Math-Net.Ru

Общероссийский математический портал

Е. В. Dynkin, A probabilistic approach to a nonlinear differential equation on a Riemannian manifold, *Теория вероятн. и ее примен.*, 1997, том 42, выпуск 2, 336–341

DOI: <https://doi.org/10.4213/tvp1806>

Использование Общероссийского математического портала Math-Net.Ru подразумевает, что вы прочитали и согласны с пользовательским соглашением
<http://www.mathnet.ru/rus/agreement>

Параметры загрузки:

IP: 54.174.149.18

26 апреля 2023 г., 12:10:04

КРАТКИЕ СООБЩЕНИЯ

© 1997 г.

DYNKIN E. B.*

A PROBABILISTIC APPROACH
TO A NONLINEAR DIFFERENTIAL EQUATION
ON A RIEMANNIAN MANIFOLD¹⁾

Мы изучаем минимальное решение задачи

$$Lu = u^\alpha \quad \text{в } D,$$

$$u = f \quad \text{на } O,$$

где $1 < \alpha \leq 2$, D — открытое подмножество риманова многообразия, O — регулярное относительно открытое подмножество ∂D и f — отображение из ∂D в $[0, \infty]$, непрерывное на O и равное нулю на $\partial D \setminus O$. Дается явная формула для такого решения в терминах (L, α) -супердиффузии.

Ключевые слова и фразы: диффузия, супердиффузия, мера выхода, минимальное положительное решение граничной задачи, регулярные точки границы, канонические римановы координаты.

1. Let L be a second order elliptic differential operator without zero order terms on a Riemannian manifold E . There is a continuous Markov process $\xi = (\xi_t, \Pi_x)$ on E with the generator L . We call it the L -diffusion. To every $1 < \alpha \leq 2$ there corresponds a Markov process $X = (X_t, P_\mu)$ in the space $\mathcal{M}(E)$ of all finite measures on E which we call an (L, α) -superdiffusion. The exit measure X_D of X from an open set D is a random measure with the probability distribution determined by the formula: for every $\mu \in \mathcal{M}(E)$ and for every positive Borel function f ,

$$P_\mu e^{-(f, X_D)} = e^{-(u, \mu)}, \quad (1.1)$$

where

$$u(x) = \Pi_x \int_0^{\tau_D} u(\xi_s)^\alpha ds = \Pi_x f(\xi_\tau) \quad (1.2)$$

(τ_D is the first exit time of ξ from D). It follows from (1.2) that u is a solution of a nonlinear differential equation

$$Lu = u^\alpha \quad \text{in } D. \quad (1.3)$$

A point $c \in \partial D$ is called *regular* if

$$\Pi_c \{\tau_D = 0\} = 1. \quad (1.4)$$

*Cornell University, Department of Mathematics, White Hall, Ithaca, NY, 14853-7901, USA.

¹⁾Partially supported by National Science Foundation, Grant DMS-9623190.

If c is regular and if f is bounded on ∂D and continuous at c , then

$$u(x) \rightarrow f(c) \quad \text{as } x \rightarrow c, \quad x \in D. \quad (1.5)$$

To simplify notation, we use, instead of (1.5), a short writing « $u = f$ at c ». We say that a subset B of ∂D is regular if all points of B are regular. An open set D is regular if ∂D is regular.

In [1] and [2] we considered the case when D is a bounded domain in the Euclidean space $\mathbb{R}^d$ and ∂D is regular. By Theorem 1.1 in [1], if f is continuous on ∂D , then²⁾

$$u(x) = -\log P_x e^{-(f, X_D)} \quad (1.6)$$

is the unique solution of (1.3) subject to the boundary condition

$$u = f \quad \text{on } \partial D. \quad (1.7)$$

By Theorem 1.2 in [1],

$$u(x) = -\log P_x \{X_D = 0\} \quad (1.8)$$

is the minimal positive solution of (1.3) such that

$$f = \infty \quad \text{on } \partial D. \quad (1.9)$$

Both statements follow from Theorem 3 in [2]: if f is a continuous map from ∂D to $[0, \infty]$, then u given by (1.6) is the minimal positive solution of (1.3) which satisfies (1.7).

In the present paper we prove a more general result:

Theorem 1.1. Suppose that D is an open subset of a Riemannian manifold E and O is a regular relatively open subset of ∂D . If $f: \partial D \rightarrow [0, \infty]$ is continuous on O and vanishes on $\partial D \setminus O$, then u given by (1.6) is the minimal positive solution of (1.3) for which

$$u = f \quad \text{on } O. \quad (1.10)$$

By applying this to $f = \infty$ on O , we get

Corollary. Function

$$u(x) = -\log P_x \{X_D(O) = 0\} \quad (1.11)$$

is the minimal solution of (1.3) subject to the condition

$$u = \infty \quad \text{on } O. \quad (1.12)$$

2. We assume that manifold E is the union of a countable family of compact sets. This implies the existence of a sequence of open relatively compact sets V_n such that $\overline{V_n} \subset V_{n+1}$ and $V_n \uparrow E$.³⁾ Denote by $d(x, y)$ the Riemannian distance between x and y and by $d(x, B)$ the infimum of $d(x, y)$ over all $y \in B$. Put $U_\rho(c) = \{y: d(c, y) < \rho\}$. We say that a neighborhood $U = U_\rho(c)$ of c is *admissible* if, for every $x \in \overline{U}$, there exists a unique geodesic segment $[c, x]$ which is contained in $\overline{U}$ and connects x with c . Set $\overline{U}$ can be identified with a closed ball in a Euclidean space in such a way that c is the center of the ball, each $[c, x]$ is a line segment, $d(c, x)$ coincides with the Euclidean distance between c and x and the Riemannian angle between $[c, x]$ and $[c, y]$ coincides with the Euclidean

²⁾We write P_x for P_{δ_x} where δ_x is the unit mass concentrated at x .

³⁾We denote by $\overline{B}$ the closure of B .

angle between these line segments. (This construction leads to the canonical Riemannian coordinates near c .)

A well-known test of regularity in a Euclidean space implies a similar test for a Riemannian manifold:

Proposition 2.1. *A point $c \in \partial D$ is regular if it can be touched from outside by a cone. More precisely, consider an admissible neighborhood U of c . Fix $0 < \beta < \pi$ and $a \in U$ distinct from c and denote by K the set of all points $x \in U$ such that $d(c, x) < d(c, a)$ and the angle between $[c, a]$ and $[c, x]$ is smaller than β . Point $c \in \partial D$ is regular if K is disjoint from D for some β and b .*

Clearly, this implies:

Proposition 2.2. *A point $c \in \partial D$ is regular if it belongs to a closed ball disjoint from D .*

The next property follows immediately from definition of regularity:

Proposition 2.3. *If $D_1 \subset D_2$ and if $c \in \partial D_1 \cap \partial D_2$ is regular for D_2 , then c is regular for D_1 .*

Proposition 2.3 implies:

Proposition 2.4. *If an open set D is the intersection of any family of regular sets, then D is regular.*

3. Lemma 3.1. *If V and W are open sets and if $\bar{V} \subset W$, then there exists a regular open set D such that $\bar{V} \subset D \subset W$.*

P r o o f. 1°. Denote by $\mathcal{H}$ the class of all open balls $U_\varepsilon(x)$ with $x \in \partial W$ and $\varepsilon < d(x, V)$. Our first objective is to construct a countable locally finite subset $\mathcal{P}$ of $\mathcal{H}$ which covers ∂W .

Consider a sequence U_n of open relatively compact sets U_n such that $\bar{U}_n \subset U_{n+1}$ and $U_n \uparrow E$. Put $\beta_n = d(U_n, E \setminus U_{n+1})$, for $n > 0$, $\beta_n = \infty$ for $n \leq 0$. Note that $\beta_n > 0$ for all n . Let $K_n = \bar{U}_n \cap \partial W$ for $n > 0$ and $K_n = \emptyset$ for $n \leq 0$. We construct a sequence $\mathcal{P}_1 \subset \mathcal{P}_2 \subset \dots \subset \mathcal{P}_n \subset \dots \subset \mathcal{H}$ with the properties:

(a) $\mathcal{P}_n$ covers K_n ;

(b) every $U \in \mathcal{P}_n \setminus \mathcal{P}_{n-1}$ is disjoint with K_{n-2} .

The family $\mathcal{P} = \mathcal{P}_1 \cup \dots \cup \mathcal{P}_n \cup \dots$ covers ∂W by (a) and it is locally finite by (b).

Denote by $\mathcal{H}_n$ the set of all $U_\varepsilon(x)$ with $x \in K_n \setminus K_{n-1}$ and ε smaller than, both, $d(x, \bar{V})$ and β_{n-2} . If $U \in \mathcal{H}_n$, then $U \in \mathcal{H}$ and $U \cap K_{n-2} = \emptyset$ because, for every $x \in K_n \setminus K_{n-1}$,

$$d(x, K_{n-2}) \geq d(E \setminus U_{n-1}, U_{n-2}) = \beta_{n-2}.$$

Set $\mathcal{P}_0 = \emptyset$. If $\mathcal{P}_{n-1}$ satisfies (a)-(b), then $\mathcal{H}'_n = \mathcal{P}_{n-1} \cup \mathcal{H}_n$ is an open cover of K_n . Take a finite subcover $\mathcal{P}'_n$ of $\mathcal{H}'_n$. Clearly, $\mathcal{P}_n = \mathcal{P}'_n \cup \mathcal{P}_{n-1}$ satisfies (a). Since $\mathcal{P}_n \setminus \mathcal{P}_{n-1} \subset \mathcal{H}_n$ it satisfies (b).

2°. Since $\mathcal{P}$ is locally finite, the union B of closed balls $\bar{U}$ over all $U \in \mathcal{P}$ is closed (see, e.g., [4, Ch. 1, Sect. 5.VII]). Put $D = (E \setminus B) \cap W$. Note that $\bar{V} \cap \bar{U} = \emptyset$ for every $U \in \mathcal{H}$. Therefore $\bar{V} \cap B = \emptyset$ and $\bar{V} \subset D$. Clearly, $D \subset W$.

It remains to prove that every point $c \in \partial D$ is regular for D . If $c \in \partial D$, then $c \in \partial W$ or $c \in \partial(E \setminus B)$. Every point of ∂W is an interior point of B and it does not belong to ∂D . Hence, $c \in \partial(E \setminus B)$. By Proposition 2.2, c is regular for $E \setminus B$. By Proposition 2.3, it is regular for D . Lemma is proved.

Corollary. *There exists a sequence of regular relatively compact open sets U_n such that $\bar{U}_n \subset U_{n+1}$ and $U_n \uparrow E$.*

P r o o f. Consider a sequence of relatively compact open sets V_n such that $\bar{V}_n \subset V_{n+1}$ and $V_n \uparrow E$ and apply Lemma 3.1 to the pair V_n, V_{n+1} . Corollary is proved.

Lemma 3.2. *If U is an admissible neighborhood of c disjoint from B , then there exists $a \in U$ such that $d(c, B) = d(c, y) + d(y, B)$ for all $y \in [c, a]$.*

P r o o f. Denote by $\ell(\Gamma)$ the length of an arc Γ . For every n , there exists an arc Γ_n connecting c with a point of B such that $\ell(\Gamma_n) < d(c, B) + 1/n$. Let $b \in U$ and let a_n be

a point on Γ_n such that $d(c, a_n) = d(c, b)$. We have

$$d(c, a_n) + d(a_n, B) \leq d(c, a_n) + d(a_n, B) \leq \ell(\Gamma_n) \leq d(c, B) + \frac{1}{n}.$$

If a is a limit point of sequence a_n , then $d(c, a) + d(a, B) \leq d(c, B)$. For every $y \in [c, a]$, $d(c, y) + d(y, a) = d(c, a)$ and therefore

$$d(c, y) + d(y, B) \leq d(c, y) + d(y, a) + d(a, B) \leq d(c, B).$$

On the other hand, $d(c, B) \leq d(c, y) + d(y, B)$. Lemma is proved.

Lemma 3.3. Suppose that O is an open subset of ∂D with the regular closure B . If $0 < \varepsilon < 1$, then

$$\tilde{D} = \{x \in D: d(x, \partial D) > \varepsilon d(x, B)\} \quad (3.1)$$

is an open set with the following properties:

- (a) for every $c \in O$, there exists $\gamma > 0$ such that $U_\gamma(c) \cap D \subset \tilde{D}$;
- (b) $\partial \tilde{D} \cap \partial D = B$;
- (c) $\tilde{D}$ is regular.

Proof. 1°. Since $\varepsilon < 1$, $\tilde{D} = \{x \in D: d(x, A) > \varepsilon d(x, B)\}$ where $A = \partial D \setminus B$. If $c \in O$, then $\beta = d(c, A) > 0$. If $x \in U_\gamma(c) \cap D$, then, for every $c' \in A$, we have $d(x, c') \geq d(c, c') - d(c, x) > \beta - \gamma$ and therefore $d(x, A) \geq \beta - \gamma$. On the other hand, $d(x, B) \leq d(x, c) < \gamma$. Hence $d(x, A)/d(x, B) \geq (\beta - \gamma)/\gamma > \varepsilon$ if $\gamma < \beta/(1 + \varepsilon)$ which proves (a).

2°. If $c \in \partial \tilde{D}$, then $d(c, \partial D) \geq \varepsilon d(c, B)$. If, in addition, $c \in \partial D$, then $0 = d(c, \partial D) = d(c, B)$. Therefore $\partial \tilde{D} \cap \partial D \subset B$ and, to prove (b), it is sufficient to show that every $c \in O$ belongs to $\partial \tilde{D}$. Clearly, c is a limit point for the complement of $\tilde{D}$. By (a), c is a limit point for $\tilde{D}$.

3°. Every point $c \in B$ is regular for $\tilde{D}$ by Proposition 2.3. We apply Proposition 2.2 to show that $c \in \tilde{D} \setminus B$ is regular for $\tilde{D}$. By (b), $c \in D$ and therefore there exists an admissible neighborhood U of c which is disjoint from ∂D . By Lemma 3.2, there is a point $a \in U$ with the property

$$d(c, \partial D) = d(c, y) + d(y, \partial D) \quad \forall y \in [c, a]. \quad (3.2)$$

Let $x \in U$ such that $d(c, x) \leq d(c, a)$ and let y be a point on $[c, a]$ such that $d(c, y) = d(c, x)$. We have

$$\frac{d_0(x, y)}{d_0(c, x)} = 2 \sin \frac{\theta}{2} < \theta, \quad (3.3)$$

where d_0 is the Euclidean metric in U and θ is the angle between $[c, x]$ and $[c, y]$.

As we know, $d_0(c, x) = d(c, x)$ for every $x \in \bar{U}$. There exists a constant k such that $d(x, y) \leq k d_0(x, y)$ for all $x, y \in U$. It follows from (3.3) that $\kappa d(x, y)/d(c, x) \leq k\theta$. By the triangle inequality and (3.2),

$$d(c, x) + d(x, \partial D) \leq d(c, y) + d(y, x) + d(x, y) + d(y, \partial D) = d(c, \partial D) + 2d(x, y)$$

and therefore

$$d(x, \partial D) \leq d(c, \partial D) - d(c, x) + 2d(x, y) = d(c, \partial D) - (1 - 2\kappa) d(c, x). \quad (3.4)$$

We have $d(c, B) \leq d(c, x) + d(x, B)$ and $d(c, \partial D) = \varepsilon d(c, B)$ (since $c \in \partial \tilde{D} \cap D$). Therefore (3.4) implies

$$d(x, \partial D) \leq \varepsilon [d(c, x) + d(x, B)] - (1 - 2\kappa) d(c, x) = \varepsilon d(x, B) - (1 - \varepsilon - 2\kappa\theta) d(c, x).$$

If $\theta < \beta = (1 - \varepsilon)/2k$, then $d(x, \partial D) \leq \varepsilon d(x, B)$ and $x \notin \bar{D}$. Lemma is proved.

Lemma 3.4. Suppose that a regular compact subset B of ∂D is the closure of a relatively open subset O . Then there exists a sequence of regular relatively compact open sets $D_n \uparrow D$ with the following properties:

- (i) $\partial D_n \cap \partial D = B$;
- (ii) if τ is the first exit time of ξ from D and τ_n is the first exit time from D_n , then three events $A = \{\xi_\tau \in O\}$, $B = \{\tau = \tau_n \text{ for all sufficiently large } n\}$ and $C = \{\xi_{\tau_k} \in O \text{ for some } k\}$ coincide;
- (iii) if a bounded positive function f on $\bar{D}$ vanishes outside O , then

$$\Pi_x f(\xi_{\tau_n}) \rightarrow \Pi_x f(\xi_\tau) \quad (3.5)$$

and

$$\langle f, X_{D_n} \rangle \uparrow \langle f, X_D \rangle \quad \text{a.s.} \quad (3.6)$$

(Writing «a.s.» means « P_x -a.s. for all $x \in E$ ».)

P r o o f. (i) By Corollary to Lemma 3.1, there exists a sequence of regular relatively compact open sets $U_n \uparrow E$ such that $\bar{U}_n \subset U_{n+1}$. Since B is compact, it is contained in U_n for all sufficiently large n . By changing notation, we can assume that $B \in U_n$ for all n .

By Lemma 3.3, open sets $\tilde{D}_n = \{x \in D: d(x, \partial D) > d(x, B)/n\}$ are regular and $\partial \tilde{D}_n \cap \partial D = B$.

By Proposition 2.4, $D_n = \tilde{D}_n \cap U_n$ are regular. Clearly, $D_n \uparrow D$. Note that $\partial U_n = (\partial \tilde{D}_n \cap \bar{U}_n) \cup (\tilde{D}_n \cap \partial U_n)$ and therefore $\partial D_n \cap \partial D = (B \cap \tilde{D}_n) \cup (\partial D \cap \tilde{D}_n \cap \partial U_n)$. We have $B \cap \bar{U}_n = B$ because $B \subset U_n$ and $\tilde{D}_n \cap \partial D = \emptyset$ because $\tilde{D}_n \subset D$.

(ii) The implications $B \Rightarrow C \Rightarrow A$ are obvious. By part (a) of Lemma 3.3, A implies: there exists $\sigma \in (0, \tau)$ such that $\xi_t \in \tilde{D}_1$ for all $t \in (\sigma, \tau)$. The path $\{\xi_t, 0 \leq t \leq \sigma\}$ is a compact subset of D and, since $D_n \uparrow D$, it is covered by some D_k . Clearly, this implies B .

(iii) It follows from (ii) that $f(\xi_{\tau_n}) = f(\xi_\tau)$ for all n bigger than N depending on the path of ξ and the dominated convergence theorem implies (3.5). By Lemma 3.1 in [3],

$$X_{D_1}(\Gamma) \leq \dots \leq X_{D_n}(\Gamma) \leq \dots \leq X_D(\Gamma) \quad \text{a.s.}$$

for all $\Gamma \subset B$. Therefore $\langle f, X_{D_n} \rangle \uparrow Z \leq \langle f, X_D \rangle$ a.s. For every positive Borel function φ , $P_x \langle f, X_D \rangle = \Pi_x f(\xi_\tau)$ and therefore (3.6) follows from (3.5). Lemma is proved.

4. We denote by $\mathcal{U}(D)$ the set of all positive solutions of equation (1.3). For every Borel function on ∂D with values in $[0, \infty]$ we put

$$V_D(f)(x) = -\log P_x e^{-\langle f, X_D \rangle} \quad \text{for } x \in D. \quad (4.1)$$

We use the following facts proved in [2, Theorems 1.1 and 1.2].

Proposition 4.1. Suppose that $u_n(x) \rightarrow u(x)$ for all $x \in D$. If $u_n \in \mathcal{U}(D)$, then $u \in \mathcal{U}(D)$. If O is a regular relatively open subset of ∂D , f is a continuous function on O , and if $u_n = f$ on O , then $u = f$ on O .

Proposition 4.2. For every f , $V_D(f) \in \mathcal{U}(D)$. If f is continuous at a regular point $c \in \partial D$, then $V_D(f) = f$ at c .

We prove, step by step, that the theorem holds if:

- (I) $\bar{O}$ is regular and compact and f is bounded and continuous on $\bar{O}$;
- (II) O is regular and f is bounded and continuous on O ;
- (III) O is regular and f is continuous on O .

In the case (I) we consider open sets D_n constructed in Lemma 3.4 and continue f to $\bar{D}$ by setting it equal to 0 on D . Clearly, f is continuous on every ∂D_n . By Proposition 4.2, $u = V_D(f)$ belongs to $\mathcal{U}_D$ and $u = f$ on ∂D . If v is an arbitrary element of $\mathcal{U}(D)$

subject to condition (1.10), then, by the comparison principle (see, e.g., [1, Theorem 0.6]), $v \geq u$ on D .

In the case (II) we consider relatively open relatively compact sets $O_k \uparrow O$ such that $\overline{O_k} \subset O_{k+1}$. Construct a sequence of continuous functions $f_k \leq f$ on ∂D such that $f_k = f$ on O_k , $f_k = 0$ on $\partial D \setminus O_{k+1}$. By (I), $u_k = V_D(f_k)$ is the minimal element of $\mathcal{U}(D)$ subject to the condition $u_k = f_k$ on O_{k+1} . We have $u_k \uparrow u$. By Proposition 4.1, $u \in \mathcal{U}(D)$ and $u = f$ on O . If v has the same properties, then $v = f \geq f_k = u_k$ on O_{k+1} . Hence, $v \geq u_k$ for every k which implies $v \geq u$.

In the case (III), we introduce functions $f_k = f \wedge k$. By (II), $u_k = V_D(f_k)$ is the minimal element of $\mathcal{U}(D)$ such that $u_k = f_k$ on O . We have $u_k \uparrow u$. Let $c \in O$. If $f(c) < \infty$, then there exist a neighborhood $\tilde{O}$ of c and k such that $f < k$ in $\tilde{O}$. Hence $u_k = f_k = f$ in $\tilde{O}$. If $f(c) = \infty$, then, for every k , $\liminf_{x \rightarrow c} u(x) \geq \liminf_{x \rightarrow c} u_k(x) = f_k(c) = k$ and therefore $u(c) = \infty$.

Acknowledgment. I am indebted to S. E. Kuznetsov for valuable discussions on the subject of this paper. Some of his suggestions have been used in the proofs of Lemmas 3.1 and 3.3.

REFERENCES

1. Dynkin E. B. A probabilistic approach to one class of nonlinear differential equations. — Probab. Theory Relat. Fields, 1991, v. 89, p. 89–115.
2. Dynkin E. B. Superdiffusions and parabolic nonlinear differential equations. — Ann. Probab., 1992, v. 20, p. 942–962.
3. Dynkin E. B., Kuznetsov S. E. Linear additive functionals of superdiffusions and related nonlinear p.d.e. — Trans. Amer. Math. Soc., 1996, v. 348, p. 1959–1987.
4. Kuratowski K. Topology. New York–London: Academic Press, 1966.

Поступила в редакцию
25.XII.1996

© 1997 г.

ИБРАГИМОВ Р.*, ШАРАХМЕТОВ Ш.*

О ТОЧНОЙ КОНСТАНТЕ В НЕРАВЕНСТВЕ РОЗЕНТАЛЯ

Пусть $\xi_1, \dots, \xi_n$ — независимые симметрично распределенные случайные величины с конечным p -м моментом, $2 < p < \infty$. В настоящей статье показано, что точная константа C_p^* в неравенстве Розенталя

$$\left\| \sum_{i=1}^n \xi_i \right\|_p \leq C_p \max \left(\left\| \sum_{i=1}^n \xi_i \right\|_2, \left(\sum_{i=1}^n \|\xi_i\|_p^p \right)^{1/p} \right)$$

имеет вид

$$C_p^* = \left(1 + \frac{2^{p/2}}{\pi^{1/2}} \Gamma\left(\frac{p+1}{2}\right) \right)^{1/p}, \quad 2 < p < 4,$$

$$C_p^* = \|\xi_1 - \xi_2\|_p, \quad p \geq 4,$$

*Ташкентский государственный университет, 700095 Ташкент, Узбекистан.