходных вероятностей эргодического отраженного случайного блуждания к соответствующей инвариантной мере.

Ключевые слова и фразы: момент попадания, счетные цепи Маркова, отраженное случайное блуждание с отражением на границе.

1. Introduction

The present paper deals mainly with the question of integrability of functions of first hitting times of compact sets (called simply first passage-times) for some countable Markov chains with non-geometric ergodicity and one-dimensional not necessarily Markovian stochastic processes with discrete parameter.

The study of the first passage problem is of interest for us because of its close relationships with the question of integrability of appropriate functions

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of the first return-times for countable irreducible Markov chains. The former question is known to be important for recurrence classification of these Markov chains and the limiting behavior of additive functionals on them (see [3]). Moreover, T. Lindvall [8], E. Nummelin, P. Tuominen [10], P. Tuominen, R. L. Tweedie [11] using the method of splitting showed that under general conditions the integrability of functions of the first return-times for positive recurrent countable Markov chains and the upper bounds on their tails provide one with quite precise upper bounds on the rate of convergence of the transition probabilities to the invariant distribution. Roughly speaking, these results for positive recurrent irreducible Markov chains show that for an appropriate class of functions f if $f(\rho)$ is integrable, then an upper bound for the rate of convergence is provided by $o(1/f'(n))$ (here ρ is the first return-time to the initial state). Recently, M. Menshikov and S. Popov in [9] showed how to derive a lower bound on the rate of convergence using lower bounds for the first return-time ρ under some additional hypotheses on conditional increments.

In the huge literature devoted to the first passage-times problem the case of Markov chains with geometric ergodicity, i.e., the case when the first return-times and first passage-times have exponential moments is well studied. For references, we invite the reader to consult the recent survey [5]. It turns out however, that in the study of driftless multidimensional Markov chains appearing for example in the queueing networks under heavy traffic one also needs to treat Markov chains with sub-geometric ergodicity. The first passage problem for such Markov chains is especially important because of the above mentioned results linking it, via the first return-times, to the upper bounds for the rate of convergence. However, on the contrary to the geometric ergodic case, first passage-times are much less investigated and to best of our knowledge there are quite a few results in this direction.

One of existing approaches to the first passage-time problem for irreducible countable Markov chains consists in reducing it to one-dimensional one in the following way: one first finds a positive function defined on the original state space in such a way that the tails of the first passage-times for the image of the original Markov chain under this function have the same asymptotic behavior as the original ones and then gets an information about «transformed» first passage-times. Obviously, by doing so the original state space can be greatly simplified but in exchange the new one-dimensional process need not to be Markovian. An example of this approach can be found in [7], where it permitted to describe the recurrence classification of simple random walks in $\mathbf{R}^d$ through studying their Euclidean norms.

In our previous papers [1], [2] following ideas of J. Lamperti [7] we have treated non-negative stochastic not necessarily Markovian processes with discrete parameter having the following characteristic property: their one-step mean drifts from x were negative and of the asymptotic order $1/x$ as $x \rightarrow \infty$. The results obtained for such processes in [7], [1], [2] provide one

with general martingale sufficient conditions on existence or non-existence of their first passage-time moments. However, in some applications (one example of them will be given in Corollary 2), one should deal with processes with negative one-step mean drifts from x of the asymptotic order $g(x)$ with arbitrary positive non-increasing function $g(x)$ as $x \rightarrow \infty$. One also needs to have results on the integrability of arbitrary functions of the first passage-times and not only their powers. The results obtained in the present paper permit one to address these questions and provide an unified approach to the integrability of functions of the first passage-times, and hence, first return-times for processes with negative drift. We also believe that they might have other applications, as for instance, in the analysis of optimal constants in Burkholder's type inequalities.

Let us turn to the structure of the paper. We start by discussing Markov chains with discrete parameter taking values in a countable unbounded subset of $\mathbf{R}^d$ and prove Theorem 1 stating that under appropriate conditions the integrability of «sub-exponential» functions of the first passage-time into some finite set F is equivalent to integrability of the same functions of the first return-time ρ to the initial state, for all initial states belonging to F . The latter fact ensures the integrability of the same functions of the first return-time ρ for all initial states z and provides with upper bounds for the rate of convergence of the transition probabilities of a positive recurrent Markov chain to the invariant measure.

In Section 3 we obtain general martingale conditions (Theorems 2-2', 3-3') on integrability of functions of the first passage-times for non-negative discrete parameter stochastic processes with negative drift. Let us notice that the technique employed in the proofs is a substantial development of the ideas of [2].

In Section 4 we apply the results of the previous section to the study of analogous questions for 2-dimensional reflected random walks in a quadrant with zero-drift in the interior. We prove in Theorems 5-6 that functions f of the first passage-times and the first return-times of the type $f(x) = x^{\alpha/2} \log^{-1} x \cdots \log_{k-1}^{-1} x \log_k^{-q} x$ in the case $\alpha > 0$ and $f(x) = \log x \log_2^{-1} x \cdots \log_k^{-1} x \log_{k+1}^{-q} x$ in the case $\alpha = 0$ with $k \geq 1$, $q > 0$ are integrable (α is a parameter which depends on geometrical data of the reflected random walks). This together with Theorem 3 of [1] give quite sharp estimates for the tails of the first passage-times. The proofs of the results of this section are based on the construction of appropriately chosen non-negative supermartingales satisfying the criteria of Section 2 (so-called Lyapunov's functions) and the properties of a certain harmonic function that was used by S. R. S. Varadhan and R. J. Williams in [12] to prove the existence and uniqueness of the Brownian motion in a wedge with oblique boundary reflection. Finally, the obtained bounds for the first passage-times and Theorem 1 permit us to get an upper bound for the rate of convergence of transition probabilities of the positive recurrent reflected random walk to the invariant measure.

2. First return-times versus first passage-times for countable Markov chains

Our principal goal in this section is to prove some results on links between the integrability of functions of passage-times into a finite set and the integrability of the same functions for the first return-times to the initial state for irreducible countable Markov chains with discrete parameter. Roughly speaking, the results of this section state that for a certain class (though quite rich) of functions, functions of the first return-time to the initial state are integrable for all initial states from a given finite subset of the state space if and only if the same functions of the first passage-time into this finite set are integrable. This permits one to treat only the latter problem which is much more easier. We also prove one result for positive recurrent Markov chains linking the integrability of functions of the first passage-times into a given set and upper bounds for the rate of convergence of transition probabilities to the invariant distribution.

Let us consider a recurrent aperiodic irreducible Markov chain $\{Z_n, n \geq 0\}$ taking values in a countable set $\mathcal{A}$. Let F be a fixed finite subset of $\mathcal{A}$.

Definition 1. Let $z \in \mathcal{A}$. We define successive passage-times into F by the Markov chain Z :

$$\rho_0^F = 0; \quad \rho_{k+1}^F = \inf\{n > \rho_k^F; Z_n \in F\}.$$

The first passage-time of F (i.e., ρ_1^F) will be simply denoted ρ^F . We also define the first return-time ρ by

$$\rho = \inf\{n > 0; Z_n = Z_0\}.$$

To formulate our principal results we adopt the following convention. Let f be a $\mathbf{R}_+$ -defined function. Then the phrase « f has the property A in a neighborhood of ∞ » should be understood as «there exists $B \geq 0$ such that the property A is verified in $[B, \infty)$ ».

Proposition 1. Let f be a positive $\mathbf{R}_+$ -defined function with the following property: there exist a positive function $\mu(x)$ ($\mu(x) \rightarrow 0$ as $x \rightarrow \infty$) and a positive constant c such that for any positive $m \geq 1, x_1 \geq 1, \dots, x_m \geq 1$

$$f\left(\sum_{k=1}^m x_k\right) \leq c e^{m\mu(m)} \sum_{k=1}^m f(x_k). \quad (1)$$

1. Suppose for any $z \in F$,

$$\mathbf{E}_z f(\rho^F) \text{ is finite.} \quad (2)$$

Then for any $z \in F$, $\mathbf{E}_z f(\rho)$ is finite.

2. Suppose in addition f is non-decreasing in a neighborhood of ∞ . If for some $z_0 \in F$, $\mathbf{E}_{z_0} f(\rho^F)$ is infinite, then $\mathbf{E}_{z_0} f(\rho)$ is also infinite.

P r o o f. 1. Let $z \in F$. We introduce the Markov chain $\{Y_n^F, n \geq 0\}$ by setting for all $n \geq 0$, $Y_n^F = Z_{\rho_n^F}$. Let us denote the transition probabilities of the new Markov chain by $\{\bar{p}_{x,y}\}_{x \in F, y \in F}$ and let $\bar{p} = \min\{\bar{p}_{x,y}; x, y \in F, \bar{p}_{x,y} > 0\}$. We also introduce the corresponding filtration generated by the variables $Y_1^F, Y_2^F, \dots$

$$\bar{\mathcal{F}}_n = \sigma\{Y_1^F, \dots, Y_n^F\}, \quad \forall n \geq 0.$$

Set $\nu(z) = \inf\{n > 0; Y_n^F = z\}$. From (1) we have,

$$\begin{aligned} \mathbf{E}_z f(\rho) &= \mathbf{E}_z f(\rho_{\nu(z)}^F) = \mathbf{E}_z \left(f \left(\sum_{k=1}^{\nu(z)} (\rho_k^F - \rho_{k-1}^F) \right) \right) \\ &\leq c \mathbf{E}_z \left(e^{\nu(z)\mu(\nu(z))} \sum_{k=1}^{\nu(z)} f(\rho_k^F - \rho_{k-1}^F) \right) \\ &= c \sum_{k=1}^{\infty} \mathbf{E}_z \left(e^{\nu(z)\mu(\nu(z))} f(\rho_k^F - \rho_{k-1}^F) 1_{(\nu(z) \geq k)} \right). \end{aligned}$$

Notice that for any $k \geq 1$,

$$1_{(\nu(z) \geq k)} \mathbf{E}_z (e^{\nu(z)\mu(\nu(z))} | \mathcal{F}_{\rho_k^F}) = 1_{(\nu(z) \geq k)} \mathbf{E}_z (e^{\nu(z)\mu(\nu(z))} | \bar{\mathcal{F}}_k).$$

Therefore, applying the strong Markov property at ρ_k^F we obtain:

$$\begin{aligned} &\mathbf{E}_z \left(\nu(z)^\mu f(\rho_k^F - \rho_{k-1}^F) 1_{(\nu(z) \geq k)} \right) \\ &= \mathbf{E}_z \left(f(\rho_k^F - \rho_{k-1}^F) 1_{(\nu(z) \geq k)} \mathbf{E}_z (e^{\nu(z)\mu(\nu(z))} | \bar{\mathcal{F}}_k) \right) \\ &= \mathbf{E}_z (1_{(\nu(z) \geq k)} \mathbf{E}_z (f(\rho_k^F - \rho_{k-1}^F) | \bar{\mathcal{F}}_k) \mathbf{E}_z (e^{\nu(z)\mu(\nu(z))} | \bar{\mathcal{F}}_k)). \end{aligned}$$

On the other hand, we easily see that for any $k \geq 1$,

$$\mathbf{E}_z (f(\rho_k^F - \rho_{k-1}^F) | \bar{\mathcal{F}}_k) = \mathbf{E}_z (f(\rho_k^F - \rho_{k-1}^F) | Y_{k-1}^F, Y_k^F),$$

on $\{\nu(z) \geq k\}$. Hence,

$$\begin{aligned} &\mathbf{E}_z (e^{\nu(z)\mu(\nu(z))} f(\rho_k^F - \rho_{k-1}^F) 1_{(\nu(z) \geq k)}) \\ &= \mathbf{E}_z (1_{(\nu(z) \geq k)} \mathbf{E}_z (f(\rho_k^F - \rho_{k-1}^F) | Y_{k-1}^F, Y_k^F) \mathbf{E}_z (e^{\nu(z)\mu(\nu(z))} | \bar{\mathcal{F}}_k)). \end{aligned}$$

But on $\{\nu(z) \geq k\}$ the strong Markov property at ρ_{k-1}^F gives us the following bound:

$$\mathbf{E}_z (f(\rho_k^F - \rho_{k-1}^F) | Y_{k-1}^F, Y_k^F) \leq \frac{\bar{f}(Y_{k-1}^F)}{\bar{p}_{Y_{k-1}^F, Y_k^F}} \leq \frac{\bar{f}(Y_{k-1}^F)}{\bar{p}},$$

where the function $\bar{f}(x)$ is defined for all $x \in F$ by $\bar{f}(x) = \mathbf{E}(f(\rho_x^F))$. Recalling our assumption (2) and joining together the last bounds we finally get that for some positive constant M which does not depend on z ,

$$\mathbf{E}_z f(\rho) \leq cM \bar{p}^{-1} \sum_{k=1}^{\infty} \mathbf{E}_z \left(e^{\nu(z)\mu(\nu(z))} 1_{(\nu(z) \geq k)} \right),$$

which is finite since the irreducible Markov chain $\{Y_n^F, n \geq 0\}$ has the finite state space and its first return-times $\nu(z)$ have exponential moments for any $z \in F$.

2. The second part of the proposition is an easy consequence of the monotonicity assumption on f and the following obvious observation:

$$\sum_{k=1}^{\nu(z)} (\rho_k^F - \rho_{k-1}^F) \geq \rho_1^F = \rho_z^F.$$

Proposition 1 is proved.

Let us introduce the following

Definition 2. Let f be a positive function defined on $[0, \infty)$. It is said to satisfy the condition (R), if there exists a positive constant $\tilde{A}_f$ such that $\limsup_{x \rightarrow \infty} f(2x)/f(x) \leq \tilde{A}_f$.

Remark 1. As is easy to see, the class of functions satisfying the condition (1) includes the functions $f(x) = x^p \log^q x$, $f(x) = \log^p x \log^q x$, $\forall p > 0, \forall q$ and the class of convex non-decreasing functions on $\mathbf{R}_+$ satisfying the condition (R) $A_f m^{\log_{A_f} A_f} f(x)$.

Let us introduce another class of functions on $\mathbf{R}_+$ (cf. [8], [10]).

Definition 3. $\mathcal{G}' = \{\text{non-decreasing in a neighborhood of } \infty \text{ functions } f \text{ such that } \log(f(x))/x \text{ is non-increasing in a neighborhood of } \infty \text{ and tends to 0 as } x \rightarrow \infty\}$.

For instance, functions $f(x) = x^p \log^r(x) \exp(sx^q)$ with $q \in (0, 1)$ and either $s > 0$ or $s = 0$ and $p > 0$ belong to $\mathcal{G}'$ and satisfy the condition (1).

Theorem 1. I. Let f be a positive non-decreasing in a neighborhood of ∞ function. If for some $z_0 \in \mathcal{A} \setminus F$, $\mathbf{E}_{z_0}(f(\rho^F))$ is infinite, then for all $z \in F$, $\mathbf{E}_z(f(\rho))$ is infinite.

II. Let f be a function satisfying the condition (1) of Proposition 1. Suppose there exists a positive function φ defined on $\mathcal{A}$ such that for all $z \in \mathcal{A} \setminus F$,

$$\mathbf{E}_z(f(\rho^F)) \leq \varphi(z). \quad (3)$$

Suppose also that for all $z \in F$, $\mathbf{E}_z(\varphi(Z_1))$ is finite. Then, for all $z \in F$,

$$\mathbf{E}_z f(\rho^F) \text{ is finite} \quad (4)$$

and, by Proposition 1,

$$\mathbf{E}_z(f(\rho)) \text{ is finite.} \quad (5)$$

Furthermore, if f' belongs to $\mathcal{G}'$, then

1. For all $z \in \mathcal{A}$, $E_z(f(\rho))$ is finite.
2. For any initial distribution λ on $\mathcal{A}$ such that

$$E_\lambda \left(f'(\rho^F) \right) \left(\equiv \int \lambda(dz) E_z(f'(\rho^F)) \right) \text{ is finite,} \quad (6)$$

the following upper bound holds:

$$\int \lambda(dz) \|P^n(\cdot) - \pi\| = o\left(\frac{1}{f'(n)}\right) \text{ as } n \rightarrow \infty, \quad (7)$$

where for any signed measure β on $\mathcal{A}$ we denote $\|\beta\| = \sum_{z \in \mathcal{A}} |\beta(z)|$.

P r o o f. Assertion I follows from the irreducibility of the Markov chain, its strong Markov property and the monotonicity assumption on f . (4) is an immediate consequence of the Markov property. As far as the second assertion of II is concerned, its first part follows from (4) and Remark 2.8 in [10] and the second part is a particular case of Theorem 2.7 in [10] (see also Corollary 1 in [8]).

3. Integrability of functions of the first passage-times for non-negative stochastic processes with negative drift

Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a filtration $\{\mathcal{F}_n\}_{n \geq 0}$. Let x be a positive number and let $\{X_n, n \geq 0\}$ be a discrete-time $\{\mathcal{F}_n\}$ -adapted non-negative stochastic process such that $X_0 = x$. For any positive number A let us denote by τ_A the first passage-time into the interval $[0, A]$, i.e.,

$$\tau_A \equiv \tau_{x,A}^X = \inf\{n \geq 0; X_n \leq A\}$$

(with convention $\inf \emptyset = +\infty$.)

The main goal of this section is to state and to prove sufficient conditions for existence of means of $f(\tau_A)$ for non-decreasing in a neighborhood of ∞ functions f formulated in terms of existence of appropriate supermartingales in the spirit of Foster's ergodicity criteria in [6]. They generalize the related results obtained by J. Lamperti in [7] and S. Aspandiarov, R. Iasnogorodski and M. Menshikov in [2] and, loosely speaking, claim the following: if the process $\{f(X_{n \wedge \tau_A}), n \geq 0\}$ is a supermartingale with negative conditional increments of the order at least $O(f'(X_{n \wedge \tau_A}))$ for sufficiently large $X_{n \wedge \tau_A}$, then the random variable $f(\tau_A)$ admits finite expectation.

Let us now proceed. We introduce the following class of positive functions:

$$\mathcal{G} = \{f \in C^2(0, \infty), f(x) > 0, \forall x > 0; \lim_{x \rightarrow \infty} f(x) = \infty; \\ f' > 0 \text{ and } \log f' \text{ is concave in a neighborhood of } \infty\}.$$

The subset of $\mathcal{G}$ of functions f such that $f' > 0$ and $\log f'$ is concave on $\mathbf{R}_+$, will be denoted by $\mathcal{G}_0$. Next, let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}})$ be another probability space. For any positive random variable X defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}})$ and for any $f \in \mathcal{G}$ we denote

$$|X|_f = \begin{cases} f^{-1}\{\tilde{\mathbf{E}}(f(X))\}, & \text{if } \tilde{\mathbf{E}}(f(X)) \text{ is finite;} \\ \infty, & \text{if not.} \end{cases}$$

We start with one preliminary though important result which might be viewed as a generalized triangle inequality.

Lemma 1. *Let $f \in \mathcal{G}_0$ and X be a positive random variable such that $\tilde{\mathbf{E}}(f(X))$ is finite. Let*

$$a_X = \sup \left\{ a \in \mathbf{R}_+; \tilde{\mathbf{E}}(f(X+a)) \text{ is finite} \right\}.$$

Then, for any $a \in [0, a_X]$,

$$|X+a|_f \leq |X|_f + a. \quad (8)$$

P r o o f. Let $f \in \mathcal{G}_0$. First, it follows from the monotonicity of f and the dominated convergence theorem that the mapping $a \rightarrow \tilde{\mathbf{E}}(f(X+a))$ is continuous on $[0, a_X]$. Next, since $\log f'$ is concave, f is increasing and

$$(f' \circ f^{-1})' = (\log f')' \circ (f^{-1}),$$

one can immediately see that $f' \circ f^{-1}$ is concave. Therefore, by Jensen's inequality,

$$\tilde{\mathbf{E}}\{f'(X+a)\} = \tilde{\mathbf{E}}\{f' \circ f^{-1}(f(X+a))\} \leq f' \circ f^{-1}\{\tilde{\mathbf{E}}(f(X+a))\}, \quad (9)$$

which shows that $f'(X+a)$ is integrable for all $a \in [0, a_X]$. It will be now shown that the mapping $a \rightarrow \tilde{\mathbf{E}}(f(X+a))$ is differentiable on $(0, a_X)$ and for any $a \in (0, a_X)$,

$$\frac{d}{da} \tilde{\mathbf{E}}\{f(X+a)\} = \tilde{\mathbf{E}}\{f'(X+a)\}. \quad (10)$$

To verify (10) we notice that the concavity of $\log f'$ implies that $(\log f')'$ is decreasing. Using this and the mean-value theorem one can easily obtain the following inequalities which are valid for all $a \in (0, a_X)$ and μ such that $|\mu| < a/2$,

$$\begin{aligned} \mu^{-1}(f(X+a+\mu) - f(X+a)) &= f'(X+a+Y\mu) \\ &= f'\left(X + \frac{a}{2}\right) \exp \left\{ \left(\frac{a}{2} + Y\mu\right) (\log f')' \left(X + \frac{a}{2} + Z\left(\frac{a}{2} + Y\mu\right)\right) \right\} \\ &\leq f'\left(X + \frac{a}{2}\right) \exp \left\{ a \left| (\log f')' \left(\frac{a}{2}\right) \right| \right\} \end{aligned}$$

(here Y, Z are some random variables taking values in $(0, 1)$). Since $f'(X+a)$ is integrable for any $a \in (0, a_X)$, then by the dominated convergence for all $a \in (0, a_X)$,

$$\frac{d}{da} \tilde{\mathbf{E}}\{f(X+a)\} = \lim_{\mu \rightarrow 0} \mu^{-1} \tilde{\mathbf{E}}(f(X+a+\mu) - f(X+a)) = \tilde{\mathbf{E}}\{f'(X+a)\}.$$

We are now ready to check that (8) holds for any $a \in [0, a_X]$. Let us consider the non-trivial case $a \in (0, a_X)$. By (9) and (10),

$$\frac{d}{da} \{|X+a|_f - a\} = \frac{\tilde{\mathbf{E}}\{f'(X+a)\}}{f' \circ f^{-1}\{\tilde{\mathbf{E}}(f(X+a))\}} - 1 \leq 0. \quad (11)$$

Since the mapping $a \rightarrow \tilde{\mathbf{E}}(f(X+a))$ is continuous at $a = 0$, then (11) implies that (8) holds for any $a \in (0, a_X)$. To finish the proof we should prove that (8) also holds for $a = a_X$. If a_X is infinite, then it is trivial. Suppose a_X is finite. Recall that the mapping $a \rightarrow |X+a|_f$ is continuous and increasing on $[0, a_X)$. Therefore, by the monotone convergence,

$$|X+a_X|_f = \lim_{a \uparrow a_X} |X+a|_f \leq |X|_f + a_X.$$

This terminates the proof.

Theorem 2. *Let $\{X_n, n \geq 0\}$ be an $\{\mathcal{F}_n\}$ -adapted stochastic process taking values in an unbounded subset of $\mathbf{R}_+$. Let f be a convex in a neighborhood of ∞ function from $\mathcal{G}$ satisfying the condition (R). Suppose for some positive constant A_0 , $\{f(X_{n \wedge \tau_{A_0}}), n \geq 0\}$ is a supermartingale and there exists $\varepsilon > 0$ such that for any $n \geq 0$,*

$$\mathbf{E}(f(X_{n+1}) - f(X_n) | \mathcal{F}_n) \leq -\varepsilon f'(X_n) \quad \text{on } \{\tau_{A_0} > n\}. \quad (12)$$

Then there exists a positive constant c such that for all $x \geq A_0$ whenever $X_0 = x$ with probability 1,

$$\mathbf{E}(f(\tau_{A_0})) \leq c f(x). \quad (13)$$

Let us now discuss the possible extensions of Theorem 2. As the reader might see, the main difficulty in applying Theorem 2 is finding of an appropriate «strong» supermartingale $\{f(X_{n \wedge \tau_A}), n \geq 0\}$ such that its conditional increments are negative and are of an order at least $f'(X_{n \wedge \tau_A})$ when $X_{n \wedge \tau_A} \rightarrow \infty$. Namely, in applications (and we will see some of them later) these increments may be of the order $f''(X_{n \wedge \tau_A})$ or even $g(X_{n \wedge \tau_A})$ with some arbitrary positive function g .

Another problem appears when one tries to use Theorem 2 in the study of multidimensional countable Markov chains. Suppose we are given a multidimensional Markov chain $\{Z_n, n \geq 0\}$ and would like to prove the integrability of a certain function of the first passage-time into a given finite set. One natural algorithm based on Theorem 2 may be as follows. We first try to

find a positive not necessarily one-to-one function F defined on the original state space such that the image-process $\{X_n, n \geq 0\}$ defined by $X = F(Z)$ satisfies the conditions of Theorem 2. The important restriction on the choice of F is that the integrability of the first passage-times for X should provide one with an adequate information for the original process Z . Suppose we can find such F . The next step would then consist in constructing a supermartingale $\{f(X_{n \wedge \tau_A}), n \geq 0\}$ satisfying the conditions of Theorem 2. As soon as such f is found, we will be able to apply Theorem 2 to get the integrability of the function of the first passage-times for the «transformed» processes X which in turn gives us the desired integrability results for the original process Z . It turns out, however, that it is quite rare to see that such f can be found. As a rule, one can only find the supermartingales of the form $U = G(Z)$ where G is another positive mapping (called Lyapunov function) of the original state space and the processes X and U are related to each other only implicitly.

We will now give a slight generalization of Theorem 2 that sometimes permits to avoid the just mentioned problems.

Theorem 2'. *Let $\{X_n, n \geq 0\}$ be an $\{\mathcal{F}_n\}$ -adapted stochastic process taking values in an unbounded subset of $\mathbf{R}_+$. Let $\{U_n, n \geq 0\}$ be another $\{\mathcal{F}_n\}$ -adapted stochastic process taking values in an unbounded subset of $\mathbf{R}_+$ such that for any n , U_n is integrable. Suppose there exist a positive real number A_0 and positive on (A_0, ∞) functions g, h such that for any $n \geq 0$, $U_n \leq h(X_n)$ and*

$$\mathbf{E}(U_{n+1} - U_n \mid \mathcal{F}_n) \leq -g(X_n) \quad \text{on } \{\tau_{A_0} > n\}. \quad (14)$$

Then for any convex in a neighborhood of ∞ function $f \in \mathcal{G}$ satisfying the condition (R) and

$$\liminf_{x \rightarrow \infty} \frac{g(x)}{f' \circ f^{-1} \circ h(x)} > 0, \quad (15)$$

there exists a positive constant c such that for all $x \geq A_0$ whenever $X_0 = x$ with probability 1,

$$\mathbf{E}(f(\tau_{A_0})) \leq c h(x). \quad (16)$$

Proof of Theorems 2 and 2'. We only give the proof of Theorem 2'. The proof of Theorem 2 can be obtained replacing U_n by $f(X_n)$ and g by $\varepsilon f'$. We start the proof with preliminary observations. Firstly, without loss of generality we can suppose that $f \in \mathcal{G}_0$ and is convex on $[0, \infty)$. In fact, if it is not the case we can reduce the proof to the case $f \in \mathcal{G}_0$ by replacing f by $\bar{f}$ defined by $\bar{f}(x) = f(x + B) - f(B)$ for some sufficiently large B . Let us fix any positive λ less than

$$1 \wedge A_0 \wedge \liminf_{x \rightarrow \infty} \frac{g(x)}{f' \circ f^{-1} \circ h(x)}.$$

Then, for all $x \geq A_0$,

$$g(x) > \lambda f' \circ f^{-1} \circ h(x).$$

Next, as is easy to see, the convexity of f on $\mathbf{R}_+$ implies that for all $x \geq \lambda$,

$$f(x) - f(x - \lambda) \leq \lambda f'(x). \quad (17)$$

We are now ready to prove (16). For each $n \geq 0$, we set:

$$\begin{aligned} Y_n &= f^{-1}(U_n); \\ V_n &= f(Y_{n \wedge \tau_{A_0}}(n \wedge \tau_{A_0})). \end{aligned}$$

We claim that the process $\{V_n, n \geq 0\}$ is a supermartingale. To prove this we notice first that the assumptions on f and the integrability of U_n for all $n \geq 0$ imply that for each $n \geq 0$, the random variable V_n is also integrable. Next, replacing, if necessary, U_n by $U_n + c$ with some constant c greater than $f(\lambda)$ we can ensure that for any $n \geq 0$, $Y_{(n+1) \wedge \tau_{A_0}} \geq \lambda$. Applying then the statement of Lemma 1 to f with the expectation replaced by the conditional expectations with respect to the σ -algebras $\mathcal{F}_n$, the random variable X replaced by the variables $Y_{(n+1) \wedge \tau_{A_0}}$ and the constant a by the variables $\lambda(\tau_{A_0} \wedge (n+1))$ we obtain recalling the $\mathcal{F}_n$ -measurability of the random variable $\tau_{A_0} \wedge (n+1)$ that for any $n \geq 0$,

$$\begin{aligned} & f^{-1} \left\{ \mathbf{E} \left(f \left(Y_{(n+1) \wedge \tau_{A_0}} + \lambda(\tau_{A_0} \wedge (n+1)) \right) \mid \mathcal{F}_n \right) \right\} \\ & \leq f^{-1} \left\{ \mathbf{E} \left(f(Y_{(n+1) \wedge \tau_{A_0}}) \mid \mathcal{F}_n \right) \right\} + \lambda(\tau_{A_0} \wedge (n+1)). \end{aligned} \quad (18)$$

On the other hand, we get from (14) and the condition $U_n \leq h(X_n)$ on $\{\tau_{A_0} > n\}$,

$$\begin{aligned} \mathbf{E}(f(Y_{(n+1) \wedge \tau_{A_0}}) \mid \mathcal{F}_n) & \leq f(Y_{\tau_{A_0}}) 1_{(\tau_{A_0} \leq n)} \\ & + \left\{ f(Y_n) - \lambda f' \circ f^{-1} \circ h(X_n) \right\} 1_{(\tau_{A_0} > n)}. \end{aligned} \quad (19)$$

Then using (17) and recalling the convexity and monotonicity of f one immediately obtains that the last expression is bounded from above by:

$$\begin{aligned} & f(Y_{\tau_{A_0}}) 1_{(\tau_{A_0} \leq n)} + \left\{ f(Y_n) - \lambda f'(Y_n) \right\} 1_{(\tau_{A_0} > n)} \\ & \leq f(Y_{n \wedge \tau_{A_0}} - \lambda 1_{(\tau_{A_0} > n)}). \end{aligned} \quad (20)$$

Putting the last bounds into (18) we finally obtain,

$$f^{-1} \left\{ \mathbf{E} \left(f \left(Y_{(n+1) \wedge \tau_{A_0}} + \lambda(\tau_{A_0} \wedge (n+1)) \right) \mid \mathcal{F}_n \right) \right\} \leq Y_{n \wedge \tau_{A_0}} + \lambda(\tau_{A_0} \wedge n).$$

This proves that the process $\{V_n, n \geq 0\}$ is a supermartingale. Hence,

$$\mathbf{E}(f(\lambda(n \wedge \tau_{A_0}))) \leq \mathbf{E}(V_n) \leq \mathbf{E}(V_0) = \mathbf{E}(U_0) = h(x),$$

and, applying the monotone convergence theorem, we get

$$\mathbf{E}(f(\lambda\tau_{A_0})) \leq h(x),$$

which together with the condition (R) readily imply (16) that terminates the proof.

As the reader might see the conditions of Theorems 2-2' exclude from consideration concave functions f . The following results permit to deal with them. As above, we first give a simple variant and then its generalization.

Theorem 3. Let $\{X_n, n \geq 0\}$ be an $\{\mathcal{F}_n\}$ -adapted stochastic process taking values in an unbounded subset of $\mathbf{R}_+$. Let $h \in C^1[0, \infty)$ be a function such that h' decreases in a neighborhood of ∞ and $h'(x) \rightarrow 0$ as $x \rightarrow \infty$. Suppose there exist positive constants A_0, ε such that h increases on $[A_0, \infty)$ and for any $n \geq 0$,

$$\mathbf{E}(h(X_{n+1}) - h(X_n) \mid \mathcal{F}_n) \leq -\varepsilon h'(X_n) \quad \text{on } \{\tau_{A_0} > n\}. \quad (21)$$

Then for any positive concave in a neighborhood of ∞ function $f \in C^2(0, \infty)$ such that $\lim_{x \rightarrow \infty} f(x) = \infty$, $\lim_{x \rightarrow \infty} f'(x) = 0$ and the integral

$$\int^{+\infty} \frac{f'(x) dx}{h \circ (h')^{-1} \circ f'(x)} \quad \text{converges,} \quad (22)$$

there exist positive constants $c, A \geq A_0$ such that for all $x > A$ whenever $X_0 = x$,

$$\mathbf{E}(f(\tau_A)) \leq c h(x).$$

Theorem 3'. Let $\{X_n, n \geq 0\}$ be an $\{\mathcal{F}_n\}$ -adapted stochastic process taking values in an unbounded subset of $\mathbf{R}_+$. Let $\{U_n, n \geq 0\}$ be another $\{\mathcal{F}_n\}$ -adapted stochastic process taking values in an unbounded subset of $\mathbf{R}_+$. Suppose there exist a positive constant A_0 and a continuous decreasing in a neighborhood of ∞ function g such that g is positive on $[A_0, \infty)$, $g(x) \rightarrow 0$ as $x \rightarrow \infty$ and for any $n \geq 0$,

$$\mathbf{E}(U_{n+1} - U_n \mid \mathcal{F}_n) \leq -g(X_n) \quad \text{on } \{\tau_{A_0} > n\}. \quad (23)$$

Suppose finally there exists an increasing in a neighborhood of ∞ function h such that the family $\{\mathbf{E}(h(X_{n \wedge \tau_{A_0}})), n \geq 0\}$ is uniformly bounded. Then for any positive concave in a neighborhood of ∞ function $f \in C^2(0, \infty)$ such that $\lim_{x \rightarrow \infty} f(x) = \infty$, $\lim_{x \rightarrow \infty} f'(x) = 0$ and the integral

$$\int^{+\infty} \frac{f'(x) dx}{h \circ g^{-1} \circ f'(x)} \quad \text{converges,} \quad (24)$$

there exists $A \geq A_0$ such that for all $x > A$ whenever $X_0 = x$,

$$\mathbf{E}(f(\tau_A)) \quad \text{is finite.} \quad (25)$$

P r o o f. We only prove Theorem 3'. Let B be a positive constant such that f is concave, h increases and g decreases on $[B, \infty)$. Let A be any fixed number greater than $A_0 \vee B$. For each $n \geq 0$, we set

$$V_n = U_{n \wedge \tau_A} + f(n \wedge \tau_A).$$

Let us define n_0 by $n_0 = \sup\{n; f'(n) \geq g(A)\} \vee B$. For each $x \geq n_0$, let $r(x) = \sup\{y \geq A; f'(x) = g(y)\}$. Obviously, this definition is not ambiguous, because g is continuous decreasing function on $[A, \infty)$ and $g(x) \rightarrow 0$ as $x \rightarrow \infty$. Then, for any $n \geq n_0$,

$$\mathbf{E}(U_{(n+1) \wedge \tau_A} - U_{n \wedge \tau_A} \mid \mathcal{F}_n) \leq -g(X_{n \wedge \tau_A}) \leq -g(X_{n \wedge \tau_A}) 1_{(X_{n \wedge \tau_A} \in (A, r(n)))}.$$

On the other hand, since the function f is concave on $[B, \infty)$, then for any $n \geq n_0$,

$$f(n+1) - f(n) \leq f'(n).$$

Moreover, since g is decreasing on $[B, \infty)$, then for any $n \geq n_0$,

$$f(n+1) - f(n) \leq f'(n) = g(r(n)) \leq g(X_{n \wedge \tau_A}) \quad \text{on } \{X_{n \wedge \tau_A} \in (A, r(n))\}.$$

Joining together the last bounds we obtain that for all $n \geq n_0$,

$$\mathbf{E}(V_{n+1} - V_n) \leq f'(n) \mathbf{P}\{X_{n \wedge \tau_A} \geq r(n)\}.$$

The Chebyshev inequality gives that for any $n \geq n_0$,

$$\mathbf{E}(V_{n+1} - V_n) \leq \frac{f'(n)}{h \circ r(n)} \left(\left(\sup_{m \geq n_0} \mathbf{E}(h(X_{m \wedge \tau_A})) \right) \right). \quad (26)$$

Recall that r increases on $[n_0, \infty)$ and $f'(x)/(h \circ r(x))$ decreases on $[n_0, \infty)$. This fact and (26) easily imply that for any $n > n_0$,

$$\begin{aligned} \mathbf{E}(f(n \wedge \tau_A)) &\leq \mathbf{E}(V_n) \leq \mathbf{E}(V_{n_0}) \\ &\quad + \left(\sup_{n \geq n_0} \mathbf{E}(h(X_{n \wedge \tau_A})) \right) \left(\int_{n_0}^{+\infty} \frac{f'(x) dx}{h \circ g^{-1} \circ f'(x)} \right) \\ &\leq U_0 + f(n_0) + \sup_{[1, n_0+1]} f(x) \\ &\quad + \left(\sup_{n \geq n_0} \mathbf{E}(h(X_{n \wedge \tau_A})) \right) \left(\int_{n_0}^{+\infty} \frac{f'(x) dx}{h \circ g^{-1} \circ f'(x)} \right). \end{aligned}$$

Since $f(x) \rightarrow \infty$ as $x \rightarrow \infty$, then the last inequality ensure that τ_A is finite with probability 1. (25) now follows from the Fatou lemma.

R e m a r k 2. The last inequality and the Fatou lemma also imply that, if in Theorem 3 we suppose additionally that for some positive function $\bar{h}$ and for each $n \geq 0$, $U_n \leq \bar{h}(X_n)$, then there exist positive constants $c, A \geq A_0$ such that for all $x > A$ whenever $X_0 = x$,

$$\mathbf{E}(f(\tau_A)) \leq c \bar{h}(x). \quad (27)$$

Let us now give first applications of the last results. They mainly deal with situations when one knows the asymptotic behavior of conditional increments of a given process X and tries to derive an information on the recurrence classification of X by constructing an appropriate supermartingale U and functions g, h related to it. Once they are found, one then tries to find an optimal function f and to apply Theorems 2-3' to get integrability of the first passage-times for X . Applications for multidimensional random walks will be given in the next section.

As first application of Theorems 2-3' we recover Theorem 1 of [2].

Corollary 1. *Let $\{X_n, n \geq 0\}$ be an $\{\mathcal{F}_n\}$ -adapted stochastic process taking values in an unbounded subset of $\mathbf{R}_+$. Suppose there exist positive constants A_0 and ε such that $\{X_{n \wedge \tau_{A_0}}^{2p}, n \geq 0\}$ is a supermartingale and for all $n \geq 0$*

$$\mathbf{E}(X_{n+1}^{2p} - X_n^{2p} | \mathcal{F}_n) \leq -\varepsilon X_n^{2p-2} \quad \text{on } \{\tau_{A_0} > n\}.$$

1) *If $p \geq 1$, then for any positive $\delta \geq 0$, there exists a positive constant $\tilde{c}$ such that for all $x \geq A_0$ whenever $X_0 = x$ with probability 1,*

$$\mathbf{E}(\tau_{A_0})^{p-\delta} \leq \tilde{c} x^{2(p-\delta)}.$$

2) *If $p < 1$, then for any $\delta > 1 - p$ there exists a constant $A \geq A_0$ such that for all $x > A$ whenever $X_0 = x$,*

$$\mathbf{E}((\tau_A)^p \log^{-\delta}(\tau_A)) \quad \text{is finite.}$$

Proof. 1) Let $p \geq 1$. Set $f(x) = x^p$, $h(x) = x^{2p-2}$, $g(x) = x^{2p-2}$ and for each $n \geq 0$, $U_n = h(X_n)$. Obviously, U, f, g, h, A_0 satisfy the conditions of Theorem 2'. So we get the desired assertion for $\delta = 0$ and, hence, by Jensen's inequality, for all $\delta \geq 0$.

2) Let $p < 1$. We set $f(x) = x^p \log^{-\delta}(x)$, $h(x) = x^{2p}$, $g(x) = x^{2p-2}$ and for each $n \geq 0$, $U_n = h(X_n)$. Let us verify (24). For all x sufficiently large, we define $u(x) = (g)^{-1} \circ f'(x)$. Then

$$(u(x))^{2p-2} = x^{p-1} \log^{-\delta}(x) \left(p - \frac{\delta}{\log(x)} \right)$$

and

$$\int^{+\infty} \frac{f'(x) dx}{h \circ g^{-1} \circ f'(x)} = \int^{+\infty} \frac{dx (p - \delta / \log(x))^{1/(1-p)}}{x \log^{\delta/(1-p)}(x)}.$$

The choice of δ assures the convergence of the last integral. Obviously, U, f, g, h, A_0 satisfy other conditions of Theorem 3' and we arrive at the desired result.

Remark 3. The just proved result improves Theorem 1 from [2], in the case $p < 1$ since the latter theorem only asserts finiteness of the moments of τ_A of the order $p - \delta$ for any $\delta > 0$.

We would like to emphasize here that idea of the last proof is quite general and gives an approach that allows one to treat the passage-time problem for the processes with general negative asymptotically small mean drift and not only of the order $1/x$ as it was in [2] or [7]. Let us illustrate this in the situation when the process X has negative asymptotically small mean drifts of the order $1/x^a$, $a \in [0, 1)$. In the particular case $a = 0$ the result to be now proved is well known (cf. [5]), although we were unable to find any references treating the general case $a > 0$.

Corollary 2. *Let $\{X_n, n \geq 0\}$ be an $\{\mathcal{F}_n\}$ -adapted stochastic process taking values in an unbounded subset of $\mathbf{R}_+$. Suppose there exists a positive constant B such that for all $n \geq 0$, $|X_{n+1} - X_n| \leq B$. If there exist constants $A_0 > B$, $\varepsilon > 0$, $a \in [0, 1)$ such that for any $n \geq 0$,*

$$\mathbf{E}(X_{n+1} - X_n | \mathcal{F}_n) \leq -\varepsilon |X_n|^{-a} \quad \text{on } \{\tau_{A_0} > n\}$$

then there exist positive constants ε_0 , $A \geq A_0$ such that for all $x \geq A$ whenever $X_0 = x$ with probability 1,

$$\mathbf{E}(f(\tau_A)) \quad \text{is finite,}$$

where

$$f(x) = x^{a/(1+a)} \exp \{ \varepsilon_0 x^{(1-a)/(1+a)} \}.$$

Proof. It suffices to apply the statement of Theorem 2' to $U_n = F(X_n)$, $h(x) = f(x)$, $g(x) = cx^{-a} \exp \{ \varepsilon_0 x^{1-a} \}$ with some positive constant c , function F given by $F(x) = \int_0^x \exp \{ \varepsilon_0 s^{1-a} \} ds$ and some small positive constant $\varepsilon_0 < (2\varepsilon)/(B(1-a))$.

Let us explain why we take such f, g, h and verify the conditions of Theorem 2'. We first find a supermartingale U_n and functions g, h satisfying the conditions of Theorem 2'. Let us look for U among the processes of the form $U = F(X)$ for some positive function F . First, the condition on $\{X_0 > A\}$, $U_{\tau_A} \rightarrow \infty$ as $A \rightarrow \infty$, implies that $F(x) \rightarrow \infty$ as $x \rightarrow \infty$. Next, let us look at the conditional increments of $\mathbf{E}(F(X_{n+1}) - F(X_n) | \mathcal{F}_n)$. We easily see using the Taylor formula and the boundedness of jumps that for all A sufficiently large and for all $n \geq 0$, whenever $X_0 = x > A$,

$$\begin{aligned} \mathbf{E}(F(X_{n+1}) - F(X_n) | \mathcal{F}_n) &\leq -\varepsilon \frac{F'(X_n)}{(X_n)^\alpha} + \frac{B^2 F''(X_n)}{2} \\ &\quad + O(F'''(X_n)) \quad \text{on } \{\tau_A > n\}. \end{aligned}$$

We see that it suffices now to find a function F such that $F'''(x) = o(F''(x))$ as $x \rightarrow \infty$ and satisfying the following differential inequality:

$$-\varepsilon \frac{F'(x)}{x^\alpha} + \frac{rB^2 F''(x)}{2} < 0, \quad \text{for all sufficiently large } x$$

(here r is a constant which is greater or less than 1 according to the sign of F''). To this end we resolve the ordinary differential equation

$$-\varepsilon \frac{F'(x)}{x^a} + \frac{B^2 F''(x)}{2} = 0$$

and find that we can take $F(x) = \int_0^x \exp\{\varepsilon_0 s^{1-a}\} ds$ with any positive constant $\varepsilon_0 < (2\varepsilon)/(B(1-a))$. The corresponding function g will then be given by $g(x) = cx^{-a} \exp\{\varepsilon_0 x^{1-a}\}$ with some positive constant c . We can now fix such c and set $h(x) = F(x)$.

Let us next find which convex functions f satisfy the conditions of Theorem 2 with the just chosen g, h . We set $u(x) = f^{-1} \circ h(x)$. Then it follows from (15) that there exists a positive constant δ such that when $x \rightarrow \infty$, $f'(u(x)) \leq \delta g(x)$. This inequality, $f'(u(x))u'(x) = h'(x)$ and the form of g, h immediately imply that there exist positive constants c_1, c_2 such that $u'(x) \geq c_1 x^a$ and $u(x) \geq c_2 x^{a+1}$ for all sufficiently large x . On the other hand, by definition and the form of h we have that there exists a positive constant c_3 such that $f(u(x))$ is asymptotically equivalent to $c_3 x^a \exp\{\varepsilon_0 x^{1-a}\}$. The last two observations show that one can take f defined as follows: $f(x) = x^{a/(1+a)} \exp\{\varepsilon_0 x^{(1-a)/(1+a)}\}$. Corollary 2 is proved.

In fact, the condition on the a.s. boundedness of the increments of $\{X_n, n \geq 0\}$ by some positive constant can be weakened. For instance, in the case $a = 0$ it suffices to assume that there exist positive constants δ_1, δ_2 and C, D such that for any $n \geq 0$, on $\{\tau_A > n\}$,

$$\begin{aligned} \mathbf{E}(e^{\delta_1(X_{n+1}-X_n)^+} | \mathcal{F}_n) &\leq C; \\ \mathbf{E}(\{(X_{n+1}-X_n)^-\}^{1+\delta_2} | \mathcal{F}_n) &\leq D. \end{aligned}$$

The proof of this claim is straightforward and is left to the reader.

The last example of applications of Theorems 2-3' of this section is borrowed from [11].

Corollary 3. Let $\{Z_n, n \geq 0\}$ be a countable Markov chain on $\mathbf{R}^d$ defined by

$$Z_{n+1} = F(Z_n) + W_{n+1}, \quad n \in \mathbf{Z}_+,$$

where

- 1) $F: \mathbf{R}^d \rightarrow \mathbf{R}^d$ is continuous;
- 2) $|F(x)| \leq |x|$ for all $x \in \mathbf{R}^d$ and for some $0 < \delta < 1$ and $c, b > 0$, $|x| \geq b$,

$$|F(x)| \leq |x| \left(1 - \frac{c}{|x|^\delta}\right);$$

- 3) (W_m) is an i.i.d. sequence of random variables taking values in a countable subset of $\mathbf{R}^d$ independent of Z_0 such that for some integer $m \geq 2$,

$$\mathbf{E}(|W|^m) < \infty.$$

Then, the first return-time ρ of the Markov chain $\{Z_n, n \geq 0\}$ has finite moment of the order m/δ and the upper bound (7) with the function $f(s) = s^{m/\delta}$ holds.

Remark 4. In the terminology adopted in [11], [10] the last assertion implies that $\{Z_n, n \geq 0\}$ is f' -ergodic with the just mentioned f which gives much stronger result than Proposition 5.2 of [11] giving only f' -ergodicity with $\bar{f}(s) = s^{m-1}$, and, in particular, the existence of the $(m-1)$ -st moment of ρ and the upper bound (7) with f replaced by $\bar{f}$.

Proof. We only outline the proof. For each $n \geq 0$, let us set $Y_n = |Z_n|$. The Minkowski inequality for conditional expectations, the triangle inequality and the conditions on F easily imply that for all $n \geq 0$,

$$(E(Y_{n+1}^m | \mathcal{F}_n))^{1/m} \leq Y_n \left(1 - \frac{c}{Y_n^\delta}\right) + (E(|W|^m))^{1/m} \quad \text{on } \{|Z_n| \geq b\}.$$

Hence, using the moment condition on W , one can get that there exist positive constants $b_1 \geq b$ and c_1, c_2 such that for all $n \geq 0$ on $\{|Z_n| \geq b_1\}$

$$E(Y_{n+1}^m | \mathcal{F}_n) \leq Y_n^m \left(1 - \frac{c_1}{Y_n^\delta}\right)^m \leq Y_n^m - c_2 Y_n^{m-\delta}.$$

Let us define the process X by $X_n = Y_n^{\delta/2}$. Then X satisfies the conditions of Corollary 1 with $p_0 = m/\delta$, $A = (b_1)^{\delta/2}$ and, hence,

$$E(f(\tau_A)) \leq c x^{2m/\delta}, \quad \text{whenever } X_0 = x > A.$$

Rewriting this statement in terms of the original process Z we obtain that whenever $|z| > b_1$, $E_z(f(\rho^F)) \leq c|z|^m$, where $F = \{z \in \mathbf{R}^d; |z| \leq b_1\}$ and, as above, ρ^F denotes the first hitting time of F by the Markov chain Z . Finally, the last condition on the finiteness of $E_z(|Z_1|^m)$ for all $z \in F$ follows from the moment condition on W , the properties of F and the definition of Z . We can now apply the result of Theorem 1 and get the desired assertion.

Remark 5. As seen from the proof, one can replace the moment condition on W by

$$\exists c_1 < c \quad \text{such that } \forall n \geq 0, \text{ess sup} \left(\frac{E(W_{n+1}^m | \mathcal{F}_n)}{Y_n^{m-m\delta}} \right) < c_1.$$

Notice also that the assertion of the last corollary can be strengthen if one impose more restrictive moment conditions on W . For example, if $E(e^{cW^\delta})$ is finite for some $c > 0$, $0 < \delta < 1$, then it can be proved that for any positive λ the function $f(s) = e^{\lambda s^\delta}$ of the first return time ρ of the Markov chain $\{Z_n, n \geq 0\}$ has finite expectation and (7) holds with such function f . To show this one can repeat the proof of the last result replacing the Minkowski inequality by Lemma 1 and Theorem 2' by Corollary 1.

4. Passage-times for 2-dimensional Markov chains in wedges with boundary reflection

4.1. Formulation of the problem and statement of the main results. Let $\tilde{G}$ be the quadrant given by $\tilde{G} = \{(x, y) \in \mathbf{R}^2; x \geq 0, y \geq 0\}$. The two sides of the quadrant are denoted by $\partial\tilde{G}_1$ and $\partial\tilde{G}_2$, where $\partial\tilde{G}_1 = \{(x, y) \in \tilde{G}; x \neq 0, y = 0\}$ and $\partial\tilde{G}_2 = \{(x, y) \in \tilde{G}; y \neq 0, x = 0\}$. The coordinate vectors of $\partial\tilde{G}_1$ and $\partial\tilde{G}_2$ will be denoted by $\vec{e}_1$ and $\vec{e}_2$. The interior of $\tilde{G}$ is referred to as $\tilde{G}^0$.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space equipped with a filtration $\{\mathcal{F}_n\}_{n \geq 0}$. We are dealing with a discrete-time homogeneous irreducible aperiodic $\{\mathcal{F}_n\}$ -adapted Markov chain $\{\tilde{Z}_n, n \geq 0\}$ defined on $(\Omega, \mathcal{F}, \mathbf{P})$, taking values in $\mathbf{Z}_+^2$ with the following one-step transition probabilities:

$$p_{(x,y) \rightarrow (x+i, y+j)} = \begin{cases} p_{i,j}^0, & (x, y) \in \tilde{G}^0 \cap \mathbf{Z}_+^2; \\ p_{i,j}^1, & (x, y) \in \partial\tilde{G}_1 \cap \mathbf{Z}_+^2; \\ p_{i,j}^2, & (x, y) \in \partial\tilde{G}_2 \cap \mathbf{Z}_+^2; \\ p_{i,j}^3, & (x, y) = (0, 0); \end{cases}$$

where $i, j \in \mathbf{Z}$, $i, j \geq -1$. In order to keep the process in $\mathbf{Z}_+^2$ it is assumed that for any i, j , $p_{i,-1}^1 = p_{-1,j}^2 = p_{i,-1}^3 = p_{-1,j}^3 = 0$. We suppose that $\gamma \equiv \sup\{\kappa; \sum_{i,j} (|i|^\kappa + |j|^\kappa) p_{i,j}^l < \infty, \forall l = 0, 1, 2\} > 2$. We also suppose that in the interior $\tilde{G}^0$ the one-step mean drift (i.e., $\mathbf{E}((\tilde{Z}_{n+1} - \tilde{Z}_n) | \mathcal{F}_n)$ conditioned on $\tilde{Z}_n \in \tilde{G}^0$) is zero and the covariance matrix A^0 corresponding to the vector $\tilde{Z}_{n+1} - \tilde{Z}_n$ conditioned on $\tilde{Z}_n \in \tilde{G}^0$ is positive definite (i.e., the process can not jump only along a fixed straight line in $\tilde{G}^0$). As far as the boundary $\partial\tilde{G}$ is concerned, we suppose the one-step mean drifts on the axes $\partial\tilde{G}_1, \partial\tilde{G}_2$ (called vectors of boundary reflection) are not collinear with the corresponding coordinate vectors $\vec{e}_1$ and $\vec{e}_2$, i.e., the boundary reflection on the axes is non-degenerate. For more details on the model we refer to [1, Subsection 3.1].

Definition 4. Let F be any Borel subset of $\mathbf{R}^2$. Let us define the first passage-time in F of the Markov chain $\{\tilde{Z}_n, n \geq 0\}$ by:

$$\tilde{T}_F \equiv \tilde{T}_F^{\tilde{Z}} = \inf\{n \geq 0; \tilde{Z}_n \in F\}.$$

In particular, if $F_A = \{z; |z| \leq A\}$, we denote by $\tilde{T}_A$:

$$\tilde{T}_A = \tilde{T}_{F_A} = \inf\{n \geq 0; |\tilde{Z}_n| \leq A\}.$$

In [2] and [1] we have established the existence of a parameter α depending on geometrical data of the Markov chain $\{\tilde{Z}_n, n \geq 0\}$ such that the following result holds.

Theorem 4. I. If $\alpha < 0$, then the Markov chain $\{\tilde{Z}_n, n \geq 0\}$ is transient. If $\alpha \geq 0$, then it is recurrent.

II. Suppose $0 \leq \alpha < \gamma$. Then the following statements hold:

1) If $\alpha > 0$, then for any $p < \alpha/2$, there exists $\tilde{A}_0 > 0$ such that for all $|\tilde{z}| > \tilde{A}_0$,

$$E_{\tilde{z}}(\tilde{T}_{\tilde{A}_0}^p) \text{ is finite.}$$

2) There exist positive constants $\tilde{A}_1, \tilde{C}_1, \tilde{c}_1$ and $\tilde{s}_0$ such that for any $\tilde{A} \geq \tilde{A}_1$ and $|\tilde{z}| > \tilde{C}_1 \tilde{A}$, the following bound holds for all $s > \tilde{s}_0 \tilde{A}^2$,

$$P_{\tilde{z}}\{\tilde{T}_{\tilde{A}} \geq s\} \geq \begin{cases} \tilde{c}_1 \tilde{A}^\alpha s^{-\alpha/2}, & \text{if } \alpha > 0; \\ \frac{\tilde{c}_1}{\log(s)}, & \text{if } \alpha = 0. \end{cases} \quad (28)$$

In particular, if $\alpha > 0$ (respectively $\alpha = 0$), then for any $k \geq 1$,

$$E_{\tilde{z}}(\tilde{T}_{\tilde{A}}^{\alpha/2} \log_0^{-1}(\tilde{T}_{\tilde{A}}) \cdots \log_k^{-1}(\tilde{T}_{\tilde{A}})) \text{ is infinite}$$

$$(\text{respectively } E_{\tilde{z}}(\log(\tilde{T}_{\tilde{A}}) \log_2^{-1}(\tilde{T}_{\tilde{A}}) \cdots \log_{k+1}^{-1}(\tilde{T}_{\tilde{A}})) \text{ is infinite})$$

(here, as elsewhere, the functions $\log_k x$ are defined by: $\log_0 x \equiv 1$, $\log_k x = \log_k(x + c_k)$, $k \geq 1$, where the constants c_k are chosen in such a way that $\log_k 0 = 1$ and we simply write $\log x$ instead of $\log_1 x$).

The principal results of this section complement Theorem 4 and are stated as follows.

Theorem 5. Let $\{\tilde{Z}_n, n \geq 0\}$ be the Markov chain defined above. Suppose $\alpha < \gamma$. Then the following two statements hold.

1. If $\alpha > 2$ (respectively $0 < \alpha \leq 2$), then for any $q > \alpha/2$ (respectively $q > 1$), there exists a positive constant $\tilde{A}_2$ such that for any $\tilde{A} \geq \tilde{A}_2$ and $|\tilde{z}| > \tilde{A}$,

$$E_{\tilde{z}}(\tilde{T}_{\tilde{A}}^{\alpha/2} \log^{-q}(\tilde{T}_{\tilde{A}})) \text{ is finite.}$$

2. If $\alpha = 0$, then for any $q > 1$, there exists a positive constant $\tilde{A}_2$ such that for any $\tilde{A} \geq \tilde{A}_2$ and $|\tilde{z}| > \tilde{A}$,

$$E_{\tilde{z}}(\log(\tilde{T}_{\tilde{A}}) \log_2^{-q}(\tilde{T}_{\tilde{A}})) \text{ is finite.}$$

Remark 6. Using the same method of the proof it can be proved that the result of Theorem 5 can be strengthened in the following way.

1. If $0 < \alpha \leq 2$, then for any $q > 1$ and for any integer $k \geq 1$, there exists a positive constant $\tilde{A}_2$ such that for any $\tilde{A} \geq \tilde{A}_2$ and $|\tilde{z}| > \tilde{A}$,

$$E_{\tilde{z}}(\tilde{T}_{\tilde{A}}^{\alpha/2} \log_0^{-1}(\tilde{T}_{\tilde{A}}) \cdots \log_{k-1}^{-1}(\tilde{T}_{\tilde{A}}) \log_k^{-q}(\tilde{T}_{\tilde{A}})) \text{ is finite.}$$

2. If $\alpha = 0$, then for any $q > 1$ and for any integer $k \geq 3$, there exists a positive constant $\tilde{A}_2$ such that for any $\tilde{A} \geq \tilde{A}_2$ and $|\tilde{z}| > \tilde{A}$,

$$E_{\tilde{z}}(\log(\tilde{T}_{\tilde{A}}) \log_2^{-1}(\tilde{T}_{\tilde{A}}) \cdots \log_{k-1}^{-1}(\tilde{T}_{\tilde{A}}) \log_k^{-q}(\tilde{T}_{\tilde{A}})) \text{ is finite.}$$

By using Theorem 1 it can be also proved that results analogous to Theorems 4–5, Remark 6 hold for the first return-time ρ of the Markov chain $\{\tilde{Z}_n, n \geq 0\}$. Namely, we will prove

Theorem 6. For any $\tilde{z} \in \mathbf{Z}_+^2$ the following statements hold.

1. If $\alpha > 2$, then

$$E_{\tilde{z}}(\rho^{\alpha/2} \log^{-q}(\rho)) \begin{array}{l} \text{is finite, if } q > \alpha/2 \\ \text{is infinite, if } q \leq 1. \end{array}$$

Moreover, for any initial distribution λ on $\mathbf{Z}_+^2$ such that

$$\int_{\mathbf{R}_+ \times \mathbf{R}_+} \lambda(d\tilde{z}) |\tilde{z}|^{\alpha-2} \log^{-1} |\tilde{z}| < \infty,$$

and for any $q > \alpha/2$ we have

$$\int \lambda(d\tilde{z}) \|P^n(\tilde{z}, \cdot) - \pi\| = o\left(\frac{\log^q n}{n^{\alpha/2-1}}\right), \quad \text{as } n \rightarrow \infty. \quad (29)$$

2. If $0 < \alpha \leq 2$, then for any integer $k \geq 1$,

$$E_{\tilde{z}}(\rho^{\alpha/2} \log^{-1}(\rho) \cdots \log_{k-1}^{-1}(\rho) \log_k^{-q}(\rho)) \begin{array}{l} \text{is finite, if } q > 1; \\ \text{is infinite, if } q \leq 1. \end{array}$$

3. If $\alpha = 0$, then for any integer $k \geq 3$,

$$E_{\tilde{z}}(\log(\rho) \log_2^{-1}(\rho) \cdots \log_{k-1}^{-1}(\rho) \log_k^{-q}(\rho)) \begin{array}{l} \text{is finite, if } q > 1; \\ \text{is infinite, if } q \leq 1. \end{array}$$

Remark 7. M. Menshikov and S. Popov in their paper [9] proved that in the ergodic case $\alpha > 2$ for any $\tilde{z}$ and any positive δ the following lower bound for the rate of convergence of the transition probabilities of the Markov chain Z holds:

$$|P^n(\tilde{z}, \tilde{z}) - \pi(\tilde{z})| \geq cn^{-(\alpha/2-1+\delta)},$$

with some positive constant $c = c(\tilde{z})$ depending on $\tilde{z}$. This bound shows that (29) is quite precise.

In order to prove the main results of this section we «transform» the original Markov chain by a linear isomorphism Φ of $\mathbf{R}^2$ chosen in such a way that the image of the covariance matrix A^0 under Φ is unit (in what follows Φ is defined as in Subsection 3.2 in [1]). Next we will prove analogues of Theorem 5–6 and Remark 6 for the transformed Markov chains by

constructing supermartingales satisfying the hypotheses of Theorems 1, 2' and 3'. This will conclude the proof.

4.1. Notation and technical results. The principal goal of this section is to recall some notation and technical results provided by [2] and [1]. Let Φ be the linear isomorphism of $\mathbf{R}^2$ introduced by formulae (13)–(14) in [1]. Set

$$\begin{aligned} G_4 &= \Phi(\mathbf{Z}_+^2), & G &= \Phi(\tilde{G}), & G^0 &= \Phi(\tilde{G}^0), \\ \partial G_1 &= \Phi(\partial \tilde{G}_1), & \partial G_2 &= \Phi(\partial \tilde{G}_2). \end{aligned} \quad (30)$$

The angle of the wedge G is denoted by ξ . The image of the Markov chain $\{\tilde{Z}_n, n \geq 0\}$ will be called «transformed» Markov chain and denoted by $\{Z_n, n \geq 0\}$. By construction, $\{Z_n, n \geq 0\}$ takes values in G_4 and the covariance matrix of the conditional distribution of the vector $Z_{n+1} - Z_n$ conditioned on $Z_n \in G^0$ is unit. Let α_1, α_2 be the angles that the mean drifts of $\{Z_n, n \geq 0\}$ from the $\partial G_1, \partial G_2$, i.e., conditional expectations of the vector $Z_{n+1} - Z_n$ conditioned on $Z_n \in \partial G_1, \partial G_2$ (in other words, images of the vectors of boundary reflection for the initial Markov chain $\{\tilde{Z}_n, n \geq 0\}$) make with the inward normals to $\partial G_1, \partial G_2$, positive angles being toward the corner. Since the initial boundary reflection is non-degenerate, $\alpha_1, \alpha_2 \in (-\pi/2, \pi/2)$. We can now give the exact expression for the parameter α appearing in the statement of the last theorems:

$$\alpha = \frac{\alpha_1 + \alpha_2}{\xi}.$$

Let F be any Borel subset of $\mathbf{R}^2$. The first passage-time of the Markov chain $\{Z_n, n \geq 0\}$ in F is defined in the same way as the first passage-time $\tilde{T}_F$ and is denoted by T_F . The passage-times T_A are defined similarly to $\tilde{T}_A$.

Before proving the main results let us give an account of notation and preliminary technical results that we used in [1]. To start with, let introduce a family $\{\psi_{\beta_1, \beta_2}, \beta_1, \beta_2 \in (-\pi/2, \pi/2)\}$ of non-negative harmonic functions on $G \setminus \{0\}$ which are defined in polar coordinates (ρ, θ) as follows. For any $\beta_1, \beta_2 \in (-\pi/2, \pi/2)$ we set $\beta = (\beta_1 + \beta_2)/\xi$ and define

$$\psi_{\beta_1, \beta_2}(\rho, \theta) = \begin{cases} \rho^\beta \cos(\beta\theta - \beta_1), & \beta_1 + \beta_2 \neq 0, \rho > 0, \theta \in [0, \xi], \\ \log \rho + |\operatorname{tg} \beta_1| \theta, & \beta_1 + \beta_2 = 0, \rho > 0, \theta \in [0, \xi], \\ 0, & \rho = 0. \end{cases}$$

A remarkable property of the functions ψ_{β_1, β_2} is that they are asymptotically equivalent to a power (or logarithm) of the usual Euclidean norm in $\mathbf{R}^2$. More precisely,

a) if $\beta \neq 0$, then for any $z \in G \setminus (0, 0)$,

$$|z|^\beta \cos(|\beta_1| \vee |\beta_2|) \leq \psi_{\beta_1, \beta_2}(z) \leq |z|^\beta; \quad (31)$$

b) if $\beta = 0$, then there exist a positive constant c_1 such that for all $|z| > 2$ we have

$$\log |z| \leq \psi_{\beta_1, \beta_2}(z) \leq c_1 \log |z|; \quad (32)$$

this property will be referred to as the «monotonicity» property of ψ_{β_1, β_2} .

We need some further notation. For $\beta_1, \beta_2 \in (-\pi/2, \pi/2)$, $\xi \in (0, \pi)$, and $A > 0$ we set $\beta = (\beta_1 + \beta_2)/\xi$ and:

$$F_{A, \beta_1, \beta_2, \xi} = \begin{cases} \{z \in G; \psi_{\beta_1, \beta_2}^{1/\beta}(z) \leq A\}, & \text{if } \beta \neq 0, \\ \{z \in G; \exp(\psi_{\beta_1, \beta_2}(z)) \leq A\}, & \text{if } \beta = 0, \end{cases} \quad (33)$$

and

$$c(\beta) = \begin{cases} \cos^{-1/\beta}(|\beta_1| \vee |\beta_2|), & \text{if } \beta \neq 0, \\ 1, & \text{if } \beta = 0. \end{cases} \quad (34)$$

In the sequel we will denote by $\{X_n, n \geq 0\}$ the process defined by:

$$X_n = \begin{cases} \psi_{\alpha_1, \alpha_2}^{1/\alpha}(Z_n), & \text{if } \alpha \neq 0, \\ \exp(\psi_{\alpha_1, \alpha_2}(Z_n)), & \text{if } \alpha = 0. \end{cases} \quad (35)$$

Throughout the rest of the paper η is a fixed positive number such that

$$\eta < \min\left(\frac{\gamma - 2}{2}, \gamma - \alpha, \frac{1}{2}\right). \quad (36)$$

Next, for each $n \geq 0$ and for any $\mathcal{F}_{n+1}$ -measurable function F , we introduce the notation $\Delta_n = Z_{n+1} - Z_n$, $\mathbf{E}_{Z_n}(F) = \mathbf{E}(F|Z_n)$.

We finally recall Lemma 2 of [1] on asymptotic behavior of conditional expectations of $f \circ \psi_{\beta_1, \beta_2}(Z_{n+1}) - f \circ \psi_{\beta_1, \beta_2}(Z_n)$ for functions f belonging to the following class $\mathcal{H}$.

Definition 5.

$$\mathcal{H} = \left\{ f: \mathbf{R}_+ \rightarrow \mathbf{R}_+; f \in C^3(0, \infty) \right.$$

$$\left. \frac{f'''(x)}{f''(x)} = O\left(\frac{1}{x}\right) \text{ and } \frac{f''(x)}{f'(x)} = O\left(\frac{1}{x}\right) \text{ as } x \rightarrow \infty, \right.$$

$$\left. \forall \nu > 0, \liminf_{x \rightarrow \infty} \left| \frac{f''(x)x^{1+\nu}}{f'(x)} \right| > 0 \text{ and } \liminf_{x \rightarrow \infty} \left| \frac{f'(x)x^{1+\nu}}{f(x)} \right| > 0, \right.$$

there exist positive $a_f > 1$ and $\tilde{A}_f$ such that

$$\limsup_{x \rightarrow \infty} \frac{|f''(a_f x)|}{|f''(x)|} \leq \tilde{A}_f \left. \right\}.$$

Lemma 2. Let β_1 and $\beta_2 \in (-\pi/2, \pi/2)$ be real numbers such that $\beta_1 + \beta_2 \geq 0$. Set $\beta = \beta_1 + \beta_2 / \xi$. Let f be a function from $\mathcal{H}$ such that in the case $\beta \neq 0$ (respectively $\beta = 0$) the function $|f''(x^\beta)|x^{2\beta-2}$ (respectively $|f''(\log x)|x^{-2}$)

is monotone on some interval $[B, \infty)$. Suppose there exist positive constants $\chi \in (0, 1)$ and c such that for all $n \geq 0$ and for all z , P_z -a.s.

$$\begin{aligned} \mathbf{E}_{Z_n} \left(|\Delta_n|^{2+\chi} \max(1, |f''(|\Delta_n|^\beta)| |\Delta_n|^{2\beta-2}) \right) &\leq c, \quad \text{if } \beta \neq 0, \\ \mathbf{E}_{Z_n} \left(|\Delta_n|^{2+\chi} \max(1, |f''(|\log |\Delta_n||)| |\Delta_n|^{-2}) \right) &\leq c, \quad \text{if } \beta = 0. \end{aligned} \quad (37)$$

Then there exist positive constants A, b, C such that for any $n \geq 0$ and for any $|z| > A$ the following two statements hold P_z -a.s.

a) On $\{Z_n \in G^0\} \cap \{|Z_n| > A\}$,

$$\left| \mathbf{E}_{Z_n} \left(f \circ \psi_{\beta_1, \beta_2}(Z_{n+1}) - f \circ \psi_{\beta_1, \beta_2}(Z_n) \right) \right| \leq b |f'' \circ \psi_{\beta_1, \beta_2}(Z_n)| |Z_n|^{2\theta-2}; \quad (38)$$

furthermore,

$$\begin{aligned} \text{sign}(f'' \circ \psi_{\beta_1, \beta_2}(Z_n)) \mathbf{E}_{Z_n} (f \circ \psi_{\beta_1, \beta_2}(Z_{n+1}) - f \circ \psi_{\beta_1, \beta_2}(Z_n)) \\ \geq C |f'' \circ \psi_{\beta_1, \beta_2}(Z_n)| |Z_n|^{2\beta-2}. \end{aligned} \quad (39)$$

b) For each $i = 1, 2$ we have on $\{Z_n \in \partial G_i\} \cap \{|Z_n| > A\}$,

$$\begin{aligned} \left| \mathbf{E}_{Z_n} (f \circ \psi_{\beta_1, \beta_2}(Z_{n+1}) - f \circ \psi_{\beta_1, \beta_2}(Z_n)) \right| \\ \leq \begin{cases} b |f' \circ \psi_{\beta_1, \beta_2}(Z_n)| \psi_{\beta_1, \beta_2}(Z_n) |Z_n|^{-2}, & \text{if } \beta_i = \alpha_i, \\ b |f' \circ \psi_{\beta_1, \beta_2}(Z_n)| |Z_n|^{\beta-1}, & \text{if not;} \end{cases} \end{aligned} \quad (40)$$

furthermore,

$$\begin{aligned} \text{sign}(f' \circ \psi_{\beta_1, \beta_2}(Z_n) \sin(\beta_i - \alpha_i)) \mathbf{E}_{Z_n} (f \circ \psi_{\beta_1, \beta_2}(Z_{n+1}) - f \circ \psi_{\beta_1, \beta_2}(Z_n)) \\ \geq C |f' \circ \psi_{\beta_1, \beta_2}(Z_n) \sin(\beta_i - \alpha_i)| |Z_n|^{\beta-1}. \end{aligned} \quad (41)$$

5. Proof of Theorem 5

Let us first outline the proof. Let, for example, $\alpha > 2$ and let q be any fixed positive number greater than $\alpha/2$. It is easy to see that in order to prove Theorem 5 it suffices to find a positive real number A_0 , a process $\{U_{n \wedge \tau_{A_0}}, n \geq 0\}$ and functions g, h satisfying the conditions of Theorem 2' with $f(x) = x^{\alpha/2} \log^{-q} x$. In fact, suppose we have done it. Then Theorem 2' shows that there exists a positive constant c such that for all $x > A_0$ whenever $X_0 = x$,

$$\mathbf{E}(f(\tau_{A_0})) \leq ch(x). \quad (42)$$

In other words, there exists $c > 0$ such that whenever $|\psi^{1/\alpha}(z)| > A_0$,

$$\mathbf{E}_z(f(T_{F_{A_0, \alpha_1, \alpha_2, \xi}})) \leq ch(\psi^{1/\alpha}(z)) \quad (43)$$

(recall that we defined $F_{A, \alpha_1, \alpha_2, \xi}$ as $F_{A, \alpha_1, \alpha_2, \xi} = \{z \in G, \psi_{\alpha_1, \alpha_2}^{1/\alpha}(z) \leq A\}$).

Let B be any fixed positive number such that $f(x)$ increases on $[B, \infty)$. Then the «monotonicity» property of $\psi_{\alpha_1, \alpha_2}$ and (43) imply that for any $A \geq A_0 c(\alpha)$ and $|z| > A$,

$$\mathbf{E}_z(f(T_A)1_{(T_A > B)}) \leq \mathbf{E}(f(T_{A_0 c(\alpha)})1_{(T_{A_0 c(\alpha)} > B)}) \leq \mathbf{E}_z(f(T_{F_{A_0, \alpha_1, \alpha_2, \xi}})).$$

This implies that there exists a positive constant c_1 such that for any $A \geq A_0 c(\alpha)$ and $|z| > A$,

$$\mathbf{E}_z(f(T_A)) \leq c h(\psi^{1/\alpha}(z)) + c_1.$$

The final step consisting in transition from the «transformed» chain $\{Z_n, n \geq 0\}$ to $\{\tilde{Z}_n, n \geq 0\}$ is immediate. In fact, since Φ is a linear isomorphism, then there exist positive constants $\tilde{c} = \tilde{c}(\Phi)$, $\tilde{C} = \tilde{C}(\Phi)$ such that for any positive A , $\{\tilde{z} \in \mathbf{R}_+^2; |\tilde{z}| \leq A\tilde{c}\} \subset \{\tilde{z} \in \mathbf{R}_+^2; |\Phi(\tilde{z})| \leq A\} \subset \{\tilde{z} \in \mathbf{R}_+^2; |\tilde{z}| \leq A\tilde{C}\}$. Hence, for any $A \geq A_0 c(\alpha)$ and $|\tilde{z}| > \tilde{C}A$,

$$\mathbf{E}_{\tilde{z}}(f(\tilde{T}_{\tilde{C}A})) \leq c h(\psi^{1/\alpha}(\Phi(\tilde{z}))) + c_1. \quad (44)$$

This finishes the verification of the assertion.

As the reader might have guessed the main problem in verification of the conditions of Theorems 2'-3' consists in constructing of suitable supermartingales $\{U_n, n \geq 0\}$. To construct them we apply the ideas of [1] where we have treated a similar problem in the submartingale setting. Namely, following [1] let us search the processes $\{U_n, n \geq 0\}$ among the processes of the following form:

$$U_n = \tilde{F} \circ (\psi_{\alpha_1, \alpha_2})(Z_n) + \tilde{e} \circ \psi_{\beta_1, \beta_2}(Z_n), \quad \forall n \geq 0,$$

where $\tilde{F}$ and $\tilde{e}$ are some functions of the form:

$$\tilde{e}(x) = e(x^{1/\beta}) \quad \text{and} \quad \tilde{F}(x) = \begin{cases} F(\exp(x)), & \text{if } \alpha = 0, \\ F(x^{1/\alpha}), & \text{if } \alpha > 0. \end{cases}$$

However, on the contrary to [1], let us impose on $\beta_1, \beta_2, \tilde{F}$ and $\tilde{e}$ the following conditions:

1) β_1 and β_2 are some fixed real numbers such that $\beta_i \in (-\pi/2, \pi/2)$ and in the case $\alpha > 0$ (respectively $\alpha = 0$) $\beta_i < \alpha_i$, for $i = 1, 2$ and $\beta_1 + \beta_2 > 0$ (respectively $\beta_i > \alpha_i$, for $i = 1, 2$);

2) the functions $\tilde{F}$ and $\tilde{e}$ satisfy the hypotheses of Lemma 2 and in a neighborhood of ∞ , $\tilde{F}'' < 0$, and $\tilde{e}' > 0$ (respectively $\tilde{e}' < 0$);

3) as $|z| \rightarrow \infty$,

$$\begin{aligned} \psi_{\alpha_1, \alpha_2}(z) |\tilde{F}' \circ \psi_{\alpha_1, \alpha_2}(z)| |z|^{-2} &= o(|\tilde{e}' \circ \psi_{\beta_1, \beta_2}(z)| |z|^{\beta-1}), \\ |\tilde{e}'' \circ \psi_{\beta_1, \beta_2}(z)| |z|^{2\beta-2} &= o(|\tilde{F}'' \circ \psi_{\alpha_1, \alpha_2}(z)| |z|^{2\alpha-2}). \end{aligned}$$

Using these assumptions and Lemma 2 applied separately to the triples $(\tilde{F}, \alpha_1, \alpha_2)$ and $(\tilde{e}, \beta_1, \beta_2)$ one can easily see that there exist positive constants $c_1, \dots, c_6$ such that for all large enough $|Z_n|$:

1) whenever $Z_n \in G^0$,

$$\begin{aligned} \mathbf{E}(U_{n+1} - U_n \mid \mathcal{F}_n) &\leq c_1 \tilde{F}'' \circ \psi_{\alpha_1, \alpha_2}(Z_n) |Z_n|^{2\alpha-2} \\ &\quad + c_2 |\tilde{e}'' \circ \psi_{\beta_1, \beta_2}(Z_n)| |Z_n|^{2\beta-2} \\ &\leq -c_3 |\tilde{F}'' \circ \psi_{\alpha_1, \alpha_2}(Z_n)| |Z_n|^{2\alpha-2}; \end{aligned}$$

2) whenever $Z_n \in \partial G_i$ for $i = 1, 2$,

$$\begin{aligned} \mathbf{E}(U_{n+1} - U_n \mid \mathcal{F}_n) &\leq c_4 \sin(\beta_i - \alpha_i) \tilde{e}' \circ \psi_{\beta_1, \beta_2}(Z_n) |Z_n|^{\beta-1} \\ &\quad + c_5 \psi_{\alpha_1, \alpha_2}(Z_n) |\tilde{F}' \circ \psi_{\alpha_1, \alpha_2}(Z_n)| |Z_n|^{-2} \\ &\leq -c_6 \tilde{e}' \circ \psi_{\beta_1, \beta_2}(Z_n) |Z_n|^{\beta-1}, \end{aligned}$$

which shows that the processes $\{U_{n \wedge \tau_A}, n \geq 0\}$ are supermartingales for all sufficiently large A .

Having briefly described the method of constructing of supermartingale $\{U_{n \wedge \tau_A}, n \geq 0\}$ we now prove Theorem 5. As in the last section the choice of functions $\tilde{F}$ and $\tilde{e}$ will be quite different in the cases $\alpha > 0$ and $\alpha = 0$.

I. Let us consider first the case $\alpha > 0$. Let ν and β_i , $i = 1, 2$, be some fixed real numbers such that:

- 1) $\beta_i \in (-\pi/2, \pi/2)$ and $\beta_i < \alpha_i$, for $i = 1, 2$;
- 2) $(\xi(\alpha - 1) \vee 0) < \nu(\beta_1 + \beta_2) < \xi\alpha$.

Let us introduce the process $\{U_n, n \geq 0\}$ by:

$$U_n = \psi_{\alpha_1, \alpha_2}(Z_n) \log^{-\mu} \psi_{\alpha_1, \alpha_2}(Z_n) + \psi_{\beta_1, \beta_2}^\nu(Z_n), \quad \forall n \geq 0,$$

where μ is some positive number that will be chosen later. Notice that since $\alpha < \gamma$ and since for all $n \geq 0$, $U_n \leq c|Z_n|^\alpha$ for some positive constant c , then the random variables U_n are integrable. Next, applying twice Lemma 2 and the «monotonicity» property of the functions $\psi_{\alpha_1, \alpha_2}$, ψ_{β_1, β_2} we immediately see that there exist positive constants $c_1, \dots, c_4, a_0$ such that for any $A \geq a_0$ and for any $n \geq 0$,

1) on $\{Z_n \in G^0\} \cap \{|Z_n| > A\}$,

$$-c_1 |Z_n|^{\alpha-2} \log^{-\mu-1}(|Z_n|) \leq \mathbf{E}_{Z_n}(U_{n+1} - U_n) \leq -c_2 |Z_n|^{\alpha-2} \log^{-\mu-1}(|Z_n|);$$

2) on $\{Z_n \in (\partial G_1 \cup \partial G_2)\} \cap \{|Z_n| > A\}$,

$$-c_3 |Z_n|^{\beta\nu-1} \leq \mathbf{E}_{Z_n}(U_{n+1} - U_n) \leq -c_4 |Z_n|^{\beta\nu-1}.$$

These bounds, the «monotonicity» property of $\psi_{\alpha_1, \alpha_2}$, ψ_{β_1, β_2} and the choice of ν, β_1, β_2 imply that there exist positive constants c_6 and $A_1 \geq a_0$ such that for any $A \geq A_1$ and for any $n \geq 0$,

$$\mathbf{E}_{Z_n}(U_{n+1} - U_n) \leq -c_6 g(X_n) \quad \text{on } \{\tau_A > n\} \quad (45)$$

with the function $g(x)$ given by $g(x) = x^{\alpha-2} \log^{-\mu-1} x$.

Besides, the choice of β_1, β_2, ν and the «monotonicity» property of $\psi_{\alpha_1, \alpha_2}$ ensures that for any z ,

$$\psi_{\beta_1, \beta_2}^\nu(z) \leq (c(\alpha))^{\beta_\nu} \psi_{\alpha_1, \alpha_2}^{\beta_\nu/\alpha}(z).$$

Let us define $h(x)$ by

$$h(x) = x^\alpha \log^{-\mu}(x^\alpha) + (c(\alpha))^{\beta_\nu} x^{\beta_\nu}. \quad (46)$$

We split further the proof.

a) Let $\alpha > 2$. We fix any positive $q > \alpha/2$ and $\mu < q - \alpha/2$. Set

$$f(x) = x^{\alpha/2} \log^{-q} x. \quad (47)$$

It will be now shown that there exists a positive constant A_0 such that the above defined functions f, g, h and the processes $\{X_n, n \geq 0\}, \{U_n, n \geq 0\}$ with such q, μ satisfy the conditions of Theorem 2'. As we have seen above this would give us the desired result on the integrability of $f(\tilde{T}_A)$ for all large enough A .

To start with, f satisfies the conditions of Theorem 2' (recall $\alpha > 2$). Next, since the jumps of the Markov chain $\{Z_n, n \geq 0\}$ are bounded from below, then it is easy to see that on $\{X_0 > A\}$ $X_{\tau_A} \rightarrow \infty$ and $U_{\tau_A} \rightarrow \infty$ as $A \rightarrow \infty$. By definition, h is positive, increasing in a neighborhood of ∞ and for all $n \geq 0$, $U_n \leq h(X_n)$. Finally, (45) shows that for all $A_0 \geq A_1$ (14) is satisfied. The last condition of Theorem 2' that should be verified is (15). Obviously, it suffices to show that in a neighborhood of ∞ , $g(x)/(f' \circ f^{-1} \circ (ch)(x)) > 1$. Since $f'(x) < (\alpha/(2x))f(x)$, then it suffices in turn to prove that in a neighborhood of ∞ , $g(x) > c\alpha h(x)/(2f^{-1} \circ (ch)(x))$. Recall that f is increasing in a neighborhood of ∞ , $\lim_{x \rightarrow \infty} f(x) = \infty$ and $\lim_{x \rightarrow \infty} h(x)/g(x) = \infty$. Therefore, it suffices to check that in a neighborhood of ∞ ,

$$ch(x) > f\left(\frac{c\alpha h(x)}{2g(x)}\right).$$

But this is an easy consequence of the definitions of f, g, h and the choice of μ . In fact, there exist positive constants c', c'' such that in a neighborhood of ∞ ,

$$ch(x) > c' x^\alpha \log^{-q+\alpha/2} x > f(c'' x^2 \log x) > f\left(\frac{c\alpha h(x)}{2g(x)}\right)$$

(here in the first inequality we use the choice of μ , $\mu < q - \alpha/2$). This concludes the proof of Theorem 5 in the case $\alpha > 2$.

b) Let now $\alpha \leq 2$. We fix any positive $q > 1$ and $\mu < q - 1$. Let again $f(x)$ be defined by (47) with just defined q, μ . Set

$$h(x) = x^\alpha \log^{-\mu}(x^\alpha). \quad (48)$$

We will verify that for all sufficiently large A_0 the conditions of Theorem 3' are satisfied with the above defined functions f, g, h and processes $\{X_n, n \geq 0\}$, $\{U_n, n \geq 0\}$. Plainly, the condition $\alpha \leq 2$ implies that there exists a positive B such that on $[B, \infty)$ f is concave, g is decreasing and h is increasing. Let us fix such B . Notice that $g(x) \rightarrow 0$ and $f'(x) \rightarrow 0$ as $x \rightarrow \infty$. Let $A_2 = A_1 \vee B$ (A_1 is a positive number appearing in (45)) and let A_0 be any number $\geq A_2$. The conditions of Theorem 3' will now be verified. The uniform boundedness of the family $\{\mathbf{E}(h(X_{n \wedge \tau_{A_0}})), n \geq 0\}$ is immediate since for any $n \geq 0$,

$$\mathbf{E}(h(X_{n \wedge \tau_{A_0}})) \leq \mathbf{E}(U_{n \wedge \tau_{A_0}}) \leq U_0.$$

The second condition (24) is non-trivial. We have to show that

$$\int^{+\infty} \frac{f'(x) dx}{h \circ g^{-1} \circ f'(x)} \text{ converges.} \quad (49)$$

To check this we define the function $u(x)$ for all large enough x by

$$u(x) = g^{-1} \circ f'(x). \quad (50)$$

This definition is not ambiguous since the functions g, f' are decreasing in a neighborhood of ∞ . To prove (49) we need to investigate the rate of the asymptotic growth of the function u as $x \rightarrow \infty$ and separate two subcases.

b1) Suppose $\alpha < 2$. By the definition of u ,

$$(u(x))^{-2+\alpha} \log^{-1-\mu}(u(x)) = x^{\alpha/2-1} \left(\frac{\alpha}{2} \log^{-q} x - \frac{qx}{x+e} \log^{-q-1} x \right).$$

Therefore, for all large enough x

$$\begin{aligned} & (2-\alpha) \log(u(x)) + (1+\mu) \log_2(u(x)+e) \\ &= \left(1 - \frac{\alpha}{2}\right) \log x + q \log_2(x+e) - \log \left\{ \frac{\alpha}{2} - \frac{qx}{(x+e)\log x} \right\}. \end{aligned}$$

This immediately shows that $u(x) \rightarrow \infty$ as $x \rightarrow \infty$ (recall that $\alpha < 2$). Moreover,

$$\lim_{x \rightarrow \infty} \frac{\log x}{2 \log u(x)} = 1.$$

Therefore, there exist positive constants c_7, c_8 such that as $x \rightarrow \infty$,

$$\begin{aligned} u(x)^{2-\alpha} x^{-1+\alpha/2} &= \log^q(x) \left(\frac{\alpha}{2} - \frac{qx}{(x+e)\log x} \right)^{-1} \log^{-1-\mu}(u(x)) \\ &\asymp c_7 \log^{q-1-\mu}(x) \end{aligned}$$

and

$$u(x) \asymp c_8 \sqrt{x} \log^{(q-1-\mu)/(2-\alpha)}(x)$$

(here as elsewhere the phrase « $f(x) \asymp g(x)$, as $x \rightarrow \infty$ » for two functions $f(x), g(x)$ defined on $\mathbf{R}$, means that $\lim_{x \rightarrow \infty} f(x)/g(x) = 1$).

Hence, there exist positive constants c_9, c_{10} as $x \rightarrow \infty$,

$$\begin{aligned} \frac{f'(x)}{h \circ g^{-1} \circ f'(x)} &\asymp c_9 x^{\alpha/2-1} (u(x))^{-\alpha} \log^{-q} x \log^{\mu} (u(x)) \\ &\asymp c_{10} x^{-1} \log^{(\alpha-2(q-\mu))/(2-\alpha)} x. \end{aligned}$$

This and the choice of μ , $q - \mu > 1$ give (49).

b2) Let now $\alpha = 2$. It suffices to verify that:

$$\int^{+\infty} \frac{e^x f'(e^x) dx}{h \circ g^{-1} \circ f'(e^x)} \text{ converges.} \quad (51)$$

In this case the function $u(x)$ can be explicitated as follows:

$$u(x) = \exp \left\{ \left(1 - \frac{qx}{(x+e)\log x} \right)^{-(1+\mu)^{-1}} \log^{q(1+\mu)^{-1}} x \right\} - e.$$

It then follows that there exists a positive constant c_{11} such that as $x \rightarrow \infty$,

$$\begin{aligned} \frac{e^x f'(e^x)}{h \circ g^{-1} \circ f'(e^x)} &\asymp e^x (u(e^x))^{-2} x^{-q} \log^{\mu} (u(e^x)) \\ &\asymp c_{11} x^{-q/(1+\mu)} \exp \{ x - 2x^{q(1+\mu)^{-1}} \}. \end{aligned}$$

Since $q > 1 + \mu$, then the integral in (51) converges.

II. Let now $\alpha = 0$ and $q > 1$. We are going again to use Theorem 3' but in this case we introduce another process $\{U_n, n \geq 0\}$ and new functions f, g, h . Let ν and β_i , $i = 1, 2$, be some fixed real numbers such that $\beta_i \in (-\pi/2, \pi/2)$ and $\beta_i > \alpha_i$, for $i = 1, 2$. Let us also fix any positive ν, μ such that $\nu > \mu$. We introduce the processes $\{U_n, n \geq 0\}$ by:

$$U_n = \psi_{\alpha_1, \alpha_2}(Z_n) \log_2^{-\mu} \psi_{\alpha_1, \alpha_2}(Z_n) + \log_3^{-\nu} \psi_{\beta_1, \beta_2}(Z_n), \quad \forall n \geq 0.$$

Applying as above Lemma 2 two times one immediately obtains that there exists a positive constant A_1 such that for all $A \geq A_1$ and for all $n \geq 0$,

$$\mathbf{E}_{Z_n}(U_{n+1} - U_n) \leq -g(X_n) \quad \text{on } \{\tau_A > n\},$$

where the function $g(x)$ is defined by

$$g(x) = c_{12} x^{-2} \log^{-1} x \log_2^{-1} x \log_3^{-\mu-1} x,$$

with some constant c_{12} that does not depend on A . Next, let us define the function $h(x)$ by:

$$h(x) = \begin{cases} \log x \log_2^{-\mu}(\log x), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Finally, let us introduce f by:

$$f(x) = \log x \log_2^{-q} x.$$

Obviously, the functions f, g, h are monotone in a neighborhood of ∞ . Again the only difficult condition of Theorem 3' to be verified is (49). To check it, we follow the ideas of the proof in the case $\alpha \in (0, 2]$. Namely, let $u(x)$ be the function defined in (50). Then,

$$\begin{aligned} & c_{12}(u(x))^{-2} \log^{-1}(u(x)) \log_2^{-1}(u(x)) \log_3^{-\mu-1}(u(x)) \\ &= \frac{1}{x+e} \left(\log_2^{-q} x - \frac{q(x+e)}{x+e^e} \frac{\log_2^{-q-1} x}{\log(x+e^e)} \right). \end{aligned}$$

This implies that as $x \rightarrow \infty$,

$$2 \log u(x) \asymp \log x, \quad \text{and} \quad \log_k u(x) \asymp \log_k x, \quad \text{for any } k \geq 2.$$

Therefore there exist positive constants c_{13}, c_{14} such that as $x \rightarrow \infty$,

$$\begin{aligned} \frac{f'(x)}{h \circ g^{-1} \circ f'(x)} &\asymp c_{13} x^{-1} \log^{-1}(u(x)) \log_2^{-q} x \log_3^{\mu}(u(x)) \\ &\asymp c_{14} x^{-1} \log^{-1} x \log_2^{-q} x \log_3^{\mu} x. \end{aligned}$$

Since $q > 1$, the last asymptotic equalities imply that the integral

$$\int^{+\infty} \frac{f'(x) dx}{h \circ g^{-1} \circ f'(x)} \quad \text{converges.}$$

Theorem 5 is proved completely.

Proof of Remark 6. Let k be any fixed integer satisfying the conditions of Remark 6. The proof being almost identical to the proof of Theorem 5, we only give the expressions for $\{U_n, n \geq 0\}$, f, g, h satisfying the conditions of Theorem 3'. Namely, for each $n \geq 0$, let us set:

$$U_n = \begin{cases} \psi_{\alpha_1, \alpha_2}(Z_n) \log_k^{-\mu} \psi_{\alpha_1, \alpha_2}(Z_n) + \psi_{\beta_1, \beta_2}^{\nu}(Z_n), & \text{if } \alpha \in (0, 2], \\ \psi_{\alpha_1, \alpha_2}(Z_n) \log_k^{-\mu} \psi_{\alpha_1, \alpha_2}(Z_n) + \log_{k+1}^{-\nu} \psi_{\beta_1, \beta_2}(Z_n), & \text{if } \alpha = 0, \end{cases}$$

with the constants $\mu, \nu, \beta_1, \beta_2$ chosen in the same way as in the corresponding parts of the proof of Theorem 5. The appropriate functions f, g, h will be then given by:

$$\begin{aligned} f(x) &= \begin{cases} x^{\alpha/2} \log^{-1} x \cdots \log_{k-1}^{-1} x \log_k^{-q} x, & \text{if } \alpha \in (0, 2], \\ \log x \log_2^{-1} x \cdots \log_{k-1}^{-1} x \log_k^{-q} x & \text{if } \alpha = 0; \end{cases} \\ g(x) &= \begin{cases} c x^{\alpha-2} \log^{-1} x \cdots \log_{k-1}^{-1} x \log_k^{-\mu-1} x, & \text{if } \alpha \in (0, 2], \\ c x^{-2} \log^{-1} x \cdots \log_k^{-1} x \log_{k+1}^{-\mu-1} x, & \text{if } \alpha = 0; \end{cases} \\ h(x) &= \begin{cases} x^{\alpha} \log_k^{-\mu}(x^{\alpha}), & \text{if } \alpha \in (0, 2], \\ \log x \log_k^{-\mu}(\log x), & \text{if } \alpha = 0, \end{cases} \end{aligned}$$

where c is some positive constant. Moreover, if we introduce the function $\bar{h}(x)$ by

$$\bar{h}(x) = \begin{cases} x^\alpha \log_k^{-\mu}(x^\alpha) + (c(\alpha))^{\beta_\nu} x^{\beta_\nu}, & \text{if } \alpha \in (0, 2], \\ \log x \log_k^{-\mu}(\log x) + 1, & \text{if } \alpha = 0, \end{cases}$$

then applying the statement of Remark 2 we can obtain bounds for $E(f(\tau_A))$ and $E(f(\tilde{T}_A))$ replacing h by $\bar{h}$ in (42) and (44). The last fact will be very useful in the proof of Theorem 6. Remark 6 is proved.

Proof of Theorem 6. It is immediate that the statements concerning non-integrability of functions of the first return-time follow from Theorem 4 and the first assertion of Theorem 1. To prove the integrability results of the theorem we are going to use other assertions of Theorem 1 and the bounds obtained in the proof of Theorem 5. Namely, we now show that there exists a finite set F such that the conditions of Theorem 1 hold for f defined in (47) and F . In fact, suppose, for instance, that $\alpha > 2$. From the proof of Theorem 5 (see (44)) we know that there exist positive constants $c, c_1, \tilde{A}_0$ such that for any $\tilde{A} \geq \tilde{A}_0$ and $|\tilde{z}| > \tilde{A}$,

$$E_{\tilde{z}}(f(\tilde{T}_{\tilde{A}})) \leq c \bar{h}(\psi_{\alpha_1, \alpha_2}^{1/\alpha}(\Phi(\tilde{z}))) + c_1, \quad (52)$$

where the function $\bar{h}(x)$ was defined in (46). Let $\tilde{A}$ be a positive number greater than $\max(\tilde{A}_0, |\tilde{z}_0|)$ which will be chosen later. Set

$$F = \{\tilde{z} \in \mathbf{Z}_+^2; |\tilde{z}| \leq \tilde{A}\}. \quad (53)$$

Since $\alpha < \gamma$, then $E_{\tilde{z}}(|\tilde{Z}_1|^\alpha)$ is finite for any $\tilde{z}$ belonging to F . This, the «monotonicity» property of function $\psi_{\alpha_1, \alpha_2}$ and the choice of ν, β_1, β_2 imply in turn that $E_{\tilde{z}}(h(\psi_{\alpha_1, \alpha_2}^{1/\alpha}(\Phi(\tilde{Z}_1))))$ is also finite for any $\tilde{z}$ belonging to F . Applying now Theorem 1 we get that for any $\tilde{z} \in F$,

$$E_{\tilde{z}}(f(\rho)) < \infty,$$

and (4) holds. We now verify the condition (6) for any initial distribution λ satisfying (1). Since $\alpha > 2$, it suffices to verify that for some $B > 0$,

$$E_\lambda(f'(\tilde{T}_{\tilde{A}}) 1_{(|\tilde{z}_0| > B)}) < \infty. \quad (54)$$

Notice that $f' \circ f^{-1}$ is concave, $f' \circ f^{-1}$ and $\bar{h}$ are increasing in some neighborhood of ∞ . Easy calculations then show that the last observation, (52), the Jensen inequality and the «monotonicity» property of $\psi_{\alpha_1, \alpha_2}$ ensure the existence of positive constants C and $\tilde{A}_1 > \max(\tilde{A}_0, \tilde{z}_0)$ such that for all $|\tilde{z}| > \tilde{A}_1$,

$$\begin{aligned} E_{\tilde{z}}(f'(\tilde{T}_{\tilde{A}_1})) &\leq f' \circ f^{-1} \left\{ c \bar{h}(\psi_{\alpha_1, \alpha_2}^{1/\alpha}(\Phi(\tilde{z}))) + c_1 \right\} \\ &\leq C |\tilde{z}|^{\alpha-2} \log^{-(2q)/\alpha}(|\tilde{z}|). \end{aligned} \quad (55)$$

Therefore, for any $B > \tilde{A}_1$ there exist positive constants C_1, C_2 such that

$$\begin{aligned} E_{\lambda}(f'(\tilde{T}_{\tilde{A}}) 1_{(|\tilde{Z}_0| > B)}) &\leq C \int_{|\tilde{z}| > B} \lambda(d\tilde{z}) |\tilde{z}|^{\alpha-2} \log^{-(2q)/\alpha}(|\tilde{z}|) \\ &\leq C_1 + C_2 \int_{\mathbf{R}_+ \times \mathbf{R}_+} \lambda(d\tilde{z}) |\tilde{z}|^{\alpha-2} \log^{-1}|\tilde{z}| < \infty, \end{aligned}$$

as was to be shown. The proof in the case $\alpha \leq 2$ is almost the same except that we have to use the functions f, g and $\bar{h}$ instead of h defined in the proof of Remark 6 to arrive at (52).

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Поступила в редакцию
17.II.1997