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A GENERALIZATION OF
THE MEJZLER-DE HAAN THEOREM

Пусть (k_n) — последовательность положительных целых чисел такая, что $k_n \rightarrow \infty$ при $n \rightarrow \infty$. Пусть $X_{n1}^*, \dots, X_{nk_n}^*$, $n \in \mathbf{N}$, — последовательность серий случайных величин такая, что для каждого n случайные величины $X_{n1}^*, \dots, X_{nk_n}^*$ независимы и имеют общую функцию распределения F_n . Обозначим $M_n^* = \max\{X_{n1}^*, \dots, X_{nk_n}^*\}$. В работе рассматривается пример последовательности серий случайных величин, которая возникает в комбинаторной задаче о времени ожидания (включая зависимый и независимый случай), где $k_n = n$ для каждого n и где предельной функцией распределения для M_n^* является $\Lambda(x) = \exp(-e^{-x})$, хотя функции распределения F_n , $n = 1, 2, \dots$, не принадлежат области притяжения $D(\Lambda)$. Мы также обобщили теорему Мейзлера и де Хаана и дали необходимые и достаточные условия на последовательность F_n , $n = 1, 2, \dots$, для того чтобы существовали последовательности $a_n > 0$ и $b_n \in \mathbf{R}$, $n \in \mathbf{N}$, такие, что $F_n^{k_n}(a_n x + b_n) \rightarrow \exp(-e^{-x})$ при $n \rightarrow \infty$ для всех действительных чисел x .

Ключевые слова и фразы: распределения экстремальных значений, последовательность серий, область притяжения, правильное изменение, двойное экспоненциальное распределение.

1. Introduction. Let (X_n) be a sequence of independent identically distributed random variables, defined on the same probability space. Let us denote $F(x) = \mathbf{P}\{X_1 \leq x\}$ and $M_n = \max\{X_1, X_2, \dots, X_n\}$. Then $\mathbf{P}\{M_n \leq x\} = F^n(x)$. A distribution function F belongs to the domain of attraction of a nondegenerate distribution function G (notation $F \in D(G)$) if there exist sequences $a_n > 0$ and $b_n \in \mathbf{R}$, $n \in \mathbf{N}$, such that the equality

$$\lim_{n \rightarrow \infty} F^n(a_n x + b_n) = G(x)$$

holds for every continuity point of G . There are three types of distribution functions with nonempty domains of attraction. The following representatives determine these three types:

$$\begin{aligned} \Lambda(x) &= \exp(-e^{-x}), & -\infty < x < +\infty; \\ \Phi_\alpha(x) &= \begin{cases} 0, & \text{if } x < 0, \\ \exp(-x^{-\alpha}), & \text{if } x \geq 0, \end{cases} \\ \Psi_\alpha(x) &= \begin{cases} \exp(-(-x)^\alpha), & \text{if } x < 0, \\ 1, & \text{if } x \geq 0, \end{cases} \end{aligned}$$

where $\alpha > 0$. The characterization of domains of attraction can be given in terms of the regular varying of tails of corresponding distribution functions. This result was first proved by Gnedenko [10]. As Gnedenko himself mentioned, the necessary and sufficient conditions for the domain of attraction of the double exponential distribution function $\Lambda(x)$ (given in [10]) were not final and simple for applications. Consequently, many other authors revisited this problem: Mejlzer [22] gave the characterization of $D(\Lambda)$ in terms of the inverse function of F ; Marcus and Pinsky [21], Balkema and de Haan [1], de Haan (see [11]–[13]) gave a simple characterization of $D(\Lambda)$ which involves only the distribution function itself.

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Let (k_n) be a sequence of positive integers such that $k_n \rightarrow \infty$ as $n \rightarrow \infty$. For each positive integer n , let $X_{n1}^*, \dots, X_{nk_n}^*$ be a sequence of k_n independent random variables with the common distribution function F_n and $M_n^* = \max\{X_{n1}^*, \dots, X_{nk_n}^*\}$. The main question we are going to deal with is the following: is it possible to find necessary and sufficient conditions for the sequence (F_n) under which there exist sequences $a_n > 0$ and $b_n \in \mathbf{R}$, $n \in \mathbf{N}$, such that the equality

$$\lim_{n \rightarrow \infty} \mathbf{P}\{M_n^* \leq a_n x + b_n\} = \lim_{n \rightarrow \infty} F_n^{k_n}(a_n x + b_n) = \exp(-e^{-x})$$

holds for every real x ? As a motivation for this question we shall consider a combinatorial waiting time problem in which the maximum of random variables in a double array arises naturally. Each row in the double array from this example is a finite sequence of dependent random variables which satisfies the condition of strict stationarity. We shall also consider the associated double array with independent identically distributed random variables in every row. In both cases the limiting distribution function for the maximum is $\Lambda(x) = \exp(-e^{-x})$, although none of marginal distribution functions F_n belongs to the domain of attraction $D(\Lambda)$.

2. Waiting time problem. Let $X_{n1}^*, \dots, X_{nn}^*$ be a sequence of independent random variables that are defined on the same probability space and have the same negative binomial distribution with parameters $r \in \mathbf{N}$ and $p = 1/n$, i.e., for every $j \in \{1, \dots, n\}$,

$$\mathbf{P}\{X_{nj}^* = k\} = \binom{k-1}{r-1} \left(1 - \frac{1}{n}\right)^{k-r} \frac{1}{n^r}, \quad k = r, r+1, \dots$$

Let us denote $M_n^* = \max\{X_{n1}^*, \dots, X_{nn}^*\}$ and

$$u_n = n(x + \ln n + (r-1) \ln \ln n - \ln(r-1)!). \quad (2.1)$$

Theorem 1. (a) For any real number x the following equality holds:

$$\lim_{n \rightarrow \infty} \mathbf{P}\{M_n^* \leq u_n\} = \exp(-e^{-x}). \quad (2.2)$$

(b) If $r = 1$, then for any real x the following asymptotic relation holds as $n \rightarrow \infty$:

$$\mathbf{P}\{M_n^* \leq u_n\} - \exp(-e^{-x}) \sim \frac{e^{-x} \exp(-e^{-x})}{2} \frac{\ln n}{n}. \quad (2.3)$$

(c) If $r \geq 2$, then the rate of convergence of M_n^* is given by the following asymptotic relation as $n \rightarrow \infty$:

$$\mathbf{P}\{M_n^* \leq u_n\} - \exp(-e^{-x}) \sim -(r-1)^2 e^{-x} \exp(-e^{-x}) \frac{\ln \ln n}{\ln n}. \quad (2.4)$$

Let $Z_1, Z_2, \dots$ be a sequence of independent random variables with a common uniform distribution over the set $\mathbf{N}_n = \{1, \dots, n\}$. For $1 \leq j \leq n$ and a fixed positive integer r , let us denote by X_{nj} the waiting time until an event of the form $\{Z_k = j\}$ occurs exactly r times as a run in the process $Z_1, Z_2, \dots$ and let $M_n = \max\{X_{n1}, \dots, X_{nn}\}$. Then $X_{n1}, \dots, X_{nn}$ are dependent random variables such that X_{nj} has the same distribution as random variable X_{nj}^* defined above, and the sequence $X_{n1}, \dots, X_{nn}$ satisfies the condition of strict stationarity. Moreover, the probability distribution of a random vector $(X_{nj_1}, \dots, X_{nj_k})$ depends only on k and not on the particular choice of integers $j_1, \dots, j_k$, where $1 \leq j_1 < \dots < j_k \leq n$. Note also that M_n is the waiting time until all events of the form $\{Z_k = j\}$, $j \in \mathbf{N}_n$, occur at least r times as a run in the process $Z_1, Z_2, \dots$.

Theorem 2. (a) The limiting distribution of random variable M_n is given by

$$\lim_{n \rightarrow \infty} \mathbf{P}\{M_n \leq u_n\} = \exp(-e^{-x}). \quad (2.5)$$

(b) For $r = 1$, we have

$$|\mathbf{P}\{M_n \leq u_n\} - \exp(-e^{-x})| = O\left(\frac{\ln n}{n}\right) \quad \text{as } n \rightarrow \infty. \quad (2.6)$$

(c) For $r \geq 2$, the following asymptotic relation holds as $n \rightarrow \infty$:

$$\mathbf{P}\{M_n \leq u_n\} - \exp(-e^{-x}) \sim -(r-1)^2 e^{-x} \exp(-e^{-x}) \frac{\ln \ln n}{\ln n}. \quad (2.7)$$

R e m a r k 1. The above mentioned waiting time problem, as many others of this kind, has been studied very much using methods of characteristic and moment generating functions in the theory of summing random variables (see, for example, [6], [4], [3], [15], [26], [8]) and an approach by imbedding in Poisson processes (see [16], [17], [2]). To obtain Theorem 2 we used the methodology of verifying Leadbetter's $D(u_n)$ and $D'(u_n)$ mixing conditions which are appropriate for an analysis of extreme values. In fact we prove that the maximum M_n of random variables $X_{n1}, \dots, X_{nn}$ has the same limiting distribution (with the same normalizing constants) as the maximum M_n^* of an associated independent sequence $X_{n1}^*, \dots, X_{nn}^*$. The limiting distribution of M_n^* is determined in Theorem 1. The rate of convergence for M_n^* and M_n in the case $r = 1$, i.e., formulae (2.3) and (2.6), was obtained in [23].

R e m a r k 2. Note also that a negative binomial distribution with parameters r and p does not belong to any of the three domains of attraction of extreme value distributions, i.e., the following statement holds: if $\xi_1, \xi_2, \dots$ is a sequence of independent random variables with the common distribution $\mathbf{P}\{\xi_j = k\} = \binom{k-1}{r-1} p^r (1-p)^{k-r}$, $k = r, r+1, \dots$, and $\mu_n = \max\{\xi_1, \dots, \xi_n\}$, then it is not possible to find constants $a_n > 0$ and b_n such that $a_n(\mu_n - b_n)$ has a nondegenerate limiting distribution.

R e m a r k 3. In connection with those parts of Theorems 1 and 2 which give the rates of convergence for M_n^* and M_n let us note the following: if $\xi_1, \xi_2, \dots$ is a sequence of independent random variables with common normal $\mathcal{N}(0, 1)$ distribution, $\mu_n = \max\{\xi_1, \dots, \xi_n\}$, $n\varphi(\alpha_n)/\alpha_n = 1$, $\beta_n = \alpha_n$ (φ is a normal $(0, 1)$ density), and $D_n = |\mathbf{P}\{\mu_n \leq x/\alpha_n + \beta_n\} - \exp(-e^{-x})|$, then

$$C_1 < \sup_{x \in \mathbf{R}} D_n \ln n \leq C_2,$$

for some positive constants C_1 and C_2 , where $C_2 \leq 3$. For such constants α_n and β_n the rate of convergence is the best possible (see [14]). If $\alpha_n = (2 \ln n)^{1/2}$ and $\beta_n = (2 \ln n)^{1/2} - 2^{-1}(2 \ln n)^{-1/2}(\ln \ln n + \ln 4\pi)$, then

$$D_n \sim \frac{e^{-x} \exp(-e^{-x})}{16} \frac{(\ln \ln n)^2}{\ln n}$$

as $n \rightarrow \infty$. Similar results for extremes of stationary normal sequences were proved by Rootzén [25].

3. Regular variation. Throughout this and the next sections we shall use the following notations: $\mathbf{R}$ is the set of real numbers; $\mathbf{R}_+$ is the set of positive real numbers. We say that $H: (c, d) \rightarrow \mathbf{R}$ is an increasing function if for every $x_1, x_2 \in (c, d)$, inequality $x_1 < x_2$ implies that $H(x_1) \leq H(x_2)$. For every increasing function $H: (c, d) \rightarrow \mathbf{R}$ we define the inverse of H in the following way:

$$H^{-1}(y) = \inf\{x: H(x) \geq y\}, \quad (3.1)$$

where $\inf \emptyset = +\infty$. Note that if the inverse of H is defined by (3.1), then the function H^{-1} is left continuous.

Let $H_0, H_1, H_2, \dots$ be a sequence of increasing real functions defined on the same domain. Then H_n converges weakly to H_0 as $n \rightarrow \infty$ if $H_n(x) \rightarrow H_0(x)$ as $n \rightarrow \infty$, for all $x \in C(H_0)$ (notation $H_n \rightarrow H_0$). We also need the concept of regular variation for positive functions which was developed by Karamata [18], [19] and extended by de Haan [11], [12].

D e f i n i t i o n 1. A Lebesgue measurable function $F: \mathbf{R}_+ \rightarrow \mathbf{R}_+$ is regularly varying at infinity if

$$\lim_{t \rightarrow \infty} \frac{F(tx)}{F(t)} = x^\rho$$

for all $x > 0$. The real number ρ is called the *exponent of regular variation*. If $\rho = 0$, then F is called *slowly varying* at infinity. We say also that F is regularly varying at 0 if $F(1/x)$ is regularly varying at infinity.

Definition 2. A measurable function $H: (c, +\infty) \rightarrow \mathbf{R}$ is Π -varying (notation $H \in \Pi$) if there exist functions $a: (c, +\infty) \rightarrow \mathbf{R}_+$ and $b: (c, +\infty) \rightarrow \mathbf{R}$ such that for all $x > 0$,

$$\lim_{t \rightarrow +\infty} \frac{H(tx) - b(t)}{a(t)} = \ln x.$$

The function a is called an *auxiliary function* for H .

Definition 3. An increasing function $H: (c, x_0) \rightarrow \mathbf{R}$ is Γ -varying (notation $H \in \Gamma$) if $H(t) \rightarrow +\infty$ as $t \uparrow x_0$ and there exists an *auxiliary function* $g: (c, x_0) \rightarrow \mathbf{R}_+$ such that for all real x ,

$$\lim_{t \uparrow x_0} \frac{H(t + xg(t))}{H(t)} = e^x.$$

The inverse of a regularly varying function is again regularly varying. Class Π is a generalization of the class of regularly varying functions, and Γ is the class of inverses of increasing unbounded functions from Π . Classes Π and Γ were first introduced by [11], [12] and (together with regular varying functions) play the central role in extreme value theory. The presentation of the theory of regular variation in the sense of Karamata as well as de Haan's extensions can be found in [11], [7], [27], [9], [5], and [24].

Theorem 3 (see [22], [11]). Let F be a distribution function, $x_0 := \sup\{t: F(t) < 1\}$ and let us denote $H(x) = 1/(1 - F(x))$ for $x < x_0$. The following three statements are equivalent:

$$F \in D(\Lambda); \quad H \in \Gamma; \quad H^{-1} \in \Pi.$$

If $g(x)$ is an auxiliary function for $H \in \Gamma$, then normalizing constants a_n and b_n in asymptotic relation $F^n(a_n x + b_n) \rightarrow \Lambda(x)$ can be chosen in the following way: $a_n = g(H^{-1}(n))$, $b_n = H^{-1}(n)$.

4. Generalization of the Mejlzer-de Haan theorem. In order to formulate results concerning the maxima of random variables in double arrays we need the following two definitions which introduce the classes $\tilde{\Pi}[(k_n)]$ and $\tilde{\Gamma}[(k_n)]$ of sequences of increasing functions.

Definition 4. Let (k_n) be a sequence of positive integers such that $k_n \rightarrow \infty$ as $n \rightarrow \infty$. A sequence of increasing functions $H_n: (c, +\infty) \rightarrow \mathbf{R}$, $n \in \mathbf{N}$, belongs to the class $\tilde{\Pi}[(k_n)]$ if there exist sequences $a_n > 0$ and $b_n \in \mathbf{R}$, $n \in \mathbf{N}$, such that for every real number x

$$\lim_{n \rightarrow \infty} \frac{H_n(k_n x) - b_n}{a_n} = \ln x.$$

In that case we write $(H_n) \in \tilde{\Pi}[(k_n)]$. The sequence (a_n) is called an *auxiliary sequence* for the sequence of functions (H_n) .

Definition 5. Let (k_n) be a sequence of positive integers such that $k_n \rightarrow \infty$ as $n \rightarrow \infty$. A sequence of increasing functions $H_n: (c, x_0) \rightarrow \mathbf{R}$, $n \in \mathbf{N}$, belongs to the class $\tilde{\Gamma}[(k_n)]$ if the following conditions are satisfied:

- (a) $\lim_{t \uparrow x_0} H_n(t) = +\infty$ for any $n \in \mathbf{N}$;
- (b) there exists a sequence of positive numbers (h_n) , $n \in \mathbf{N}$, such that for every real number x ,

$$\lim_{n \rightarrow \infty} \frac{H_n(H_n^{-1}(k_n) + xh_n)}{H_n(H_n^{-1}(k_n))} = e^x.$$

In that case we shall write $(H_n) \in \tilde{\Gamma}[(k_n)]$. The sequence (h_n) is called an *auxiliary sequence* for the sequence of functions (H_n) .

Now we shall formulate theorems which determine relations between classes $\tilde{\Pi}[(k_n)]$ and $\tilde{\Gamma}[(k_n)]$ and result that generalizes the Mejlzer-de Haan theorem.

Theorem 4. If $(H_n) \in \tilde{\Gamma}[(k_n)]$ and (h_n) is an auxiliary sequence for (H_n) , then $(H_n^{-1}) \in \tilde{\Pi}[(k_n)]$ with auxiliary sequence $(a_n) = (h_n)$.

Theorem 5. If $(H_n) \in \tilde{\Pi}[(k_n)]$ and (a_n) is an auxiliary sequence for (H_n) , then $(H_n^{-1}) \in \tilde{\Gamma}[(k_n)]$ with auxiliary sequence $(h_n) = (a_n)$.

Theorem 6. Let (k_n) be a sequence of positive integers such that $k_n \rightarrow \infty$ as $n \rightarrow \infty$. Let (F_n) be a sequence of distribution functions such that

$$\sup\{t: F_n(t) < 1\} = x_0 \leq +\infty,$$

where x_0 is the common right endpoint of distribution functions F_n , $n \in \mathbf{N}$, and

$$H_n(x) = \frac{1}{1 - F_n(x)} \quad \text{for } x < x_0.$$

The following three statements are equivalent:

(a) There exist sequences $a_n > 0$ and $b_n \in \mathbf{R}$, $n \in \mathbf{N}$, such that for every real number x the following equality holds:

$$\lim_{n \rightarrow \infty} F_n^{k_n}(a_n x + b_n) = \exp(-e^{-x}).$$

(b) $(H_n) \in \tilde{\Gamma}[(k_n)]$.

(c) $(H_n^{-1}) \in \tilde{\Pi}[(k_n)]$.

Moreover, the norming constants a_n and b_n (and auxiliary sequence h_n) can be chosen as follows:

$$b_n = H_n^{-1}(k_n), \quad (4.1)$$

$$a_n = h_n = H_n^{-1}(k_n e) - H_n^{-1}(k_n). \quad (4.2)$$

5. Proof of Theorems 1 and 2. First of all we give the following lemma, which is a minor modification of Theorem 1.5.1 of [20].

Lemma 1. Let $X_{n1}^*, \dots, X_{nn}^*$, $n \in \mathbf{N}$, be a double array of random variables which are independent and identically distributed in each row. Let $F_n(x) = \mathbf{P}\{X_{nj}^* \leq x\}$ be the common distribution function for random variables in the n -th row, $M_n^* = \max\{X_{n1}^*, \dots, X_{nn}^*\}$, and (u_n) a sequence of real numbers. Then the equality

$$\lim_{n \rightarrow \infty} n(1 - F_n(u_n)) = \tau$$

holds for $0 \leq \tau \leq +\infty$ if and only if

$$\lim_{n \rightarrow \infty} \mathbf{P}\{M_n^* \leq u_n\} = e^{-\tau}.$$

Proof of Theorem 1(a). Let F_n be the common distribution function of random variables X_{nj} and X_{nj}^* , $1 \leq j \leq n$, and m a positive integer. Note that

$$\begin{aligned} 1 - F_n(m) &= \mathbf{P}\{X_{nj} > m\} = \sum_{k=m+1}^{\infty} \binom{k-1}{r-1} \left(1 - \frac{1}{n}\right)^{k-r} \frac{1}{n^r} \\ &= \frac{1}{n^r(r-1)!} \sum_{k=m+1}^{\infty} (k-1)(k-2) \cdots (k-r+1) q^{k-r} \\ &= \frac{1}{n^r(r-1)!} \frac{d^{r-1}}{dq^{r-1}} \frac{q^m}{1-q} \Big|_{q=1-1/n} \end{aligned}$$

First, we shall determine the asymptotic behavior of the tail $1 - F_n(m)$ as $n \rightarrow \infty$ and $m/n \rightarrow \infty$. Using Leibniz's formula we get

$$\begin{aligned} 1 - F_n(m) &= \frac{1}{n^r(r-1)!} \sum_{k=0}^{r-1} \binom{r-1}{k} \frac{d^{r-1-k} q^m}{dq^{r-1-k}} \frac{d^k}{dq^k} \frac{1}{1-q} \Big|_{q=1-1/n} \\ &= \frac{1}{n^r(r-1)!} \left\{ m(m-1) \cdots (m-r+2) \frac{q^{m-r+1}}{1-q} + q^m \frac{(r-1)!}{(1-q)^r} \right. \\ &\quad \left. + \sum_{k=1}^{r-2} \binom{r-1}{k} m(m-1) \cdots (m-r+k+2) \frac{q^{m-r+k+1} k!}{(1-q)^{k+1}} \right\} \Big|_{q=1-1/n} \\ &= \frac{m^{r-1}}{n^{r-1}(r-1)!} \left(1 - \frac{1}{n}\right)^{m-r+1} (1 + R(m, n)), \end{aligned}$$

where $R(m, n) \rightarrow 0$ as $n \rightarrow \infty$ and $m/n \rightarrow \infty$. A positive integer m will be determined from the condition

$$\frac{m^{r-1}}{n^{r-1}(r-1)!} \left(1 - \frac{1}{n}\right)^{m-r+1} (1 + R(m, n)) \sim \frac{1}{n} e^{-x}, \quad n \rightarrow \infty, \quad (5.1)$$

which is equivalent to

$$(r-1) \ln m - (r-1) \ln n - \ln(r-1)! + (m-r+1) \ln\left(1 - \frac{1}{n}\right) + \ln(1 + R(m, n)) \sim -x - \ln n \quad \text{as } n \rightarrow \infty. \quad (5.2)$$

The asymptotic relation (5.2) holds if

$$(r-1) \ln m - (r-1) \ln n - \ln(r-1)! - \frac{m}{n} + x + \ln n \rightarrow 0, \quad n \rightarrow \infty.$$

We shall determine a positive integer m to be equal to $[u_n]$, where the sequence $u_n = u_n(x)$ satisfies the relation

$$\frac{u_n}{n} - x + \ln(r-1)! + (r-2) \ln n - (r-1) \ln u_n \rightarrow 0, \quad n \rightarrow \infty. \quad (5.3)$$

It is easy to verify that for $u_n = n(x + \ln n + (r-1) \ln \ln n - \ln(r-1)!)$ the asymptotic relation (5.3) holds. It follows from Lemma 1 that the limiting distribution of the maximum M_n^* is given by the equality $\lim_{n \rightarrow \infty} \mathbf{P}\{M_n^* \leq u_n\} = \exp(-e^{-x})$.

Let $m \leq n$ be a positive integer. For $s_1, \dots, s_m \in \{0, 1, \dots, r-1\}$, let us denote

$$p_{nm}(s_1, \dots, s_m) = \mathbf{P}\left(\bigcap_{j=1}^m \left\{\sum_{i=1}^{[u_n]} I\{Z_i = j\} = s_j\right\}\right),$$

$$p_{nm} = \mathbf{P}\{X_{n1} > u_n, \dots, X_{nm} > u_n\} = \sum_{s_1=0}^{r-1} \dots \sum_{s_m=0}^{r-1} p_{nm}(s_1, \dots, s_m).$$

Lemma 2. *Let m be a fixed positive integer and $s = s_1 + \dots + s_m$. Then the following asymptotic relation holds as $n \rightarrow \infty$:*

$$p_{nm}(s_1, \dots, s_m) = \frac{e^{-mx}((r-1)!)^m}{s_1! \dots s_m!} \left(\frac{u_n}{n}\right)^s \times \frac{1}{n^m (\ln n)^{m(r-1)}} \left\{1 - \frac{m^2 \ln n}{2n} + o\left(\frac{\ln n}{n}\right)\right\}. \quad (5.4)$$

P r o o f. Let us denote $r_n = u_n - [u_n]$, $s = s_1 + \dots + s_m$. Then

$$p_{nm}(s_1, \dots, s_m) = \binom{[u_n]}{s_1} \binom{[u_n] - s_1}{s_2} \dots \binom{[u_n] - s_1 - \dots - s_{m-1}}{s_m} \frac{1}{n^s} \left(1 - \frac{m}{n}\right)^{[u_n] - s}$$

$$= z \frac{(u_n - r_n) \dots (u_n - r_n - s + 1)}{s_1! \dots s_m!} \frac{1}{n^s} \exp\left\{(u_n - r_n - s) \ln\left(1 - \frac{m}{n}\right)\right\}$$

and using (2.1) we obtain (5.4).

Lemma 3. *Let m be a fixed positive integer. Then we have as $n \rightarrow \infty$:*

$$p_{nm} - p_{n, m-1} p_{n1} = -\frac{e^{-mx}}{n^{m+1}} \{(m-1) \ln n + o(\ln n)\}, \quad (5.5)$$

$$p_{nm} = \frac{e^{-mx}}{n^m} (1 + o(1)). \quad (5.6)$$

P r o o f. Equality (5.5) is a straightforward consequence of (5.4). Note that for a fixed m and all sufficiently large n the probability $p_{nm}(s_1, \dots, s_m)$ takes the greatest possible value if $s_1 = \dots = s_m = r-1$. Consequently, equality (5.6) follows also from (5.4).

P r o o f o f T h e o r e m 1(c). Let us denote $\tau = e^{-x}$ and $\tau_n = n(1 - F_n(u_n))$. Then we have

$$\mathbf{P}\{M_n^* \leq u_n\} - e^{-\tau} = \mathbf{P}\{M_n^* \leq u_n\} - e^{-\tau_n} + e^{-\tau_n} - e^{-\tau}. \quad (5.7)$$

Using [20, Theorem 2.4.2] we obtain that

$$\mathbf{P}\{M_n^* \leq u_n\} - e^{-\tau_n} \sim \frac{e^{-2x} \exp(-e^{-x})}{2} \frac{1}{n}, \quad n \rightarrow \infty. \quad (5.8)$$

The tail $1 - F_n(u_n)$ is given by

$$1 - F_n(u_n) = \mathbf{P}\{X_{nj}^* > u_n\} = \mathbf{P}\{X_{nj} > u_n\} = \sum_{s=0}^{r-1} p_{n1}(s),$$

where

$$p_{n1}(s) = \frac{e^{-x}(r-1)!}{s! n (\ln n)^{r-1}} \left(\frac{u_n}{n} \right)^s \left\{ 1 - \frac{\ln n}{2n} + o\left(\frac{\ln n}{n}\right) \right\}.$$

The case $r = 1$ was considered in [23]. Now, let us consider the case $r \geq 2$. Since $u_n/n = \ln n + (r-1) \ln \ln n + o(\ln \ln n)$, we obtain the following asymptotic relations as $n \rightarrow \infty$:

$$\begin{aligned} \left(\frac{u_n}{n} \right)^s &= (\ln n)^s \left\{ 1 + \frac{s(r-1) \ln \ln n}{\ln n} + o\left(\frac{\ln \ln n}{\ln n}\right) \right\}, \\ p_{n1}(s) &= \frac{e^{-x}(r-1)!}{s! n (\ln n)^{r-1-s}} \left\{ 1 + \frac{s(r-1) \ln \ln n}{\ln n} + o\left(\frac{\ln \ln n}{\ln n}\right) \right\}, \\ \tau_n &= n \sum_{s=0}^{r-1} p_{n1}(s) = e^{-x} \left\{ 1 + \frac{(r-1)^2 \ln \ln n}{\ln n} + o\left(\frac{\ln \ln n}{\ln n}\right) \right\}, \end{aligned} \quad (5.9)$$

and finally

$$e^{-\tau_n} - e^{-\tau} = -(r-1)^2 e^{-x} \exp(-e^{-x}) \frac{\ln \ln n}{\ln n} + o\left(\frac{\ln \ln n}{\ln n}\right). \quad (5.10)$$

The asymptotic relation (2.4) now follows from (5.7), (5.8), and (5.10).

Lemma 4. Let x be a real number and k and l positive integers such that $k+l \leq n$. Then Leadbetter's $D(u_n)$ mixing condition is satisfied, i.e., there exists a constant $C(x)$ such that the following inequalities hold:

$$\begin{aligned} &\left| \mathbf{P}\left(\bigcap_{j=1}^{k+l} \{X_{nj} \leq u_n\}\right) - \mathbf{P}\left(\bigcap_{j=1}^k \{X_{nj} \leq u_n\}\right) \mathbf{P}\left(\bigcap_{j=k+1}^{k+l} \{X_{nj} \leq u_n\}\right) \right| \\ &\leq C(x) \min\{k, l\} \frac{\ln n}{n^2} \leq C(x) \frac{\ln n}{n}. \end{aligned} \quad (5.11)$$

P r o o f. Let us denote

$$\begin{aligned} \Delta_n(k, 1) &= \mathbf{P}\left(\bigcap_{j=1}^{k+1} \{X_{nj} \leq u_n\}\right) - \mathbf{P}\left(\bigcap_{j=1}^k \{X_{nj} \leq u_n\}\right) \mathbf{P}\{X_{n,k+1} \leq u_n\}, \\ D_j &= \{X_{nj} > u_n\}, \quad \text{for } j = 1, 2, \dots, k; \quad A = \{X_{n,k+1} > u_n\}. \end{aligned}$$

Then we have

$$\begin{aligned} \Delta_n(k, 1) &= \mathbf{P}(\overline{D_1} \cdots \overline{D_k} \overline{A}) - \mathbf{P}(\overline{D_1} \cdots \overline{D_k}) \mathbf{P}(\overline{A}) \\ &= 1 - \mathbf{P}(D_1 \cup \dots \cup D_k \cup A) - \left(1 - \mathbf{P}(D_1 \cup \dots \cup D_k)\right) \left(1 - \mathbf{P}(A)\right) \\ &= \mathbf{P}((D_1 \cup \dots \cup D_k) \cap A) - \mathbf{P}(D_1 \cup \dots \cup D_k) \mathbf{P}(A) \\ &= \mathbf{P}(D_1 A \cup \dots \cup D_k A) - \mathbf{P}(D_1 \cup \dots \cup D_k) \mathbf{P}(A). \end{aligned}$$

Using the inclusion-exclusion principle we obtain that

$$\Delta_n(k, 1) = \sum_{m=2}^{k+1} (-1)^m \binom{k}{m-1} (p_{nm} - p_{n,m-1} p_{n1}). \quad (5.12)$$

Using (5.5) and (5.12) we get inequality (5.11), first for integers k and $l = 1$, and then for arbitrary k and l , where $k + l \leq n$.

Lemma 5. *If u_n is given by (2.1), then the condition $D'(u_n)$ is satisfied, i.e.,*

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} n \sum_{j=2}^{\lfloor n/k \rfloor} \mathbf{P}\{X_{n1} > u_n, X_{nj} > u_n\} = 0. \quad (5.13)$$

P r o o f. Using equality (5.6) we obtain the relations

$$n \sum_{j=2}^{\lfloor n/k \rfloor} \mathbf{P}\{X_{n1} > u_n, X_{nj} > u_n\} = n \left(\left\lfloor \frac{n}{k} \right\rfloor - 1 \right) \frac{e^{-2x}}{n^2} (1 + o(1)) \sim \frac{e^{-2x}}{k}$$

as $n \rightarrow \infty$, from which equality (5.13) follows immediately as $k \rightarrow \infty$, and this completes the proof of Lemma 5.

P r o o f o f T h e o r e m 2. (a) It follows from Theorem 1(a), Lemma 4, Lemma 5, and [20, Theorem 3.5.2] that the maximum M_n has the same limiting distribution as $n \rightarrow \infty$ (with the same normalizing constants) as the maximum M_n^* , i.e., (2.5) holds.

(c) Let us estimate the difference $|\mathbf{P}\{M_n \leq u_n\} - \mathbf{P}\{M_n^* \leq u_n\}|$. Using the inclusion-exclusion principle we obtain that

$$\mathbf{P}\{M_n \leq u_n\} = \sum_{m=0}^n (-1)^m \binom{n}{m} p_{nm}, \quad (5.14)$$

where $p_{n0} = 1$. We also have the following equality:

$$\mathbf{P}\{M_n^* \leq u_n\} = \{1 - (1 - F_n(u_n))\}^n = \sum_{m=0}^n (-1)^m \binom{n}{m} (1 - F_n(u_n))^m, \quad (5.15)$$

where $1 - F_n(u_n) = \sum_{s_1=0}^{r-1} p_{n1}(s_1) = p_{n1}$. Using (5.14) and (5.15) we obtain

$$\mathbf{P}\{M_n \leq u_n\} - \mathbf{P}\{M_n^* \leq u_n\} = \sum_{m=2}^n (-1)^m \binom{n}{m} \left\{ p_{nm} - \left(\sum_{s_1=0}^{r-1} p_{n1}(s_1) \right)^m \right\}. \quad (5.16)$$

It is easy to see that the following equality holds:

$$p_{nm} - \left(\sum_{s_1=0}^{r-1} p_{n1}(s_1) \right)^m = \sum_{s_1=0}^{r-1} \cdots \sum_{s_m=0}^{r-1} \left\{ p_{nm}(s_1, \dots, s_m) - p_{n1}(s_1) \cdots p_{n1}(s_m) \right\}. \quad (5.17)$$

Now, using (5.4) and (5.9) we get that for each fixed m the following asymptotic relation holds as $n \rightarrow \infty$:

$$\begin{aligned} & p_{nm}(s_1, \dots, s_m) - p_{n1}(s_1) \cdots p_{n1}(s_m) \\ &= -\frac{e^{-mx} ((r-1)!)^m}{s_1! \cdots s_m!} \frac{(\ln n)^s}{n^{m+1} (\ln n)^{m(r-1)}} \left\{ \frac{m(m-1)}{2} \ln n + o(\ln n) \right\}, \end{aligned} \quad (5.18)$$

where $s = s_1 + \cdots + s_m$. If n is sufficiently large, then we get from (5.18) that the value of $|p_{nm}(s_1, \dots, s_m) - p_{n1}(s_1) \cdots p_{n1}(s_m)|$ is the greatest possible if $s_1 = \cdots = s_m = r-1$. Consequently we obtain from (5.17) that for fixed m the following relation holds as $n \rightarrow \infty$:

$$p_{nm} - \left(\sum_{s_1=0}^{r-1} p_{n1}(s_1) \right)^m \sim -\frac{e^{-mx}}{n^{m+1}} \left\{ \frac{m(m-1)}{2} \ln n + o(\ln n) \right\}. \quad (5.19)$$

It follows from (5.16) and (5.19) that

$$\left| \mathbf{P}\{M_n \leq u_n\} - \mathbf{P}\{M_n^* \leq u_n\} \right| = O\left(\frac{\ln n}{n}\right) \quad \text{as } n \rightarrow \infty. \quad (5.20)$$

Finally, (2.7) follows from (2.4) and (5.20) and the proof of Theorem 2 is completed.

R e m a r k 4. The proof of equality (2.5) based on Lemmas 4 and 5 represents an application of extreme value theory. Note also that equality (2.5) follows immediately from (2.2) and (5.20).

6. Proof of Theorems 4, 5, and 6.

Lemma 6. Let $H: (a, b) \rightarrow \mathbf{R}$ be an increasing function, $H^-(x) = \lim_{t \uparrow x} H(t)$ and H^{-1} the inverse of H given by (3.1). Then the following statements hold:

- (a) $x < H^{-1}(y) \implies H(x) < y$;
- (b) $x > H^{-1}(y) \implies H(x) > y$;
- (c) $(H^-)^{-1} = H^{-1}$;
- (d) $(H^{-1})^{-1}(x) = H^-(x) = \lim_{t \uparrow x} H(t)$;
- (e) if $H_0, H_1, H_2, \dots$ is a sequence of increasing functions such that $H_n \rightarrow H_0$, then $H_n^{-1} \rightarrow H_0^{-1}$.

P r o o f. The statements of Lemma 6 are straightforward consequences of the definition of functions H^- and H^{-1} and the definition of the weak convergence of a sequence of increasing functions.

Lemma 7. If $(H_n) \in \tilde{\Gamma}[(k_n)]$ with auxiliary sequence (h_n) , then the following equality holds:

$$\lim_{n \rightarrow \infty} \frac{H_n(H_n^{-1}(k_n))}{k_n} = 1.$$

P r o o f. Let $\varepsilon > 0$. Using Lemma 6(a), (b) we get

$$\frac{H_n(H_n^{-1}(k_n) - \varepsilon h_n)}{H_n(H_n^{-1}(k_n))} \leq \frac{k_n}{H_n(H_n^{-1}(k_n))} \leq \frac{H_n(H_n^{-1}(k_n) + \varepsilon h_n)}{H_n(H_n^{-1}(k_n))}. \quad (6.1)$$

Since $(H_n) \in \tilde{\Gamma}[(k_n)]$ we obtain that

$$\frac{H_n(H_n^{-1}(k_n) - \varepsilon h_n)}{H_n(H_n^{-1}(k_n))} \rightarrow e^{-\varepsilon}, \quad \frac{H_n(H_n^{-1}(k_n) + \varepsilon h_n)}{H_n(H_n^{-1}(k_n))} \rightarrow e^{\varepsilon}$$

as $n \rightarrow \infty$, and now the statement of the lemma follows from (6.1).

P r o o f o f T h e o r e m 4. If $(H_n) \in \tilde{\Gamma}[(k_n)]$ with auxiliary sequence (h_n) , then for every $x \in \mathbf{R}$,

$$\lim_{n \rightarrow \infty} \frac{H_n(H_n^{-1}(k_n) + x h_n)}{H_n(H_n^{-1}(k_n))} = e^x. \quad (6.2)$$

Let $y = e^x$ ($x = \ln y$, $y > 0$). Then equality (6.2) can be written in the form

$$\lim_{n \rightarrow \infty} \frac{H_n(H_n^{-1}(k_n) + h_n \ln y)}{H_n(H_n^{-1}(k_n))} = y. \quad (6.3)$$

Using Lemma 6(e), we get from (6.3) that

$$\lim_{n \rightarrow \infty} \frac{H_n^{-1}(y H_n(H_n^{-1}(k_n))) - H_n^{-1}(k_n)}{h_n} = \ln y. \quad (6.4)$$

Using Lemma 7 and equality (6.4) we obtain that for every $y > 0$,

$$\lim_{n \rightarrow \infty} \frac{H_n^{-1}(k_n y) - H_n^{-1}(k_n)}{h_n} = \ln y,$$

i.e., $(H_n^{-1}) \in \tilde{\Pi}[(k_n)]$ with auxiliary sequence (h_n) .

P r o o f o f T h e o r e m 5. Since $(H_n) \in \widetilde{\Pi}[(k_n)]$ with auxiliary sequence (a_n) , we obtain that for every $x > 0$,

$$\lim_{n \rightarrow \infty} \frac{H_n(k_n x) - H_n(k_n)}{a_n} = \ln x. \quad (6.5)$$

If $H_n^-(k_n) = \lim_{t \uparrow x} H_n(t)$, then (6.5) can be written in the form

$$\lim_{n \rightarrow \infty} \left(\frac{H_n(k_n x) - H_n^-(k_n)}{a_n} + \frac{H_n^-(k_n) - H_n(k_n)}{a_n} \right) = \ln x. \quad (6.6)$$

First, we shall prove that $(H_n^-(k_n) - H_n(k_n))/a_n \rightarrow 0$ as $n \rightarrow \infty$. Note that the following inequalities hold: $H_n(k_n(1 - \varepsilon)) \leq H_n^-(k_n) \leq H_n(k_n(1 + \varepsilon))$ and

$$\frac{H_n(k_n(1 - \varepsilon)) - H_n(k_n)}{a_n} \leq \frac{H_n^-(k_n) - H_n(k_n)}{a_n} \leq \frac{H_n(k_n(1 + \varepsilon)) - H_n(k_n)}{a_n}. \quad (6.7)$$

Using (6.5) and inequalities (6.7) we get

$$\ln(1 - \varepsilon) \leq \liminf_{n \rightarrow \infty} \frac{H_n^-(k_n) - H_n(k_n)}{a_n} \leq \limsup_{n \rightarrow \infty} \frac{H_n^-(k_n) - H_n(k_n)}{a_n} \leq \ln(1 + \varepsilon). \quad (6.8)$$

Since ε can be chosen arbitrarily we obtain from (6.8) that

$$\lim_{n \rightarrow \infty} \frac{H_n^-(k_n) - H_n(k_n)}{a_n} = 0. \quad (6.9)$$

Using (6.6) and (6.9) we get

$$\lim_{n \rightarrow \infty} \frac{H_n(k_n x) - H_n^-(k_n)}{a_n} = \ln x, \quad x > 0. \quad (6.10)$$

Let us denote $y = \ln x$. Using Lemma 6(e) and (6.10) we obtain that

$$\lim_{n \rightarrow \infty} \frac{H_n^{-1}(H_n^-(k_n) + a_n y)}{k_n} = e^y,$$

and since by Lemma 6(d), $H_n^-(k_n) = (H_n^{-1})^{-1}(k_n)$, it follows that $(H_n^{-1}) \in \widetilde{\Gamma}[(k_n)]$.

P r o o f o f T h e o r e m 6. (a) $\Rightarrow$ (c): If (a) is satisfied, then for every $x \in \mathbf{R}$,

$$\lim_{n \rightarrow \infty} k_n \ln F_n(a_n x + b_n) = e^{-x}. \quad (6.11)$$

It follows from (6.11) that $F_n(a_n x + b_n) \rightarrow 1$ as $n \rightarrow \infty$, and since $\ln t \sim -1 + t$ as $t \rightarrow 1$, we can rewrite (6.11) in the following form:

$$\begin{aligned} k_n(1 - F_n(a_n x + b_n)) &\longrightarrow e^{-x}, & \frac{1}{k_n(1 - F_n(a_n x + b_n))} &\longrightarrow e^x, \\ \frac{H_n(a_n x + b_n)}{k_n} &\longrightarrow e^x. \end{aligned} \quad (6.12)$$

If we denote $y = e^x$ ($x = \ln y$, $y > 0$), then we get from (6.12) that

$$\frac{H_n(a_n \ln y + b_n)}{k_n} \longrightarrow y,$$

and using Lemma 6(e) we obtain $(H_n^{-1}(k_n y) - b_n)/a_n \rightarrow \ln y$, i.e., $(H_n^{-1}) \in \widetilde{\Pi}[(k_n)]$.

(c) $\Rightarrow$ (b). Let $(H_n^{-1}) \in \widetilde{\Pi}[(k_n)]$. Using Theorem 4 we get $((H_n^{-1})^{-1}) \in \widetilde{\Gamma}[(k_n)]$. It follows from Lemma 6(d) that

$$H_n^-(x) = (H_n^{-1})^{-1}(x) = \lim_{t \uparrow x} H_n(t) \quad (6.13)$$

and $(H_n^-) \in \tilde{\Gamma}[(k_n)]$. We shall prove that $(H_n) \in \tilde{\Gamma}[(k_n)]$. Using the definition of class $\tilde{\Gamma}[(k_n)]$ we obtain that there exists a sequence of positive numbers (h_n) such that for every real number x

$$\lim_{n \rightarrow \infty} \frac{H_n^-((H_n^-)^{-1}(k_n) + xh_n)}{H_n^-((H_n^-)^{-1}(k_n))} = e^x. \quad (6.14)$$

Using Lemma 6(c) and equality (6.14) we get

$$\lim_{n \rightarrow \infty} \frac{H_n^-(H_n^{-1}(k_n) + xh_n)}{H_n^-(H_n^{-1}(k_n))} = e^x. \quad (6.15)$$

For every $\varepsilon > 0$ the following inequalities hold:

$$e^x \leq \liminf_{n \rightarrow \infty} \frac{H_n(H_n^{-1}(k_n) + xh_n)}{H_n^-(H_n^{-1}(k_n))} \leq \limsup_{n \rightarrow \infty} \frac{H_n(H_n^{-1}(k_n) + xh_n)}{H_n^-(H_n^{-1}(k_n))} \leq e^{x+\varepsilon}. \quad (6.16)$$

Since $\varepsilon > 0$ is arbitrary, it follows from (6.16) that

$$\lim_{n \rightarrow \infty} \frac{H_n(H_n^{-1}(k_n) + xh_n)}{H_n^-(H_n^{-1}(k_n))} = e^x. \quad (6.17)$$

For $x = 0$ we get from (6.17) that

$$\lim_{n \rightarrow \infty} \frac{H_n(H_n^{-1}(k_n))}{H_n^-(H_n^{-1}(k_n))} = 1,$$

and consequently equality (6.17) can be written as follows:

$$\lim_{n \rightarrow \infty} \frac{H_n(H_n^{-1}(k_n) + xh_n)}{H_n(H_n^{-1}(k_n))} = e^x,$$

i.e., $(H_n) \in \tilde{\Gamma}[(k_n)]$.

(b) $\Rightarrow$ (a): From $(H_n) \in \tilde{\Gamma}[(k_n)]$ it follows that

$$\lim_{n \rightarrow \infty} \frac{H_n(H_n^{-1}(k_n) + xh_n)}{H_n(H_n^{-1}(k_n))} = e^x. \quad (6.18)$$

Using Lemma 7 we obtain that $H_n(H_n^{-1}(k_n)) \sim k_n$ as $n \rightarrow \infty$ and equality (6.18) can be written in the form

$$\lim_{n \rightarrow \infty} \frac{H_n(H_n^{-1}(k_n) + xh_n)}{k_n} = e^x. \quad (6.19)$$

Let us denote

$$b_n = H_n^{-1}(k_n) \quad \text{and} \quad a_n = h_n.$$

From (6.19) it follows that

$$\lim_{n \rightarrow \infty} \frac{H_n(a_n x + b_n)}{k_n} = e^x. \quad (6.20)$$

Equality (6.20) is equivalent to the following:

$$\lim_{n \rightarrow \infty} k_n(1 - F_n(a_n x + b_n)) = e^{-x}, \quad \lim_{n \rightarrow \infty} k_n \ln F_n(a_n x + b_n) = -e^{-x},$$

$$\lim_{n \rightarrow \infty} F_n^{k_n}(a_n x + b_n) = \exp(-e^{-x}),$$

and the statement (a) follows.

Finally, if $(H_n^{-1}) \in \tilde{\Pi}[(k_n)]$ with auxiliary sequence (a_n) , then

$$\lim_{n \rightarrow \infty} \frac{H_n^{-1}(k_n x) - b_n}{a_n} = \ln x. \quad (6.21)$$

For $x = 1$ and $x = e$ we get from (6.21) that constants a_n and b_n can be chosen such that equalities (4.1) and (4.2) hold.

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A LARGE DEVIATIONS UPPER BOUND FOR THE KERNEL MODE ESTIMATOR¹⁾

Доказывается верхняя граница больших уклонений для ядерной оценки моды, соответствующей строгому максимуму плотности вероятностей. Скорость сходимости неожиданно оказывается более быстрой, чем можно было ожидать, судя по скорости слабой сходимости изучаемой оценки.

Ключевые слова и фразы: плотность, мода, ядерная оценка, принцип больших уклонений.

1. Introduction. Let $X_1, \dots, X_n$ be a sequence of independent and identically distributed real-valued random variables with unknown continuous probability density f . A mode of $f(x)$ is a value of x which maximizes f . More precisely, we define the functional

$$L: g \mapsto \inf \left\{ y, g(y) = \sup_{x \in \mathbf{R}} g(x) \right\} \quad \text{for } g: \mathbf{R} \rightarrow \mathbf{R}$$

and assume $|L(f)| < +\infty$; since f is continuous, the real number θ which satisfies $\theta = L(f)$ is a mode of f .

The kernel estimator of θ , introduced by Parzen [9], is defined as

$$\theta_n = L(f_n),$$

where f_n is the well-known Rosenblatt–Parzen kernel estimator of the density f ; recall that, for all $x \in \mathbf{R}$,

$$f_n(x) = \frac{1}{nh_n} \sum_{k=1}^n K\left(\frac{X_k - x}{h_n}\right),$$

where $(h_n)_{n \geq 1}$ is a sequence of positive real numbers such that $\lim_{n \rightarrow +\infty} h_n = 0$, $\lim_{n \rightarrow +\infty} nh_n = +\infty$, and the kernel K is a continuous function satisfying $\lim_{|x| \rightarrow \infty} K(x) = 0$, $\int K(x) dx = 1$.

The properties of K ensure the existence of θ_n . Moreover, K being continuous, $\sup_{x \in I} f_n(x)$ is a random variable for any interval I ; the fact that, for any $\beta \in \mathbf{R}$,

$$\{\theta_n \leq \beta\} = \left\{ \sup_{x \in]-\infty, \beta]} f_n(x) = \sup_{x \in \mathbf{R}} f_n(x) \right\}$$

ensures θ_n to be a random variable.

The convergence in probability of (θ_n) to θ was established by Parzen [9] and Yamato [13], its almost sure convergence was proved by Nadaraja [7], Van Ryzin [12], and

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