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MAXIMAL ϕ -INEQUALITIES FOR
NONNEGATIVE SUBMARTINGALES

Пусть $(M_n)_{n \geq 0}$ — неотрицательный субмартигал и $M_n^* \stackrel{\text{def}}{=} \max_{0 \leq k \leq n} M_k$, $n \geq 0$, — ассоциированная последовательность максимумов. Для неубывающих выпуклых функций $\phi: [0, \infty) \rightarrow [0, \infty)$ с $\phi(0) = 0$ (функций Орлича) доказываются различные оценки $\mathbf{E}\phi(M_n^*)$ через $\mathbf{E}\Phi_a(M_n)$, где (для $a \geq 0$)

$$\Phi_a(x) \stackrel{\text{def}}{=} \int_a^x \int_a^s \frac{\phi'(r)}{r} dr ds, \quad x > 0.$$

Особый интерес представляет случай $\phi(x) = x$, для которого вариационные соображения приводят к неравенству

$$\mathbf{E}M_n^* \leq \left(1 + \left(\mathbf{E} \int_1^{M_n \vee 1} \ln x dx \right)^{1/2} \right)^2.$$

Показано, что полученная оценка лучше классической оценки Дуба $e(e-1)^{-1}(1 + \mathbf{E}M_n \ln^+ M_n)$, как только $\mathbf{E}(M_n - 1)^+ \geq e - 2 \approx 0.718$.

Ключевые слова и фразы: неотрицательный субмартигал; последовательность максимумов; функции Орлича; функции Янга; представление Шоке; неравенства, содержащие выпуклую функцию.

1. Introduction. Let $(M_n)_{n \geq 0}$ be a nonnegative submartingale and $M_n^* \stackrel{\text{def}}{=} \max_{0 \leq k \leq n} M_k$, $n \geq 0$, the associated maxima. Moment inequalities for the sequence $(M_n^*)_{n \geq 0}$ in terms of $(M_n)_{n \geq 0}$ are usually based on Doob's maximal inequality which states that

$$\mathbf{P}\{M_n^* > t\} \leq \frac{1}{t} \int_{\{M_n^* > t\}} M_n d\mathbf{P} \quad (1.1)$$

for $n \geq 0$ and $t > 0$. If $p > 1$, a combination of (1.1) with Hölder's inequality shows

$$\mathbf{E}M_n^{*p} \leq \left(\frac{p}{p-1} \right)^p \mathbf{E}M_n^p \quad (1.2)$$

for $n \geq 0$ [4, p. 255, 256], the constant being sharp (see [3, p. 14]). In case $p = 1$ one finds with (1.1) that

$$\mathbf{E}M_n^* \leq \frac{e}{e-1} (1 + \mathbf{E}M_n \ln^+ M_n) \quad (1.3)$$

for $n \geq 0$. Clearly, these results apply to $(|M_n|)_{n \geq 0}$ if $(M_n)_{n \geq 0}$ is a martingale.

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Orlicz and Young functions. Inequalities (1.2) and (1.3) are only special cases within a whole class of convex function inequalities based on Doob's inequality which is the main topic of this paper. Let $\mathfrak{C}$ denote the class of *Orlicz functions*, that is unbounded, nondecreasing convex functions $\phi: [0, \infty) \rightarrow [0, \infty)$ with $\phi(0) = 0$. If the right derivative ϕ' is also unbounded, then ϕ is called a *Young function* and we denote by $\mathfrak{C}'$ the subclass of such functions. Given any probability space $(\Omega, \mathfrak{A}, \mathbf{P})$, each $\phi \in \mathfrak{C}$ induces the semi-Banach space $(\mathfrak{L}^\phi(\mathbf{P}), \|\cdot\|_\phi)$ of ϕ -integrable random variables X on $(\Omega, \mathfrak{A}, \mathbf{P})$, where

$$\|X\|_\phi \stackrel{\text{def}}{=} \inf\{\lambda > 0: \mathbf{E}\Phi(\lambda^{-1}|X|) \leq 1\}$$

defines the underlying semi-norm. It is called an *Orlicz space* and equals the space of α -times integrable functions $(\mathfrak{L}^\alpha(\mathbf{P}), \|\cdot\|_\alpha)$ in case $\phi(x) = x^\alpha$ for some $\alpha \in [1, \infty)$.

Since

$$\phi(x) = \int_0^x \phi'(s) ds \leq x\phi'(x)$$

by convexity, the numbers

$$p = p_\phi \stackrel{\text{def}}{=} \inf_{x>0} \frac{x\phi'(x)}{\phi(x)} \quad \text{and} \quad p^* = p_\phi^* \stackrel{\text{def}}{=} \sup_{x>0} \frac{x\phi'(x)}{\phi(x)} \quad (1.4)$$

are both in $[1, \infty]$. The function ϕ is called *moderate* [5, p. 162] if $p^* < \infty$ or, equivalently [7, Theorem 3.1.1], if for some (and then all) $\lambda > 1$ there exists a finite constant c_λ such that

$$\phi(\lambda x) \leq c_\lambda \phi(x), \quad x \geq 0. \quad (1.5)$$

This property is shared by all $\phi \in \mathfrak{C}$ which are also regularly varying at infinity at order $\alpha \geq 1$ [2], thus including $\phi(x) = x^\alpha$ for $\alpha \in [1, \infty)$. Examples of nonmoderate Orlicz functions are $\phi(x) = \exp(x^\alpha) - 1$ for any $\alpha \geq 1$.

Given a Young function ϕ , the right continuous inverse $\psi'(x) \stackrel{\text{def}}{=} \inf\{y: \phi'(y) > x\}$ of ϕ' is also unbounded and thus $\psi(x) \stackrel{\text{def}}{=} \int_0^x \psi'(s) ds$ is again an element of $\mathfrak{C}'$, called the *conjugate Young function* to ϕ . Obviously, this conjugation is reflexive. A simple geometric argument shows [5, p. 163] that

$$x\phi'(x) = \phi(x) + \int_0^{\phi'(x)} \psi'(s) ds = \phi(x) + \psi(\phi'(x)), \quad x \geq 0, \quad (1.6)$$

and, by reflexivity, the same identity holds true with the roles of ϕ and ψ interchanged. With the help of (1.6), and the inequalities $\psi(x) \geq (1/p_\psi^*)x\psi'(x)$ and $\psi'(\phi'(x)) \geq x$ we infer as in [5, p. 169] that

$$x\phi'(x) = \phi(x) + \psi(\phi'(x)) \geq \phi(x) + \frac{1}{p_\psi^*} \phi'(x) \psi'(\phi'(x)) \geq \phi(x) + \frac{1}{p_\psi^*} x\phi'(x)$$

for $x \geq 0$ and thus

$$p_\phi = \inf_{x>0} \frac{x\phi'(x)}{\phi(x)} \geq \frac{p_\psi^*}{p_\psi^* - 1}.$$

The reverse inequality can also be shown [7, Theorem 3.1.1] so that we have the identity

$$p_\phi = \frac{p_\psi^*}{p_\psi^* - 1}, \quad \text{or equivalently} \quad \frac{1}{p_\phi} + \frac{1}{p_\psi^*} = 1. \quad (1.7)$$

As further results stated in [7, Theorem 3.1.1] we mention that for any $\phi \in \mathfrak{C}$ with $p = p_\phi$ the assertions

$$\phi(\lambda x) \geq \lambda^p \phi(x) \quad \text{for all } \lambda > 1 \quad \text{and} \quad x > 0, \quad (1.8)$$

$$\frac{\phi(x)}{x^p} \nearrow \quad (1.9)$$

hold true, and that for moderate ϕ with $p^* = p_\phi^*$ furthermore

$$\phi(\lambda x) \leq \lambda^{p^*} \phi(x) \quad \text{for all } \lambda > 1 \text{ and } x > 0; \quad (1.10)$$

$$\frac{\phi(x)}{x^{p^*}} \searrow. \quad (1.11)$$

The following two inequalities, the first of which may also be found in [5, p. 169], are easily deduced from another inequality stated as (2.12) in the next section. This latter inequality emerges as a special case of one of our results, see Corollary 2.1, but can also be derived by different arguments based on Doob's inequality and the Choquet representation of a function in $\mathfrak{C}$. For the interested reader this is briefly demonstrated in Appendix 1.

Proposition 1.1. *Let ϕ be an Orlicz function with $p = p_\phi > 1$, and let $(M_n)_{n \geq 0}$ be a nonnegative submartingale. Then*

$$\|M_n^*\|_\phi \leq \frac{p}{p-1} \|M_n\|_\phi \quad (1.12)$$

for each $n \geq 0$. If ϕ is also moderate, i.e., $p^* = p_\phi^* < \infty$, then furthermore

$$\mathbf{E}\phi(M_n^*) \leq \left(\frac{p}{p-1}\right)^{p^*} \mathbf{E}\phi(M_n) \quad (1.13)$$

for each $n \geq 0$.

2. Main results. Inequalities of type (1.13) are also the content of our main results to be presented in this section. They are based upon integration of a variational variant of Doob's inequality (1.1) to be proved in Proposition 3.1, namely

$$\mathbf{P}\{M_n^* > t\} \leq \frac{\lambda}{(1-\lambda)t} \int_t^\infty \mathbf{P}\left\{\frac{M_n}{\lambda} > s\right\} ds = \frac{\lambda}{(1-\lambda)t} \mathbf{E}\left(\frac{M_n}{\lambda} - t\right)^+ \quad (2.1)$$

for all $n \geq 0$, $t > 0$, and $\lambda \in (0, 1)$. Under additional constraints on ϕ we will see that by good choices of λ the constant in (1.13) can be improved considerably. We will also derive an inequality for $\mathbf{E}M_n^*$ in terms of $\mathbf{E}M_n \ln^+ M_n$ which in many situations strictly beats (1.3).

The following two subclasses of $\mathfrak{C}$ will be of interest hereafter. We shall denote by $\mathfrak{C}^*$, the set of all differentiable $\phi \in \mathfrak{C}$ whose derivative is concave or convex, and by $\mathfrak{C}_0$ the set of $\phi \in \mathfrak{C}$ such that $\phi'(x)/x$ is integrable at 0 and thus in particular $\phi'(0) = 0$. Put $\mathfrak{C}_0^* \stackrel{\text{def}}{=} \mathfrak{C}_0 \cap \mathfrak{C}^*$.

Given $\phi \in \mathfrak{C}$ and $a \geq 0$, define

$$\Phi_a(x) \stackrel{\text{def}}{=} \int_a^x \int_a^s \frac{\phi'(r)}{r} dr ds, \quad x > 0, \quad (2.2)$$

and note that $\Phi_a \mathbf{1}_{[a, \infty)} \in \mathfrak{C}$ for $a > 0$. If $\phi \in \mathfrak{C}_0$ the same holds true for $\Phi \stackrel{\text{def}}{=} \Phi_0$, whereas $\Phi \equiv \infty$ otherwise. The function Φ will be of great importance in our subsequent analysis. If $\phi \in \mathfrak{C}_0$, then Φ is obviously again an element from this class. If in addition ϕ' is concave or convex the same holds true for $\Phi'(x) = \int_0^x \phi'(r)/r dr$, hence $\phi \in \mathfrak{C}_0^*$ implies $\Phi \in \mathfrak{C}_0^*$. Use $\Phi''(x) = \phi'(x)/x$ to see that ϕ and Φ are related through the differential equation

$$x\Phi'(x) - \Phi(x) = \phi(x), \quad x \geq 0, \quad (2.3)$$

under the initial conditions $\phi(0) = \phi'(0) = \Phi(0) = \Phi'(0) = 0$. If $\phi(x) = x^p$ for some $p > 1$, then $\Phi(x) = x^p/(p-1)$, in particular, $\Phi = \phi$ in case $\phi(x) = x^2$. If $\phi(x) = x$, then $\Phi \equiv \infty$, but we have $\Phi_1(x) = x \ln x - x + 1$. Further properties of Φ and its relation to ϕ are collected in Lemma 3.1 at the beginning of Section 3, where it will be seen particularly that Φ and ϕ grow at the same order of magnitude unless ϕ or its conjugate are nonmoderate.

Theorem 2.1. *Let $(M_n)_{n \geq 0}$ be a nonnegative submartingale and $\phi \in \mathfrak{C}$. Then*

$$\mathbf{E}\phi(M_n^*) \leq \phi(a) + \frac{\lambda}{1-\lambda} \mathbf{E}\Phi_a\left(\frac{M_n}{\lambda}\right) \quad (2.4)$$

for all $a \geq 0$, $\lambda \in (0, 1)$, and $n \geq 0$, in particular $(\lambda = \frac{1}{2})$

$$\mathbf{E}\phi(M_n^*) \leq \phi(a) + \mathbf{E}\Phi_a(2M_n) \quad (2.5)$$

for all $a > 0$ and $n \geq 0$. If $(M_n)_{n \geq 0}$ is a positive martingale with $M_{n+1} \leq cM_n$ for some $c > 0$ and all $n \geq 0$, and if $\mathbf{E}\Phi_a(M_0) < \infty$, then

$$\mathbf{E}\phi(M_n^*) \geq c^{-1}(\mathbf{E}\Phi_a(M_n) - \mathbf{E}\Phi_a(M_0)) \quad (2.6)$$

for all $n \geq 0$.

Of course, inequality (2.43) with $a = 0$ is of interest only when $\Phi_0 < \infty$, thus for $\phi \in \mathfrak{C}_0$. The conditions on $(M_n)_{n \geq 0}$ implying (2.6) were given by Gundy [6] to demonstrate that the bound in (1.3) cannot be much improved, see (2.17) below and also [8, p. 71, 72].

If $\phi(x) = x^p$ for some $p > 1$, then $\Phi(x) = x^p/(p-1)$ implies in (2.4) with $a = 0$

$$\mathbf{E}M_n^{*p} \leq \frac{\lambda^{1-p}}{(1-\lambda)(p-1)} \mathbf{E}M_n^p \quad (2.7)$$

for all $n \geq 0$ and $\lambda \in (0, 1)$. Elementary calculus shows that

$$\lambda^*(p) \stackrel{\text{def}}{=} \arg \min_{\lambda \in (0,1)} \frac{\lambda^{1-p}}{1-\lambda} = \frac{p-1}{p}.$$

With this value of λ in (2.7) Doob's $\mathcal{L}^p$ -inequality (1.2) comes out again. For an extension of it consider nonnegative increasing functions ϕ on $[0, \infty)$ such that $\phi^{1/\gamma}$ is also convex for some $\gamma > 1$. As before let $(M_n)_{n \geq 0}$ be a nonnegative submartingale. Then the same holds true for $(\phi^{1/\gamma}(M_n))_{n \geq 0}$, the associated sequence of maxima being $(\phi^{1/\gamma}(M_n^*))_{n \geq 0}$. Consequently, (1.2) implies

$$\mathbf{E}\phi(M_n^*) \leq \left(\frac{\gamma}{\gamma-1}\right)^\gamma \mathbf{E}\phi(M_n) \quad (2.8)$$

for each $n \geq 0$. Interesting examples are $\phi(x) = x^p \ln^r(1+x)$ for $p > 1$ and $r \geq 0$ (choose $\gamma = p$), as well as $\phi(x) = e^{rx}$ for $r > 0$. For the latter example any $\gamma > 0$ will do and since $(\gamma/(\gamma-1))^\gamma$ decreases to e as $\gamma \rightarrow \infty$, we obtain

$$\mathbf{E}e^{rM_n^*} \leq e \mathbf{E}e^{rM_n} \quad (2.9)$$

for all $n \geq 0$ and $r > 0$.

We will show in the next section that $\Phi(x) \leq \phi(x)/(p-1)$ for each $\phi \in \mathfrak{C}$ with $p = p_\phi > 1$ ($\Rightarrow \phi \in \mathfrak{C}_0$). With this at hand the subsequent corollary follows from Theorem 2.1.

Corollary 2.1. *Given the situation of Theorem 2.1, let $\phi \in \mathfrak{C}$ be such that $p = p_\phi > 1$. Then*

$$\mathbf{E}\phi(M_n^*) \leq \frac{\lambda}{(1-\lambda)(p-1)} \mathbf{E}\phi\left(\frac{M_n}{\lambda}\right) \quad (2.10)$$

for all $\lambda \in (0, 1)$ and $n \geq 0$. If ϕ is also moderate ($p^* < \infty$), then

$$\mathbf{E}\phi(M_n^*) \leq \frac{p^*-1}{p-1} \left(\frac{p^*}{p^*-1}\right)^{p^*} \mathbf{E}\phi(M_n) \quad (2.11)$$

for each $n \geq 0$.

A comparison of the constants going with $\mathbf{E}\phi(M_n)$ in (2.11) and (1.13) shows that the one in (2.11) is strictly better unless $p = p^*$ (see Lemma A.2 in Appendix 2 for a rigorous proof). For large p, p^* and $\beta \stackrel{\text{def}}{=} p^*/p$ we also have that

$$\frac{p^* - 1}{p - 1} \left(\frac{p^*}{p^* - 1} \right)^{p^*} \approx \beta e,$$

while $(p/(p-1))^{p^*} \approx e^\beta$.

Putting $q = p/(p-1)$ and choosing $\lambda = 1/q$ in (2.10), we obtain

$$\mathbf{E}\phi(M_n^*) \leq \mathbf{E}\phi(qM_n) \quad (2.12)$$

for each $n \geq 0$ and any $\phi \in \mathfrak{C}$ with $p > 1$. It is this inequality which will easily give the assertions of Proposition 1.1 (see Section 3).

By a similar argument as the one leading to (2.8), Doob's inequality (1.2) can be used to get refinements of (1.13) and (2.11) for functions $\phi \in \mathfrak{C}^*$. However, the main tool for this is not (2.1) but rather a suitable Choquet representation of ϕ exploited in a similar manner as in [1] for the special case of $\phi \in \mathfrak{C}^*$ with concave derivative.

Theorem 2.2. *Given the situation of Theorem 2.1, let $\phi \in \mathfrak{C}$, $k \geq 1$ and let $\phi^{(k)}$ be the k -th order derivative of ϕ . Then $\phi^{(k)} \in \mathfrak{C} \Rightarrow \phi \in \mathfrak{C}^*$ implies*

$$\mathbf{E}\phi(M_n^*) \leq \left(\frac{k+1}{k} \right)^{k+1} \mathbf{E}\phi(M_n) \quad (2.13)$$

as well as

$$\|M_n^*\|_\phi \leq \frac{k+1}{k} \|M_n\|_\phi \quad (2.14)$$

for each $n \geq 0$.

In case $\phi(x) = x^p$ for integral $p > 1$ we have $\phi^{(p-1)} \in \mathfrak{C}$ and thus that (2.13) coincides with (1.2).

We finally turn to the case $\phi(x) = x$ for which $\Phi \equiv \infty$ but $\Phi_1(x) = x \ln x - x + 1$ as mentioned earlier. A more general version of inequality (2.4) proved in Section 3 (Proposition 3.1) will be used to derive the following alternative to Doob's $\mathfrak{L}_1$ -inequality (1.3).

Theorem 2.3. *If $(M_n)_{n \geq 0}$ is a nonnegative submartingale, then*

$$\mathbf{E}M_n^* \leq b + \frac{b}{b-1} \mathbf{E} \left(\int_1^{M_n \vee 1} \ln x \, dx \right) \quad (2.15)$$

for all $b > 1$ and $n \geq 1$. The value of b which minimizes the right-hand side equals

$$b^* \stackrel{\text{def}}{=} 1 + \left(\mathbf{E} \left(\int_1^{M_n \vee 1} \ln x \, dx \right) \right)^{1/2}$$

and gives

$$\mathbf{E}M_n^* \leq \left(1 + \left(\mathbf{E} \int_1^{M_n \vee 1} \ln x \, dx \right)^{1/2} \right)^2. \quad (2.16)$$

If $(M_n)_{n \geq 0}$ is a positive martingale with $M_{n+1} \leq cM_n$ for some $c > 0$ and all $n \geq 0$, and if $\mathbf{E}M_0 \ln^+ M_0 < \infty$, then

$$\mathbf{E}M_n^* \geq \frac{1}{c} (\mathbf{E}M_n \ln^+ M_n - \mathbf{E}M_0 \ln^+ M_0) \quad (2.17)$$

for all $n \geq 0$.

Since $\int_1^x \ln y \, dy = x \ln^+ x - (x - 1)$ for $x \geq 1$, inequality (2.15) may be restated as

$$\mathbf{E}M_n^* \leq b + \frac{b}{b-1} (\mathbf{E}M_n \ln^+ M_n - \mathbf{E}(M_n - 1)^+) \quad (2.18)$$

for all $n \geq 1$ and $b > 1$. For the special choices $b = \mathbf{E}(M_n - 1)^+ + 1$ and $b = e$, this yields for each $n \geq 1$

$$\mathbf{E}M_n^* \leq \frac{1 + \mathbf{E}(M_n - 1)^+}{\mathbf{E}(M_n - 1)^+} \mathbf{E}M_n \ln^+ M_n \quad (2.19)$$

and

$$\mathbf{E}M_n^* \leq e + \frac{e}{e-1} (\mathbf{E}M_n \ln^+ M_n - \mathbf{E}(M_n - 1)^+), \quad (2.20)$$

respectively, where the right-hand side of (2.19) is to be interpreted as 1 if $M_n \leq 1$ a.s. Note that even the choice $b = e$, though only suboptimal, leads to a better bound for $\mathbf{E}M_n^*$ than in (1.3) whenever $\mathbf{E}(M_n - 1)^+ \geq e - 2 \approx 0.718$.

The previous upper bounds for $\mathbf{E}M_n^*$ may be further improved if $m \stackrel{\text{def}}{=} \mathbf{E}M_0 > 1$. To see this note that $(\widehat{M}_n)_{n \geq 0} \stackrel{\text{def}}{=} (1, M_0/m, M_1/m, \dots)$ forms again a nonnegative submartingale whose maxima $\widehat{M}_n^*$ satisfy

$$\widehat{M}_n^* = \frac{M_{n-1}^*}{m} \vee 1 \quad \text{for } n \geq 1.$$

Consequently, (2.15) and (2.18) for $(\widehat{M}_n)_{n \geq 0}$ give the following corollary.

Corollary 2.2. *If $(M_n)_{n \geq 0}$ is a nonnegative submartingale with $m = \mathbf{E}M_0 > 1$, then*

$$\begin{aligned} \mathbf{E}M_n^* &\leq \mathbf{E}\widehat{M}_{n+1}^* \leq b + \frac{b}{b-1} \mathbf{E} \int_1^{(M_n/m) \vee 1} \ln x \, dx \\ &\leq b + \frac{b}{b-1} \left(\mathbf{E} \left(\frac{M_n}{m} \ln^+ \frac{M_n}{m} \right) - \mathbf{E} \left(\frac{M_n}{m} - 1 \right)^+ \right) \end{aligned} \quad (2.21)$$

for all $b > 1$ and $n \geq 1$, in particular,

$$\mathbf{E}M_n^* \leq \left(1 + \left(\mathbf{E} \int_1^{(M_n/m) \vee 1} \ln x \, dx \right)^{1/2} \right)^2 \quad (2.22)$$

when choosing the minimizing b^* (compare (2.16)).

3. Proofs. We begin with a collection of some useful properties of the function $\Phi = \Phi_0$ in (2.2) associated with an element ϕ of $\mathcal{C}_0$.

Lemma 3.1. *For each $\phi \in \mathcal{C}_0$ with $p = p_\phi > 1$ the function Φ satisfies*

$$\Phi(x) \leq \frac{1}{p-1} \phi(x), \quad x \geq 0. \quad (3.1)$$

If ϕ is moderate, i.e., $p^* = p_\phi^* < \infty$, then

$$\Phi(x) \geq \frac{1}{p^*-1} \phi(x), \quad x \geq 0. \quad (3.2)$$

Finally, for each $\phi \in \mathcal{C}_0$ the inequality

$$\liminf_{x \rightarrow \infty} \frac{\Phi(x)}{x \ln x} > 0 \quad (3.3)$$

holds true.

The final inequality shows that $\lim_{x \rightarrow \infty} (\Phi(x)/x) = \infty$ whenever $\phi \in \mathfrak{C}_0$ is a function «close to the identity» in the sense that $\phi(x) = o(x \ln x)$ as $x \rightarrow \infty$. Another class of examples comprises $\phi(x) = rx \ln^{r-1}(1+x) - \ln^r(1+x)$ for $r \geq 1$ in which case $\Phi(x) = (1+x) \ln^r(1+x)$ (use the differential equation (2.3)).

P r o o f. Using $x\phi'(x) \geq p\phi(x)$, we obtain by partial integration

$$\int_0^s \frac{\phi'(r)}{r} dr = \frac{\phi(s)}{s} + \int_0^s \frac{\phi(r)}{r^2} dr \leq \frac{\phi(s)}{s} + \frac{1}{p} \int_0^s \frac{\phi'(r)}{r} dr$$

and thus

$$\int_0^s \frac{\phi'(r)}{r} dr \leq \frac{p}{p-1} \frac{\phi(s)}{s} \leq \frac{1}{p-1} \phi'(s) \quad \text{for } s \geq 0.$$

An integration of this inequality with respect to s obviously implies (3.1). The second inequality (3.2) is obtained analogously when utilizing that $p^* < \infty$ implies $x\phi'(x) \leq p^*\phi(x)$ for all $x \geq 0$. As to (3.3), choose $a > 0$ such that $\phi'(a) > 0$. Then

$$\Phi(x) \geq \Phi_a(x) = \int_a^x \int_a^s \frac{\phi'(1)}{r} dr ds = \phi'(a) \left(x \ln \frac{x}{a} - x + a \right)$$

for all $x \geq a$.

The proofs of Theorems 2.1 and 2.3 are based on the following more general proposition.

Proposition 3.1. *Let $(M_n)_{n \geq 0}$ be a nonnegative submartingale and let $\phi \in \mathfrak{C}$. Then inequality (2.1) holds true and furthermore*

$$\mathbf{E}\phi(M_n^*) \leq \phi(b) + \frac{\lambda}{1-\lambda} \int_{\{M_n/\lambda > b\}} \left(\Phi_a\left(\frac{M_n}{\lambda}\right) - \Phi_a(b) - \Phi'_a(b) \left(\frac{M_n}{\lambda} - b \right) \right) d\mathbf{P} \quad (3.4)$$

for all $n \geq 0$, $a, b > 0$, and $\lambda \in (0, 1)$. If $\phi'(x)/x$ is integrable at 0, i.e., $\phi \in \mathfrak{C}_0$, then (3.4) remains valid for $b = 0$.

P r o o f. Doob's maximal inequality gives

$$\begin{aligned} \mathbf{P}\{M_n^* > t\} &\leq \frac{1}{t} \int_{\{M_n^* > t\}} M_n d\mathbf{P} = \frac{1}{t} \int_0^\infty \mathbf{P}\{M_n^* > t, M_n > s\} ds \\ &= \frac{1}{t} \int_0^{\lambda t} \mathbf{P}\{M_n^* > t\} ds + \frac{1}{t} \int_{\lambda t}^\infty \mathbf{P}\{M_n > s\} ds \\ &\leq \lambda \mathbf{P}\{M_n^* > t\} + \frac{\lambda}{t} \int_t^\infty \mathbf{P}\left\{\frac{M_n}{\lambda} > s\right\} ds \end{aligned} \quad (3.5)$$

and thus

$$\mathbf{P}\{M_n^* > t\} \leq \frac{\lambda}{(1-\lambda)t} \int_t^\infty \mathbf{P}\left\{\frac{M_n}{\lambda} > s\right\} ds$$

for all $n \geq 0$, $t > 0$, and $\lambda \in (0, 1)$, which is (2.1). With this result we further infer

$$\begin{aligned} \mathbf{E}\phi(M_n^*) &\leq \phi(b) + \int_b^\infty \phi'(t) \mathbf{P}\{M_n^* > t\} dt \\ &\leq \phi(b) + \frac{\lambda}{1-\lambda} \int_b^\infty \frac{\phi'(t)}{t} \int_t^\infty \mathbf{P}\left\{\frac{M_n}{\lambda} > s\right\} ds dt \\ &= \phi(b) + \frac{\lambda}{1-\lambda} \int_b^\infty \left(\int_b^s \frac{\phi'(t)}{t} dt \right) \mathbf{P}\left\{\frac{M_n}{\lambda} > s\right\} ds \\ &= \phi(b) + \frac{\lambda}{1-\lambda} \int_b^\infty (\Phi'_a(s) - \Phi'_a(b)) \mathbf{P}\left\{\frac{M_n}{\lambda} > s\right\} ds \\ &= \phi(b) + \frac{\lambda}{1-\lambda} \int_{\{M_n/\lambda > b\}} \left(\Phi_a\left(\frac{M_n}{\lambda}\right) - \Phi_a(b) - \Phi'_a(b) \left(\frac{M_n}{\lambda} - b \right) \right) d\mathbf{P} \end{aligned}$$

for all $n \geq 0$, $b, t > 0$, $\lambda \in (0, 1)$, and $a > 0$, where $a = 0$ may also be chosen if $\phi'(x)/x$ is integrable at 0.

P r o o f o f T h e o r e m 2.1. Inequality (2.4) follows directly from Proposition 3.1 if we choose $b = a$ and recall that $\Phi_a(a) = \Phi'_a(a) = 0$. Hence we are left with the proof of (2.6). If $(M_n)_{n \geq 0}$ is a positive martingale satisfying $M_{n+1} \leq cM_n$ for all $n \geq 0$ and some $c > 0$, then

$$\mathbf{P}\{M_n^* > t\} \geq \frac{1}{ct} \int_{\{M_n^* > t\}} M_n d\mathbf{P} - \frac{1}{ct} \int_{\{M_0 > t\}} M_0 d\mathbf{P}$$

for all $n \geq 0$ and $t > 0$, see [8, p. 72]. Consequently,

$$\begin{aligned} \mathbf{P}\{M_n^* > t\} &\geq \frac{1}{ct} \int_{\{M_n > t\}} M_n d\mathbf{P} - \frac{1}{ct} \int_{\{M_0 > t\}} M_0 d\mathbf{P} \\ &= \frac{1}{ct} \int_t^\infty (\mathbf{P}\{M_n > s\} - \mathbf{P}\{M_0 > s\}) ds \\ &\quad + \frac{1}{c} (\mathbf{P}\{M_n > t\} - \mathbf{P}\{M_0 > t\}) \end{aligned} \quad (3.6)$$

for all $n \geq 0$ and $t > 0$. Assuming $\mathbf{E}\Phi_a(M_0) < \infty$, and also $\mathbf{E}\phi(M_n^*) < \infty$ (there is nothing to prove otherwise), (2.6) now follows upon integration over $(0, \infty)$ with respect to t of both sides of (3.6) multiplied with $\phi'(t)$. We must only note that

$$\int_0^\infty \phi'(t) (\mathbf{P}\{M_n > t\} - \mathbf{P}\{M_0 > t\}) dt = \mathbf{E}\phi(M_n) - \mathbf{E}\phi(M_0) \geq 0$$

because ϕ is convex and $\mathbf{E}\phi(M_k) \leq \mathbf{E}\phi(M_n) < \infty$ for $0 \leq k \leq n$.

We continue with a short proof of Proposition 1.1 stated in the Introduction.

P r o o f o f P r o p o s i t i o n 1.1. Let $\phi \in \mathfrak{C}$ with $p = p_\phi > 1$, put $q \stackrel{\text{def}}{=} p(p-1)^{-1}$ and recall from (2.12) that $\mathbf{E}\phi(M_n^*) \leq \mathbf{E}\phi(qM_n)$ for each $n \geq 0$. Setting $\gamma_n \stackrel{\text{def}}{=} \|M_n\|_\phi$, this inequality implies

$$\mathbf{E}\phi\left(\frac{M_n^*}{q\lambda_n}\right) \leq \mathbf{E}\phi\left(\frac{M_n}{\lambda_n}\right) = 1$$

as well as (use also (1.10))

$$\mathbf{E}\phi(M_n^*) \leq \mathbf{E}\phi(qM_n) \leq q^{p^*} \mathbf{E}\phi(M_n).$$

The second inequality proves (1.13) while the first one yields $\|M_n^*\|_\phi \leq q\lambda_n$ and thus assertion (1.12).

P r o o f o f T h e o r e m 2.2. The basic tool for the proof of Theorem 2.2 is to convert the assumption of smooth convexity ($\phi^{(k)} \in \mathfrak{C}$) into a suitable Choquet decomposition of ϕ . Each $\phi \in \mathfrak{C}$ can be written as

$$\phi(x) = \int_{[0, \infty)} (x-t)^+ Q_\phi(dt),$$

where $Q_\phi(dt) = \phi'(0)\delta_0 + \phi'(dt)$. Hence, if $\phi' \in \mathfrak{C}$, then

$$\begin{aligned} \phi(x) &= \int_0^x \phi'(y) dy = \int_0^x \int_{[0, \infty)} (y-t)^+ Q_{\phi'}(dt) dy \\ &= \int_{[0, \infty)} \int_0^x (y-t)^+ dy Q_{\phi'}(dt) = \int_{[0, \infty)} \frac{((x-t)^+)^2}{2} Q_{\phi'}(dt). \end{aligned}$$

An inductive argument now gives that

$$\phi(x) = \int_{[0, \infty)} \frac{((x-t)^+)^{k+1}}{(k+1)!} Q_{\phi^{(k)}}(dt) \quad (3.7)$$

for each $\phi \in \mathfrak{C}$ with $\phi^{(k)} \in \mathfrak{C}$. Put $\varphi_{k,t}(x) \stackrel{\text{def}}{=} ((x-t)^+)^{k+1}/(k+1)!$ for $k \geq 1$ and $t \in [0, \infty)$. Note that $\varphi_{k,t}^{1/(k+1)}$ is convex for each t . Thus we infer with the help of (3.7) and the argument which proved (2.8) that

$$\begin{aligned} \mathbf{E}\phi(M_n^*) &= \int_{[0,\infty)} \mathbf{E}\varphi_{k,t}(M_n^*) Q_{\phi^{(k)}}(dt) \leq \left(\frac{k+1}{k}\right)^{k+1} \int_{[0,\infty)} \mathbf{E}\varphi_{k,t}(M_n) Q_{\phi^{(k)}}(dt) \\ &= \left(\frac{k+1}{k}\right)^{k+1} \mathbf{E}\phi(M_n) \end{aligned}$$

for each $n \geq 0$. Replacing M_n^* with M_n^*/γ_n in the previous estimation, where $\gamma_n \stackrel{\text{def}}{=} (k+1)k^{-1}\|M_n\|_{\phi}$, further gives (2.14) by a similar argument as in the proof of Proposition 1.1.

P r o o f o f T h e o r e m 2.3. If $\phi(x) = x$, then $\Phi_1(x) = x \ln x - x + 1$ and $\Phi'_1(x) = \ln x$ for $x > 0$. Inequality (3.4) with these functions reads

$$\begin{aligned} \mathbf{E}M_n^* &\leq b + \frac{\lambda}{1-\lambda} \int_{\{M_n > \lambda b\}} \left(\frac{M_n}{\lambda} \ln \frac{M_n}{\lambda} - \frac{M_n}{\lambda} + b - (\ln b) \frac{M_n}{\lambda} \right) d\mathbf{P} \\ &= b + \frac{1}{1-\lambda} \int_{\{M_n > \lambda b\}} (M_n \ln M_n - M_n(\ln \lambda + \ln b + 1) + \lambda b) d\mathbf{P} \end{aligned}$$

for all $b > 0$ and $\lambda \in (0, 1)$. Choose $b > 1$ and $\lambda = 1/b$ to obtain (2.15) in its equivalent form (2.18). Inequality (2.16) follows by elementary calculus. Finally, if $(M_n)_{n \geq 0}$ is a positive martingale with $M_{n+1} \leq cM_n$ for some $c > 0$ and all $n \geq 0$, and if $\mathbf{E}M_0 \ln^+ M_0 < \infty$, then a use of the first inequality in (3.6) gives after partial integration

$$\begin{aligned} \mathbf{E}M_n^* &\geq \int_1^\infty \frac{1}{ct} \left(\int_{\{M_n > t\}} M_n d\mathbf{P} - \int_{\{M_0 > t\}} M_0 d\mathbf{P} \right) dt \\ &= \frac{1}{c} \mathbf{E} \left(M_n \int_1^{M_n \vee 1} \frac{1}{t} dt - M_0 \int_1^{M_0 \vee 1} \frac{1}{t} dt \right) = \frac{1}{c} (\mathbf{E}M_n \ln^+ M_n - \mathbf{E}M_0 \ln^+ M_0) \end{aligned}$$

for all $n \geq 1$ and hence the asserted inequality (2.17).

Appendix 1. Inequality (2.12) from which Proposition 1.1 was deduced may be viewed as a specialization of inequality (A.2) below to the pair $(X, Y) = (M_n, M_n^*)$. The purpose of this short appendix is to provide a proof of (A.2) based upon a Choquet representation of the involved convex function ϕ .

Lemma A.1. *Let X, Y be nonnegative random variables satisfying Doob's inequality*

$$t \mathbf{P}\{Y \geq t\} \leq \mathbf{E}1_{\{Y \geq t\}} X \quad (\text{A.1})$$

for all $t \geq 0$. Then

$$\mathbf{E}\phi(Y) \leq \mathbf{E}\phi(qX) \quad (\text{A.2})$$

holds for each Orlicz function ϕ , where $q \stackrel{\text{def}}{=} q_\phi = p_\phi/(p_\phi - 1)$.

P r o o f. If $p = p_\phi = 1$, thus $q = \infty$, there is nothing to prove. So let $p > 1$ and put $V = qX$. Write ϕ in Choquet decomposition, that is,

$$\phi(x) = \int_{[0,\infty)} (x-t)^+ \phi'(dt), \quad x \geq 0.$$

Then we obtain

$$\begin{aligned} \mathbf{E}\phi(V) - \mathbf{E}\phi(Y) &= \mathbf{E} \left(\int_{[0,\infty)} (V-t)^+ - (Y-t)^+ \phi'(dt) \right) \\ &= \mathbf{E} \left(\int_{[0,V]} (V-t) \phi'(dt) - \int_{[0,Y]} (Y-t) \phi'(dt) \right) \end{aligned}$$

$$\begin{aligned}
&= \mathbf{E} \left(\int_{[0,V]} (V-t) \phi'(dt) - \int_{[0,Y]} (V-t) \phi'(dt) \right) \\
&\quad + \mathbf{E} \left(\int_{[0,Y]} (V-t) \phi'(dt) - \int_{[0,Y]} (Y-t) \phi'(dt) \right) \\
&= \mathbf{E} \left(\mathbf{1}_{\{V \geq Y\}} \int_{(Y,V]} (V-t) \phi'(dt) + \mathbf{1}_{\{V < Y\}} \int_{(V,Y]} (t-V) \phi'(dt) \right) \\
&\quad + \mathbf{E} \left(\int_{[0,Y]} (V-Y) \phi'(dt) \right) = I_1 + I_2.
\end{aligned}$$

It is easily seen that $I_1 \geq 0$. So it remains to prove that

$$I_2 = \mathbf{E} V \phi'(V) - \mathbf{E} Y \phi'(Y) \geq 0.$$

To this end write

$$\mathbf{E} Y \phi'(Y) \geq p \mathbf{E} \phi(Y) = p \mathbf{E} \int_{[0,\infty)} (Y-t)^+ \phi'(dt) = p \mathbf{E} Y \phi'(Y) - p \mathbf{E} \int_{\{Y \geq t\}} t \phi'(dt)$$

which, after rearranging terms and using (A.1), leads to

$$\mathbf{E} Y \phi'(Y) \leq q \mathbf{E} \int_{\{Y \geq t\}} t \phi'(dt) \leq q \mathbf{E} \int_{\{Y \geq t\}} X \phi'(dt) = \mathbf{E} V \phi'(Y)$$

and thus the desired conclusion.

Appendix 2. We claimed in Section 2 that

$$\frac{y-1}{x-1} \left(\frac{y}{y-1} \right)^y \leq \left(\frac{x}{x-1} \right)^y \quad (\text{A.3})$$

holds true for all $y \geq x > 1$ and that the inequality is strict unless $x = y$. Inequality (A.3) may be rewritten as

$$\left(\frac{x-1}{y-1} \right)^{y-1} \leq \left(\frac{x}{y} \right)^y.$$

Taking logarithms we arrive at

$$(y-1)(\ln(x-1) - \ln(y-1)) \leq y(\ln x - \ln y).$$

The desired conclusion is now an immediate consequence of the following lemma.

Lemma A.2. For each $y > 1$, the function $f_y: (1, y] \rightarrow \mathbf{R}$,

$$f_y(x) \stackrel{\text{def}}{=} (y-1)(\ln(x-1) - \ln(y-1)) - y(\ln x - \ln y),$$

is strictly increasing with $f_y(y) = 0$.

P r o o f. It suffices to note that

$$f'_y(x) = \frac{y-1}{x-1} - \frac{y}{x} > 0$$

for all $x \in (1, y)$.

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ON ESTIMATION OF A LOCATION PARAMETER IN PRESENCE OF AN ANCILLARY COMPONENT¹⁾

Если (X, Y) есть наблюдение случайного вектора с функцией распределения $F(x - \theta, y)$, $\sigma^2 = \mathbf{D}X$, $\rho = \text{corr}(X, Y)$ и I — информация Фишера о параметре θ в (X, Y) , то $I \geq \{\sigma^2(1 - \rho^2)\}^{-1}$.

Равенство достигается при выполнении условий, тесно связанных с условиями линейности оценки Питмэна для θ по выборке из совокупности $F(x - \theta, y)$. Эти утверждения обобщают результаты, полученные ранее для случая, когда наблюдается только компонента X .

Ключевые слова и фразы: информация Фишера, оценка Питмэна.

1. Introduction. Let $(x_1, \dots, x_n)$ be a sample of size $n \geq 3$ from a population with distribution function $F(x - \theta)$ depending on a location parameter θ . If $\mathbf{D}x_i = \sigma^2 < \infty$, the Pitman estimator of θ with respect to the quadratic loss,

$$\hat{\theta}(x_1, \dots, x_n) = \bar{x} - \mathbf{E}_0(\bar{x} \mid x_1 - \bar{x}, \dots, x_n - \bar{x}),$$

is linear in observations, i.e.,

$$\mathbf{E}_0(\bar{x} \mid x_1 - \bar{x}, \dots, x_n - \bar{x}) = \text{const}, \quad (1)$$

if and only if F is Gaussian. This result known as the KLR-theorem, was proved in [2] and marked the beginning of our fruitful collaboration with Yuri Vladimirovich Linnik who would have turned now 90.

We dedicate this paper to the memory of Linnik, the teacher of one of us and a good friend of the other.

As is well known (see, e.g., [3, Chapter 13]), the Fisher information on θ contained in an observation with distribution function $F(x - \theta)$, satisfies the inequality

$$I \geq \sigma^{-2} \quad (2)$$

with the equality sign again holding only for Gaussian F . In [1, Chapter 4] a definition of the Fisher information on a location parameter is given for an arbitrary F , not necessarily absolutely continuous.

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