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ON ESTIMATION OF A LOCATION PARAMETER IN PRESENCE OF AN ANCILLARY COMPONENT¹⁾

Если (X, Y) есть наблюдение случайного вектора с функцией распределения $F(x - \theta, y)$, $\sigma^2 = \mathbf{D}X$, $\rho = \text{corr}(X, Y)$ и I — информация Фишера о параметре θ в (X, Y) , то $I \geq \{\sigma^2(1 - \rho^2)\}^{-1}$.

Равенство достигается при выполнении условий, тесно связанных с условиями линейности оценки Питмэна для θ по выборке из совокупности $F(x - \theta, y)$. Эти утверждения обобщают результаты, полученные ранее для случая, когда наблюдается только компонента X .

Ключевые слова и фразы: информация Фишера, оценка Питмэна.

1. Introduction. Let $(x_1, \dots, x_n)$ be a sample of size $n \geq 3$ from a population with distribution function $F(x - \theta)$ depending on a location parameter θ . If $\mathbf{D}x_i = \sigma^2 < \infty$, the Pitman estimator of θ with respect to the quadratic loss,

$$\hat{\theta}(x_1, \dots, x_n) = \bar{x} - \mathbf{E}_0(\bar{x} \mid x_1 - \bar{x}, \dots, x_n - \bar{x}),$$

is linear in observations, i.e.,

$$\mathbf{E}_0(\bar{x} \mid x_1 - \bar{x}, \dots, x_n - \bar{x}) = \text{const}, \quad (1)$$

if and only if F is Gaussian. This result known as the KLR-theorem, was proved in [2] and marked the beginning of our fruitful collaboration with Yuri Vladimirovich Linnik who would have turned now 90.

We dedicate this paper to the memory of Linnik, the teacher of one of us and a good friend of the other.

As is well known (see, e.g., [3, Chapter 13]), the Fisher information on θ contained in an observation with distribution function $F(x - \theta)$, satisfies the inequality

$$I \geq \sigma^{-2} \quad (2)$$

with the equality sign again holding only for Gaussian F . In [1, Chapter 4] a definition of the Fisher information on a location parameter is given for an arbitrary F , not necessarily absolutely continuous.

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In this paper, we consider the setup when a bivariate observation (X, Y) has an informative component X and ancillary Y with

$$\mathbf{P}_\theta\{X \leq x, Y \leq y\} = F(x - \theta, y).$$

The setup was first considered in [5].

Conditions for linearity of the Pitman estimator of θ are obtained as well as an inequality extending (2),

$$I \geq \frac{1}{\sigma^2(1 - \rho^2)}, \quad \rho = \text{corr}(X, Y), \quad (3)$$

the conditions for the equality sign in (3) being closely related to those for linearity of the Pitman estimator.

An open problem of interest is formulated.

A lower bound for the Fisher information. Let a bivariate random vector (X, Y) have density $f(x - \theta, y)$ with $\theta \in \mathbf{R}$ as a parameter. Assuming the second moments finite and $f(x, y)$ differentiable with respect to x , set

$$\mathbf{E}X = \mu_1, \quad \mathbf{E}Y = \mu_2, \quad \mathbf{D}X = \sigma_1^2, \quad \mathbf{D}Y = \sigma_2^2, \quad \text{corr}(X, Y) = \rho,$$

$$J = J(x - \theta, y) = \frac{\partial \ln f(x - \theta, y)}{\partial \theta},$$

$$I = \mathbf{E}_\theta J^2 = \iint |J(x - \theta, y)|^2 f(x - \theta, y) dx dy = \iint |J(x, y)|^2 f(x, y) dx dy.$$

Theorem 1. *The following inequality for the Fisher information I on θ contained in (X, Y) holds:*

$$I \geq \frac{1}{\sigma_1^2(1 - \rho^2)} \quad (4)$$

with the equality sign if and only if

$$f(x, y) = \exp\{A_0 x^2 + A_1 x + Bxy + C(y)\}, \quad (5)$$

where $A_0, A_1, B, C(y)$ are such that

$$\mathbf{D}X = \sigma_1^2, \quad \text{corr}(X, Y) = \rho, \quad \mathbf{D}Y < \infty.$$

P r o o f. If $I = +\infty$, (4) trivially holds. Hence, one may assume $I < \infty$.

Let $\hat{J}(x, y) = \hat{c}_1(x - \mu_1) + \hat{c}_2(y - \mu_2)$ be the best linear approximation of $J(x, y)$ in $L^2(f)$, i.e.,

$$\hat{J}(x, y) = \arg \min_{c_1, c_2} \iint [J(x, y) - c_1(x - \mu_1) - c_2(y - \mu_2)]^2 f(x, y) dx dy.$$

From $\mathbf{E}_0\{(X - \mu_1)J(X, Y)\} = -1$, $\mathbf{E}_0\{(Y - \mu_2)J(X, Y)\} = 0$ one gets

$$\hat{c}_1 = -\frac{1}{\sigma_1^2(1 - \rho^2)}, \quad \hat{c}_2 = \frac{\rho}{\sigma_1\sigma_2(1 - \rho^2)}, \quad \hat{J} = -\frac{(x - \mu_1) - \rho\sigma_1(y - \mu_2)/\sigma_2}{\sigma_1^2(1 - \rho^2)}$$

and

$$\mathbf{E}(\hat{J}(X, Y))^2 = \frac{1}{\sigma_1^2(1 - \rho^2)}. \quad (6)$$

Since $I = \mathbf{E}(J(X, Y))^2 \geq \mathbf{E}(\hat{J}(X, Y))^2$, (6) proves (4).

The equality sign in (4) holds if and only if

$$J(x, y) = \hat{c}_1(x - \mu_1) + \hat{c}_2(y - \mu_2).$$

Integrating this relation leads to (5). Theorem 1 is proved.

Note that the lower bound (4) does not depend on the variance of the ancillary component Y .

Bivariate Gaussian densities are a special case of (5) as are product densities of a Gaussian X and arbitrary Y .

Introduce now the characteristic function

$$\varphi(t, s) = \mathbf{E}_0 \exp(itX + isY)$$

of (X, Y) and marginal characteristic functions $\varphi_1(t)$ of X and $\varphi_2(s)$ of Y .

Since

$$\begin{aligned} \iint e^{itx+isy} \frac{\partial \ln f(x, y)}{\partial x} f(x, y) dx dy &= -it\varphi(t, s), \\ \iint x e^{itx+isy} f(x, y) dx dy &= -i \frac{\partial \varphi(t, s)}{\partial t}, \\ \iint y e^{itx+isy} f(x, y) dx dy &= -i \frac{\partial \varphi(t, s)}{\partial s}, \end{aligned}$$

the linearity of $J(x, y)$ in terms of the characteristic function is equivalent to

$$t\varphi(t, s) = \widehat{c}_1 \frac{\partial \varphi(t, s)}{\partial t} + \widehat{c}_2 \frac{\partial \varphi(t, s)}{\partial s} + i(\widehat{c}_1 \mu_1 + \widehat{c}_2 \mu_2) \varphi(t, s). \quad (7)$$

We will need this relation in the next section.

Linearity of the Pitman estimator. Let $(x_1, y_1), \dots, (x_n, y_n)$ be a sample from a bivariate population with distribution function $F(x - \theta, y)$ with $\int x^2 dF(x) < \infty$. The Pitman estimator of θ is

$$\hat{\theta}(x_1, y_1; \dots; x_n, y_n) = \bar{x} - \mathbf{E}_0(\bar{x} \mid x_1 - \bar{x}, \dots, x_n - \bar{x}; y_1, \dots, y_n).$$

Let us look first at the straightforward extension of the KLR-theorem.

Theorem 2. *If $\hat{\theta} = \bar{x} + \text{const}$ and $n \geq 3$, then x_i is Gaussian and*

$$\varphi(t, s) = \varphi_1(t) \varphi_2(s), \quad t \in \mathbf{R}, \quad |s| < \epsilon, \quad (8)$$

where $\epsilon = \max\{s: |\varphi_2(u)| > 0, |u| < s\}$. If $\varphi_2(s) \neq 0$, then x_i, y_i are independent.

P r o o f. To prove Gaussianity of x_i , note that the relation

$$\mathbf{E}_0(\bar{x} \mid x_1 - \bar{x}, \dots, x_n - \bar{x}; y_1, \dots, y_n) = \text{const} = c, \quad \text{say} \quad (9)$$

implies

$$\mathbf{E}_0(\bar{x} \mid x_1 - \bar{x}, \dots, x_n - \bar{x}) = c$$

that as shown in the KLR-theorem holds for $n \geq 3$ if and only if x_i is Gaussian.

The relation (9) is equivalent to

$$\begin{aligned} \mathbf{E}_0(\bar{x} \exp\{i[t_1(x_1 - \bar{x}) + \dots + t_n(x_n - \bar{x}) + s_1 y_1 + \dots + s_n y_n]\}) \\ = c \mathbf{E}_0(\exp\{i[t_1(x_1 - \bar{x}) + \dots + t_n(x_n - \bar{x}) + s_1 y_1 + \dots + s_n y_n]\}) \end{aligned} \quad (10)$$

holding for all real $t_1, \dots, t_n, s_1, \dots, s_n$.

From (10) one gets

$$\sum_{i=1}^n \left(\frac{\partial \varphi(\tau_i, s_i)}{\partial \tau_i} \prod_{j \neq i} \varphi(\tau_j, s_j) \right) = c \prod_{i=1}^n \varphi(\tau_i, s_i) \quad \text{if} \quad \sum_{i=1}^n \tau_i = 0. \quad (11)$$

Let $\eta > 0, \epsilon > 0$ be such that $\varphi(t, s) \neq 0$ for $|t| < \eta, |s| < \epsilon$. On dividing both sides of (11) by $\prod_{i=1}^n \varphi(\tau_i, s_i)$ one gets

$$\sum_{i=1}^n g(\tau_i, s_i) = c \quad \text{if} \quad \sum_{i=1}^n \tau_i = 0, \quad |s| < \epsilon, \quad |\tau_i| < \eta, \quad i = 1, \dots, n, \quad (12)$$

where $g(t, s) = \partial \ln \varphi(t, s) / \partial t$. On fixing $s_1 = \dots = s_n = s$, (12) becomes the Cauchy classical equation

$$\sum_1^n g(\tau_i, s) = c \quad \text{if} \quad \sum_1^n \tau_i = 0, \quad |s| < \epsilon, \quad |\tau_i| < \eta, \quad i = 1, \dots, n. \quad (13)$$

Since $g(\tau, s)$ is continuous for $|\tau| < \eta$, $|s| < \epsilon$, the only solution of (13) for $n \geq 3$ is

$$g(\tau, s) = A(s) \tau + B$$

for some $A(s)$, B . It is easy to see that $A(s) = A = \text{const}$. Indeed, if $a' = A(s') \neq A(s'') = a''$, then by setting $\tau_1 = \dots = \tau_{n-1} = \tau$, $\tau_n = -(n-1)\tau$; $s_1 = \dots = s_{n-1} = s'$, $s_n = s''$ one gets from (12)

$$(n-1)a'\tau - (n-1)a''\tau = \text{const}$$

which is a contradiction.

Hence, $g(t, s) = As + B$ and

$$\varphi(t, s) = \exp \left\{ \frac{At^2}{2} + Bt + C(s) \right\} = \varphi_1(t) \varphi_2(s), \quad |t| < \eta, \quad |s| < \epsilon. \quad (14)$$

Since $\varphi_1(t) \neq 0$, $t \in \mathbf{R}$, (14) holds for $t \in \mathbf{R}$, $|s| < \epsilon$. If one assumes *a priori* that $\varphi_2(s) \neq 0$, then (14) holds for all real t and s so that x_i, y_i are independent. Theorem 2 is proved.

Note in passing that if $\varphi(t, s)$ is the characteristic function of (X, Y) and

$$\varphi(t, s) = \varphi_1(t) \varphi_2(s), \quad |t| < \eta, \quad |s| < \epsilon,$$

X and Y are not necessarily independent. Here is an example due to Larry Shepp. Let $(\varphi_{11}(t), \varphi_{12}(t))$ and $(\varphi_{21}(s), \varphi_{22}(s))$ be two pairs of characteristic functions such that

$$\begin{aligned} \varphi_{11}(t) &= \varphi_{12}(t) = \varphi_1(t), & |t| \leq 1; & \quad \varphi_{11}(t) \neq \varphi_{12}(t), & |t| > 1, \\ \varphi_{21}(s) &= \varphi_{22}(s) = \varphi_2(s), & |s| \leq 1; & \quad \varphi_{21}(s) \neq \varphi_{22}(s), & |s| > 1. \end{aligned}$$

Such pairs exist; one may take Pólya characteristic functions (see, e.g., [4, Chapter 4]).

The function

$$\varphi(t, s) = \frac{1}{2} [\varphi_{11}(t) \varphi_{21}(s) + \varphi_{12}(t) \varphi_{22}(s)],$$

as a convex combination of characteristic functions, is the characteristic function of a random vector (X, Y) . For $|t| \leq 1$, $|s| \leq 1$,

$$\varphi(t, s) = \varphi_1(t) \varphi_2(s)$$

but the factorization does not hold for all t and s so that X, Y are dependent.

Assume from now on that $\int y^2 dF(x) < \infty$.

Theorem 3. *If $\hat{\theta}(x_1, y_1; \dots; x_n, y_n) = \bar{x} - c_0 - C\bar{y}$ and $n \geq 3$, the characteristic function $\varphi(t, s)$ of (x_i, y_i) satisfies relation (7) for $|t| < \eta$, $|s| < \epsilon$ for some $\eta > 0$, $\epsilon > 0$. If $\varphi(t, s) \neq 0$, then it satisfies (7) for all real t, s and if, moreover, the distribution of (x_i, y_i) is given by a differentiable in x density $f(x - \theta, y)$, then $f(x, y)$ is necessarily (5). If $\varphi(t, s)$ satisfies (7) for all real t, s , then the Pitman estimator of θ is linear.*

P r o o f. The first condition of Theorem 3 is equivalent to

$$\mathbf{E}_0(\bar{x} \mid x_1 - \bar{x}, \dots, x_n - \bar{x}; y_1, \dots, y_n) = c_0 + C\bar{y}.$$

To simplify notation, let us assume (without loss of generality) that $\mathbf{E}_0 X = \mathbf{E}_0 Y = 0$ in which case $c_0 = 0$. Proceeding as in the proof of Theorem 2, one gets from

$$\mathbf{E}_0(x_1 - \bar{x}, \dots, x_n - \bar{x}; y_1, \dots, y_n) = C\bar{y}$$

the relation (actually, an analytical equivalent of the previous relation)

$$\sum_{i=1}^n \left(\frac{\partial \varphi(t_i, s_i)}{\partial t_i} \prod_{j \neq i} \varphi(t_j, s_j) \right) C \sum_{i=1}^n \left(\frac{\partial \varphi(t_i, s_i)}{\partial s_i} \prod_{j \neq i} \varphi(t_j, s_j) \right), \quad (15)$$

whence

$$\sum_1^n \left(\frac{\partial \ln \varphi(\tau_i, s_i)}{\partial \tau_i} - C \frac{\partial \ln \varphi(\tau_i, s_i)}{\partial s_i} \right) = 0, \quad \sum_1^n \tau_i = 0, \quad |\tau_i| < \eta, \quad |s_i| < \epsilon. \quad (16)$$

As in Theorem 2, for $n \geq 3$ the only solution of (16) is

$$\frac{\partial \ln \varphi(t, s)}{\partial t} - C \frac{\partial \ln \varphi(t, s)}{\partial s} = Dt, \quad |t| < \eta, \quad |s| < \epsilon. \quad (17)$$

On differentiating (17) in t and s and setting $t = s = 0$, one sees that $C = \rho\sigma_1/\sigma_2$, $D = -\sigma_1^2(1 - \rho^2)$ so that the only difference between (17) and (7) is that the former holds for $|t| < \eta$, $|s| < \epsilon$ while the latter holds for all real t, s .

If $\varphi(t, s) \neq 0$, (17) holds for all real t, s . If additionally (x_i, y_i) has a differentiable in x density $f(x - \theta, y)$, then, as shown in Theorem 1, $f(x, y)$ must be of the form (5).

Conversely, if $\varphi(t, s)$ satisfies (7) for all real t, s , then simple calculations show that (15) holds for all real $t_1, \dots, t_n; s_1, \dots, s_n$ and hence the Pitman estimator is linear in $\bar{x}$ and $\bar{y}$.

2. Discussion and an open problem. As shown in Section 1, if x_i is Gaussian, x_i and y_i are dependent and the characteristic function of y_i does not vanish, $\bar{x}$ can be improved (with respect to the quadratic loss) as an estimator of θ from a sample $(x_1, y_1), \dots, (x_n, y_n)$ of size $n \geq 3$ from a population $F(x - \theta, y)$.

Does this principle hold true in general? In other words, if the distribution of x_i is not necessarily Gaussian and x_i, y_i are dependent, can the Pitman estimator $\bar{x} - \mathbf{E}_0(\bar{x} | x_1 - \bar{x}, \dots, x_n - \bar{x})$ based only on the informative components $x_1, \dots, x_n$ of a sample $(x_1, y_1), \dots, (x_n, y_n)$ from $F(x - \theta, y)$ be always improved? If $\int x^2 dF(x) < \infty$, the problem is whether the relation

$$\mathbf{E}_0(\bar{x} | x_1 - \bar{x}, \dots, x_n - \bar{x}) = \mathbf{E}_0(\bar{x} | x_1 - \bar{x}, \dots, x_n - \bar{x}; y_1, \dots, y_n) \quad (18)$$

implies independence of x_i and y_i .

An information analog of this problem is when the Fisher information $I(X, Y)$ on θ contained in a vector (X, Y) with density $f(x - \theta, y)$ equals the Fisher information $I(X)$ contained in X . It is easily seen that X and Y must be independent.

Under mild conditions, in large samples

$$\mathbf{D}\hat{\theta}(x_1, \dots, x_n) = \frac{1}{nI(X)} (1 + o(1))$$

while

$$\mathbf{D}\hat{\theta}(x_1, y_1; \dots; x_n, y_n) = \frac{1}{nI(X, Y)} (1 + o(1)).$$

Based on this, one can expect that the answer to the above question is yes under some conditions on the sample size and the characteristic or/and density function but relations of the type (18) seem to be a challenge.

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