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$$\begin{aligned}
&\Rightarrow N(t+x_T) \geq M(t) + 1 \quad \forall t \in [0, T+x_T] \wedge \\
&\quad N(t-x_T) \leq M(t) \quad \forall t \in [x_T, T+x_T] \\
&\Rightarrow S_{N(t+x_T)} > t \quad \forall t \in [0, T+x_T] \wedge S_{N(t-x_T)} \leq t \quad \forall t \in [x_T, T+x_T] \\
&\Rightarrow S_{N(t+x_T)} - (t+x_T) > -x_T \quad \forall t \in [0, T+x_T] \wedge \\
&\quad S_{N(t-x_T)} - (t-x_T) \leq x_T \quad \forall t \in [x_T, T+x_T].
\end{aligned}$$

Из последних неравенств и неравенства $S_{N(v)} - v \geq -x_T$ при $v \in [0, x_T]$ следует (14).
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BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY CÀDLÀG MARTINGALES

Обратные стохастические дифференциальные уравнения (ОСДУ) возникают во многих финансовых задачах. Хотя число статей, рассматривающих общие финансовые рынки, все возрастает, теория ОСДУ получила развитие только для броуновской постановки. Мы рассматриваем ОСДУ, управляемые $\mathbf{R}^d$ -значным càdlàg мартингалом, и изучаем свойства решений в случае генератора, удовлетворяющего (возможно, неравномерному) условию Липшица.

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Ключевые слова и фразы: обратные семимартингалы уравнения, регулярность решений, устойчивость решений, липшицев генератор, стохастическое исчисление.

1. Introduction. Backward Stochastic Differential Equations (BSDEs) have been introduced by Bismut in 1973 [1] with a linear generator. The existence and uniqueness of the solutions in the case of nonlinear generators satisfying Lipschitz conditions was proved by Pardoux and Peng [16] in the case of a Brownian driver. These results were generalized by Kobylanski [10] to generators with quadratic growth.

The interest in BSDEs has grown rapidly in the last decades, this is due to the fact that BSDEs arise naturally in many financial problems, e.g., in valuating and hedging contingent claims, or in the theory of recursive utility, see [7] by El Karoui, Peng and Quenez. It is well known that in a complete financial market the dynamics of the portfolio which replicates a claim can be described by a BSDE with a linear generator. On the contrary, when the market is incomplete, the problem of pricing, which in general leads to an interval of arbitrage free prices, cannot be characterized in terms of linear BSDEs (see, e.g., [8]); also in utility maximization problems the BSDEs characterizing the solutions have generators with quadratic growth (among others, it is worth mentioning [17] by Rouge and El Karoui).

Although there exists a growing number of papers considering general financial markets, i.e., where the price processes are modelled by semimartingales in general filtration (requiring only a no-arbitrage condition), the theory of BSDEs is only rarely used beyond the Brownian setting.

A backward equation in a semimartingale setting was first derived by Chitashvili [3], as a stochastic version of the Bellman equation in an optimal control problem. The reader is referred also to [4]. More recently, semimartingale BSDEs have been used to characterize the solution of optimal portfolio problems based on utility functions: either giving the solution of the dual problem in terms of optimal martingale measures (see, e.g., [12] and [14]) or directly, for instance, in the case of mean variance and in exponential hedging (see [2], [13], [15]).

For semimartingale BSDEs, there exist very few results, also for generators with linear growth ([6], by El Karoui and Huang in 1997, as far as we are aware). In the present paper, we want to study, on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, equipped with a complete, quasi-left continuous, right continuous filtration $(\mathcal{F}_t)_{t \in [0, T]}$, a backward stochastic differential equation of the form

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) d\langle M \rangle_s - \int_t^T Z_s dM_s - N_T + N_t, \quad t \in [0, T], \quad (1)$$

driven by the square integrable (càdlàg) martingale $(M_t)_{t \in [0, T]}$, where T is a fixed time horizon and $\mathcal{F}_T = \mathcal{F}$.

By $[M]$ and $\langle M \rangle$ we denote, respectively, the quadratic variation process and its dual predictable projection; we point out that $\langle M \rangle$ is continuous by the assumption on the filtration (see, for instance, [5, Chapter VII]). The coefficients f and ξ are the parameters of the equation: f is called the generator of (1) and the random variable ξ is the terminal condition.

The solution is a process $(Y_t, Z_t, N_t)_{t \in [0, T]}$ satisfying equation (1) $\mathbf{P}$ -a.s., with Z predictable, N a square integrable martingale orthogonal to M , with $N_0 = 0$, and verifying some integrability conditions that we will make more precise in the next section.

Under some suitable conditions, we prove the existence and uniqueness of the solutions of (1) in a proper space (the integrability conditions on the coefficients will guarantee proper similar conditions for the solution) when f is a (possibly nonuniform) Lipschitz generator.

In Section 2, we provide the main result and some other properties of the solutions (such as a priori estimates, comparison principles, stability) in the particular case of a uniformly Lipschitz generator and in dimension 1. For the sake of clarity, we give a detailed description of the one-dimensional case. Techniques and ideas are analogous for the general case.

Then, in Section 3, we explain how our proofs can be extended to the multidimensional case with nonuniform Lipschitz generator, similar to the one considered in [6]. We use a slightly different nonuniform Lipschitz condition (however, this is required for a correct proof in the multidimensional case), but the main improvement with respect to [6] is the proof of existence and uniqueness of the solution under weaker integrability conditions on the parameters.

2. Existence and uniqueness for Lipschitz generators. As we told in the introduction, in this section we study equation (1) in dimension 1, with a uniformly Lipschitz generator. We recall that we work on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with a complete, quasi-left continuous, right continuous filtration $(\mathcal{F}_t)_{t \in [0, T]}$. All the properties will be given with respect to this filtration. We will adopt notations similar to ones used in [7] (for the Brownian case).

To describe the parameters and the solution of the equation we introduce the following spaces:

$L^2_\beta = L^2_\beta(\mathcal{F}_T)$, $\beta > 0$, the space of the $\mathcal{F}_T$ -measurable random variables ξ with $\|\xi\|_\beta^2 = \mathbf{E}[e^{\beta(M)_T} |\xi|^2] < \infty$;

$\mathcal{H}^1_T = \mathcal{H}^1_T(\mathbf{R})$ the space of the $\mathbf{R}$ -valued adapted processes ϕ such that $\|\phi\|_{\mathcal{H}^1_T} = \mathbf{E}(\int_0^T |\phi_t|^2 d\langle M \rangle_t)^{1/2} < \infty$;

$\mathcal{H}^2_T = \mathcal{H}^2_T(\mathbf{R})$ the space of the $\mathbf{R}$ -valued adapted processes ϕ such that $\|\phi\|_T^2 = \mathbf{E} \int_0^T |\phi_t|^2 d\langle M \rangle_t < \infty$;

$\mathcal{H}^2_{T,\beta} = \mathcal{H}^2_{T,\beta}(\mathbf{R})$, $\beta > 0$, the space of the adapted processes ϕ such that $\|\phi\|_{T,\beta}^2 = \mathbf{E} \int_0^T e^{\beta(M)_t} |\phi_t|^2 d\langle M \rangle_t < \infty$ ($\mathcal{H}^2_{T,\beta}$ is obviously a subspace of $\mathcal{H}^2_T$).

We also denote by $\mathcal{L}^2$ the space of L^2 -bounded martingales and by $\mathcal{L}^2_\beta$ the subspace of $\mathcal{L}^2$ containing all martingales N such that $\|N\|_\beta^2 := \mathbf{E} \int_0^T e^{\beta(M)_t} d\langle N \rangle_t < +\infty$. (Notice also that $\|N\|_\beta^2 = \mathbf{E} \int_0^T e^{\beta(M)_t} d\langle N \rangle_t$, by the remark that follows.)

It is clear from the definition that each element of these spaces is really an equivalence class, even if we will call it a process. Notice also that $\mathcal{H}^2_{T,\beta_1} \subset \mathcal{H}^2_{T,\beta_2} \subset \mathcal{H}^2_T$, for $\beta_1 > \beta_2 > 0$.

As in [7], we say that the coefficients (f, ξ) of equation (1) are β -standard if (i) ξ is in the space L^2_β ; (ii) $f: \Omega \times [0, T] \times \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$ is a $\mathcal{F} \otimes \mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbf{R}) \otimes \mathcal{B}(\mathbf{R})$ -measurable mapping such that $f(\cdot, 0, 0) \in \mathcal{H}^2_{T,\beta}$ and f is uniformly Lipschitz; we call L the minimum real constant such that

$$|f(\omega, t, y_0, z_0) - f(\omega, t, y_1, z_1)| \leq L(|y_0 - y_1| + |z_0 - z_1|) \quad \text{for } \mathbf{P}\text{-a.e. } \omega$$

for all y_0, y_1, z_0, z_1, t .

The solution is an $\mathbf{R}^3$ -valued process $(Y_t, Z_t, N_t)_{t \in [0, T]}$ satisfying equation (1) $\mathbf{P}$ -a.s. such that Y is in $\mathcal{H}^2_{T,\beta}$, Z is a predictable process in $\mathcal{H}^2_{T,\beta}$, and N is a martingale in $\mathcal{L}^2_\beta$ orthogonal to M with $N_0 = 0$. Our aim is to prove the existence and uniqueness of this solution when the coefficients (f, ξ) are β -standard, for suitable β (Theorem 2.1); the admissible values for β will depend on the Lipschitz constant of the generator. The basic ideas of the proof can be sketched as follows: a result in [6] is exploited to solve the case of a generator with null Lipschitz constant; then, we prove a «good» a priori estimate by resorting to a suitable Itô's formula; this allows us to reach the desired result by a fixed point theorem.

R e m a r k. We point out that, since $([M] - \langle M \rangle)$ is a martingale, if Z is a predictable element of $\mathcal{H}^2_{T,\beta}$, we have (see [9, Theorem 11.7])

$$\mathbf{E} \int_0^T e^{\beta(M)_t} |Z_t|^2 d[M]_t = \mathbf{E} \int_0^T e^{\beta(M)_t} |Z_t|^2 d\langle M \rangle_t < \infty.$$

Moreover, it is worth remarking that the set of uniformly bounded processes is always contained in $\mathcal{H}^2_T$, since M is square integrable, while it is contained in $\mathcal{H}^2_{T,\beta}$, $\beta > 0$, if and only if $\mathbf{E} e^{\beta(M)_T} < \infty$.

Indeed, $de^{\beta(M)_t} = \beta e^{\beta(M)_t} d\langle M \rangle_t$, so $\mathbf{E} \int_0^T e^{\beta(M)_t} d\langle M \rangle_t = \beta^{-1}(\mathbf{E} e^{\beta(M)_T} - 1)$.

We start considering a particular form of (1), with a generator g independent of the solution

$$Y_t = \xi + \int_t^T g_s d\langle M \rangle_s - \int_t^T Z_s dM_s - N_T + N_t, \quad t \in [0, T]. \quad (2)$$

The following proposition resembles Proposition 6.1 of [6], adapted to our context.

Proposition 2.1. *For ξ in L_β^2 and g in $\mathcal{H}_{T,\beta}^2$, equation (2) has a unique solution in $\mathcal{H}_{T,\beta}^2 \times \mathcal{H}_{T,\beta}^2 \times \mathcal{L}_\beta^2$.*

P r o o f. If the solution exists, the process $n_t = \int_0^t Z_s dM_s + N_t$ is the martingale component of Y , so using the fact that Y is adapted and $Y_t = \xi + \int_t^T g_s d\langle M \rangle_s + n_t - n_T$ we get

$$Y_t = \mathbf{E} \left[\xi + \int_t^T g_s d\langle M \rangle_s \mid \mathcal{F}_t \right], \quad t \in [0, T]. \quad (3)$$

Remark that $(\int_{t_1}^{t_2} A_s d\langle M \rangle_s)^2 \leq a^{-1} (e^{-a\langle M \rangle_{t_1}} - e^{-a\langle M \rangle_{t_2}}) \int_{t_1}^{t_2} A_s^2 e^{a\langle M \rangle_s} d\langle M \rangle_s$ a.s., for any process A and positive constant a . So

$$\left| \int_t^T g_s d\langle M \rangle_s \right| \leq a^{-1/2} e^{-a\langle M \rangle_t/2} \left(\int_t^T g_s^2 e^{a\langle M \rangle_s} d\langle M \rangle_s \right)^{1/2}. \quad (4)$$

Since ξ is in L_β^2 and g in $\mathcal{H}_{T,\beta}^2$, the previous assures that we can define the process Y as in (3) and that it is an adapted process in $L^2(\Omega, \mathcal{F}, \mathbf{P})$. Then, by the martingale representation theorem, there exist a unique predictable process Z in $\mathcal{H}_T^2$ and a unique L^2 -bounded martingale N orthogonal to M which verify (2). Therefore, for any ξ in L_β^2 and g in $\mathcal{H}_{T,\beta}^2$, the process (Y, Z, N) is uniquely determined. We have to show that it is in $\mathcal{H}_{T,\beta}^2 \times \mathcal{H}_{T,\beta}^2 \times \mathcal{L}_\beta^2$.

Now, by (4) with $a = \beta/2$ and Jensen's inequality

$$Y_t^2 \leq 2\mathbf{E}[\xi^2 \mid \mathcal{F}_t] + 4\beta^{-1} e^{-\beta\langle M \rangle_t/2} \mathbf{E} \left[\int_t^T g_s^2 e^{\beta/2\langle M \rangle_s} d\langle M \rangle_s \mid \mathcal{F}_t \right].$$

Then recall that, for a positive measurable process K and an increasing process A , we have that $\mathbf{E} \int_0^T K_t dA_t = \mathbf{E} \int_0^T \mathbf{E}[K_t \mid \mathcal{F}_t] dA_t$ (see, e.g., [5, Chapter VI, Theorem 57]). Then

$$\begin{aligned} \|Y\|_{T,\beta}^2 &\leq 2\mathbf{E} \int_0^T \left[e^{\beta\langle M \rangle_t} \xi^2 + \frac{2}{\beta} e^{\beta\langle M \rangle_t/2} \int_t^T g_s^2 e^{\beta\langle M \rangle_s/2} d\langle M \rangle_s \right] d\langle M \rangle_t \\ &\leq 2\mathbf{E} \left(\xi^2 \int_0^T e^{\beta\langle M \rangle_t} d\langle M \rangle_t \right) + \frac{4}{\beta} \mathbf{E} \int_0^T g_s^2 e^{\beta\langle M \rangle_s/2} \left(\int_0^s e^{\beta\langle M \rangle_t/2} d\langle M \rangle_t \right) d\langle M \rangle_s \\ &\leq \frac{2}{\beta} \|\xi\|_\beta^2 + \frac{8}{\beta^2} \|g\|_{T,\beta}^2. \end{aligned} \quad (5)$$

We show that the processes Z and N are in $\mathcal{H}_{T,\beta}^2$ and $\mathcal{L}_\beta^2$, respectively. We compute the differential $d(e^{\beta\langle M \rangle_t}([n]_T - [n]_t)) = \beta e^{\beta\langle M \rangle_t}([n]_T - [n]_t) d\langle M \rangle_t - e^{\beta\langle M \rangle_t} d[n]_t$. Then integrating in time and taking expectation $\mathbf{E} \int_0^T e^{\beta\langle M \rangle_s} d[n]_s = \mathbf{E}[n]_T + \beta \mathbf{E} \int_0^T e^{\beta\langle M \rangle_s} ([n]_T - [n]_s) d\langle M \rangle_s$, that is,

$$\|Z\|_{T,\beta}^2 + \|N\|_\beta^2 = \|Z\|_{T,0}^2 + \|N\|_0^2 + \beta \mathbf{E} \left[\int_0^T e^{\beta\langle M \rangle_t} \mathbf{E}([n]_T - [n]_t \mid \mathcal{F}_t) d\langle M \rangle_t \right].$$

We still have to arrange the third term. Since $(n^2 - [n])$ is a martingale, we have

$$\begin{aligned} \mathbf{E}([n]_T - [n]_t \mid \mathcal{F}_t) &= \mathbf{E}((n_T - n_t)^2 \mid \mathcal{F}_t) = \mathbf{E} \left[\left(\xi - Y_t + \int_t^T g_s d\langle M \rangle_s \right)^2 \mid \mathcal{F}_t \right] \\ &\stackrel{(3)}{=} \mathbf{E} \left[\left(\xi + \int_t^T g_s d\langle M \rangle_s \right)^2 \mid \mathcal{F}_t \right] - \left(\mathbf{E} \left[\xi + \int_t^T g_s d\langle M \rangle_s \mid \mathcal{F}_t \right] \right)^2 \\ &\leq 2\mathbf{E} \left[\xi^2 + \left(\int_t^T g_s d\langle M \rangle_s \right)^2 \mid \mathcal{F}_t \right], \end{aligned}$$

so we conclude with some easy computations similarly as in (5). Proposition 2.1 is proved.

Lemma 2.1. Assume $(Y, Z, N) \in \mathcal{H}_{T,\beta}^2 \times \mathcal{H}_{T,\beta}^2 \times \mathcal{L}_\beta^2$ is a solution of (1) for some β -standard parameters (f, ξ) . Then the random variable $\sup_{t \in [0, T]} (e^{\beta \langle M \rangle_t / 2} Y_t)$ is in L^2 . Moreover, the process $\int_0^\bullet e^{\beta \langle M \rangle_s} Y_{s-} dH_s$ is a martingale for any martingale H in $\mathcal{L}_\beta^2$.

P r o o f. Inequality (4) with $a = \beta$ and $g_s = f(s, Y_s, Z_s)$ and Doob's inequality imply

$$\begin{aligned} \mathbf{E} \left[\sup_{t \in [0, T]} e^{\beta \langle M \rangle_t} Y_t^2 \right] &\leq 2 \mathbf{E} \left[\sup_t \left(\mathbf{E} \left[\xi e^{\beta \langle M \rangle_T / 2} \mid \mathcal{F}_t \right] \right)^2 \right] \\ &\quad + \frac{2}{\beta} \mathbf{E} \left[\sup_t \left(\mathbf{E} \left[\left(\int_0^T f^2(s, Y_s, Z_s) e^{\beta \langle M \rangle_s} d\langle M \rangle_s \right)^{1/2} \mid \mathcal{F}_t \right] \right)^2 \right] \\ &\leq 8 \|\xi\|_\beta^2 + 8\beta^{-1} \mathbf{E} \int_0^T f^2(s, Y_s, Z_s) e^{\beta \langle M \rangle_s} d\langle M \rangle_s \\ &\leq 8 \|\xi\|_\beta^2 + 24\beta^{-1} \left(\|f(\cdot, 0, 0)\|_{T,\beta}^2 + L^2 \|Y\|_{T,\beta}^2 + L^2 \|Z\|_{T,\beta}^2 \right). \end{aligned}$$

This completes the proof of the first part. Now, for the integral process, we recall that, if K is a L^2 -bounded martingale and since the filtration is quasi-left continuous for any predictable process A such that $(\mathbf{E} \int_0^T A_s^2 d\langle K \rangle_s)^{1/2} < \infty$, then the process $\int_0^\bullet A_s dK_s$ is a martingale. So we have to prove that this condition holds for $A = e^{\beta \langle M \rangle} Y_-$ and $K = H$. Indeed, we have

$$\begin{aligned} \mathbf{E} \left(\int_0^T e^{2\beta \langle M \rangle_s} Y_{s-}^2 d\langle H \rangle_s \right)^{1/2} &\leq \mathbf{E} \left[\left(\int_0^T e^{\beta \langle M \rangle_s} d\langle H \rangle_s \right)^{1/2} \sup_{t \in [0, T]} e^{\beta \langle M \rangle_t / 2} Y_t \right] \\ &\leq \left\| \sup_{t \in [0, T]} e^{\beta \langle M \rangle_t / 2} Y_t \right\|_{L^2} \|H\|_\beta \end{aligned}$$

which is finite by previous considerations. Lemma 2.1 is proved.

Proposition 2.2 (A priori estimates). Let (f_1, ξ_1) and (f_2, ξ_2) be two β -standard parameters, with $\beta > 2L + L^2$, and suppose that there exist the corresponding solutions (Y^1, Z^1, N^1) and (Y^2, Z^2, N^2) of (1), respectively, in the space $\mathcal{H}_{T,\beta}^2 \times \mathcal{H}_{T,\beta}^2 \times \mathcal{L}_\beta^2$. For any couple of processes X^1 and X^2 , let δX be the difference process $X^1 - X^2$ and let $\delta_2 f_t = (f_1 - f_2)(t, Y_t^2, Z_t^2)$. Then there are two positive constants K and μ , depending on L and β as in (7), such that

$$\|\delta Y\|_{T,\beta}^2 + \|\delta Z\|_{T,\beta}^2 + \|\delta N\|_\beta^2 \leq K \left[\|\xi_1 - \xi_2\|_\beta^2 + \mu^{-2} \|\delta_2 f\|_{T,\beta}^2 \right].$$

P r o o f. We have $\delta Y = Y^1 - Y^2$, where

$$Y_t^i = \xi_i + \int_t^T f_i(s, Y_s^i, Z_s^i) d\langle M \rangle_s - \int_t^T Z_s^i dM_s - N_T^i + N_t^i, \quad i = 1, 2.$$

Writing Itô's formula for $e^{\beta \langle M \rangle_s} \delta Y_s^2$, integrating in time and taking expectation we find

$$\begin{aligned} &\mathbf{E}(e^{\beta \langle M \rangle_T} \delta Y_T^2) - \mathbf{E} \delta Y_0^2 \\ &= \mathbf{E} \int_0^T e^{\beta \langle M \rangle_s} \left\{ \beta |\delta Y_s|^2 d\langle M \rangle_s - 2\delta Y_s (f_1(s, Y_s^1, Z_s^1) - f_2(s, Y_s^2, Z_s^2)) d\langle M \rangle_s \right. \\ &\quad \left. + \delta Z_s^2 d\langle M \rangle_s + d[\delta N]_s + 2\delta Y_{s-} \delta Z_s dM_s + 2\delta Y_{s-} d\delta N_s \right\}. \end{aligned}$$

The last two terms vanish because the time integrals are martingales (just apply Lemma 2.1 with $H_t = \int_0^t \delta Z_s dM_s + \delta N_t$). So, if we rearrange some terms, we deduce that

$$\begin{aligned} &\mathbf{E} \delta Y_0^2 + \mathbf{E} \int_0^T e^{\beta \langle M \rangle_s} (\beta \delta Y_s^2 + \delta Z_s^2) d\langle M \rangle_s + \mathbf{E} \int_0^T e^{\beta \langle M \rangle_s} d[\delta N]_s \\ &= \mathbf{E}[e^{\beta \langle M \rangle_T} \delta Y_T^2] + 2\mathbf{E} \int_0^T e^{\beta \langle M \rangle_s} \delta Y_s (f_1(s, Y_s^1, Z_s^1) - f_2(s, Y_s^2, Z_s^2)) d\langle M \rangle_s \end{aligned}$$

$$\begin{aligned}
&\leq \mathbf{E}[e^{\beta\langle M \rangle_T} \delta Y_T^2] + 2\mathbf{E} \int_0^T e^{\beta\langle M \rangle_s} |\delta Y_s| (L|\delta Y_s| + L|\delta Z_s| + |\delta_2 f_s|) d\langle M \rangle_s \\
&\leq \mathbf{E}[e^{\beta\langle M \rangle_T} \delta Y_T^2] \mathbf{E} \int_0^T e^{\beta\langle M \rangle_s} \left(\delta Y_s^2 (2L + L\lambda^2 + \mu^2) + \delta Z_s^2 \frac{L}{\lambda^2} + \delta_2 f_s^2 \frac{1}{\mu^2} \right) d\langle M \rangle_s, \quad (6)
\end{aligned}$$

for any $\lambda, \mu > 0$.

Rearranging the previous equation (notice $\mathbf{E}\delta Y_0^2 < \infty$, always by Lemma 2.1), we have

$$\begin{aligned}
&(\beta - 2L - L\lambda^2 - \mu^2) \|\delta Y\|_{T,\beta}^2 + (1 - L\lambda^{-2}) \|\delta Z\|_{T,\beta}^2 + \|\delta N\|_{\beta}^2 \\
&\leq \mathbf{E}[e^{\beta\langle M \rangle_T} |\delta Y_T|^2] + \mu^{-2} \|\delta_2 f\|_{T,\beta}^2.
\end{aligned}$$

Now we conclude by choosing λ, μ , and K such that

$$1 - L\lambda^{-2} > 0, \quad \beta - 2L - L\lambda^2 - \mu^2 > 0,$$

$$K = \max \left\{ (1 - L\lambda^{-2})^{-1}, (\beta - 2L - L\lambda^2 - \mu^2)^{-1} \right\}. \quad (7)$$

We remark that, for $\beta > 2L + L^2$, it is always possible to choose λ and μ that verify (7). Proposition 2.2 is proved.

An easy consequence of the inequalities in Proposition 2.2 is a kind of continuity of the solutions with respect to the parameters. We say that (f_α, ξ_α) , $\alpha \in \mathbf{R}$, is a continuous family of β -standard parameters if for $\alpha \rightarrow \alpha_0$, we have

$$\xi_\alpha \rightarrow \xi_{\alpha_0} \quad \text{in } L_\beta^2, \quad f_\alpha(\cdot, Y_\bullet^{\alpha_0}, Z_\bullet^{\alpha_0}) \rightarrow f_{\alpha_0}(\cdot, Y_\bullet^{\alpha_0}, Z_\bullet^{\alpha_0}) \quad \text{in } \mathcal{H}_{T,\beta}^2.$$

Indeed, it is immediate to verify the following.

Corollary 2.1 (Stability). *Let (f_α, ξ_α) , $\alpha \in \mathbf{R}$, be a continuous family of β -standard parameters of the BSDE such that all the functions f_α are equi-Lipschitz with Lipschitz constant L . Suppose also that for each equation there exists a unique solution $(Y_\alpha, Z_\alpha, N_\alpha)$ in $\mathcal{H}_{T,\beta}^2 \times \mathcal{H}_{T,\beta}^2 \times \mathcal{L}_\beta^2$ for some $\beta > 2L + L^2$. Then, the map $\mathbf{R} \ni \alpha \mapsto (Y_\alpha, Z_\alpha, N_\alpha) \in \mathcal{H}_{T,\beta}^2 \times \mathcal{H}_{T,\beta}^2 \times \mathcal{L}_\beta^2$ is continuous.*

We now prove the main result: the existence and uniqueness theorem.

Theorem 2.1. *Let (f, ξ) be β -standard parameters, with $\beta > \beta_0(L)$, where*

$$\beta_0(L) = \begin{cases} 2\sqrt{2}L & \text{if } L \leq \frac{\sqrt{2}}{2}, \\ 2L^2 + 1 & \text{if } L > \frac{\sqrt{2}}{2}. \end{cases}$$

Then equation (1) has a unique solution (Y, Z, N) in the space $\mathcal{H}_{T,\beta}^2 \times \mathcal{H}_{T,\beta}^2 \times \mathcal{L}_\beta^2$.

Moreover, the random variable $\sup_t (e^{\beta\langle M \rangle_t/2} |Y_t|)$ is in $L^2(\Omega, \mathcal{F}, \mathbf{P})$.

P r o o f. We prove the thesis by using a fixed point theorem. Let us define the map

$$\psi: \mathcal{H}_{T,\beta}^2 \times \mathcal{H}_{T,\beta}^2 \times \mathcal{L}_\beta^2 \longrightarrow \mathcal{H}_{T,\beta}^2 \times \mathcal{H}_{T,\beta}^2 \times \mathcal{L}_\beta^2, \quad \psi(y, z, n) = (Y, Z, N),$$

where (Y, Z, N) solves

$$Y_t = \xi + \int_t^T f(s, y_s, z_s) d\langle M \rangle_s - \int_t^T Z_s dM_s - N_T + N_t. \quad (8)$$

By Proposition 2.1, ψ is well defined. We want to prove that ψ is a contraction on the space $\mathcal{H}_{T,\beta}^2 \times \mathcal{H}_{T,\beta}^2 \times \mathcal{L}_\beta^2$. We consider $(Y^i, Z^i, N^i) = \psi(y^i, z^i, n^i)$, $i = 1, 2$. According to Proposition 2.2 (notice that $\beta > \beta_0(L) \geq 2L + L^2$), we have, for all positive constants K and μ^2 such that $\beta > \mu^2$ and $K = \max\{1, (\beta - \mu^2)^{-1}\}$,

$$\begin{aligned}
\|\delta N\|_{\beta}^2 + \|\delta Y\|_{T,\beta}^2 + \|\delta Z\|_{T,\beta}^2 &\leq \frac{K}{\mu^2} \|f(\cdot, y_\bullet^1, z_\bullet^1) - f(\cdot, y_\bullet^2, z_\bullet^2)\|_{T,\beta}^2 \\
&\leq 2 \frac{K}{\mu^2} L^2 (\|\delta y\|_{T,\beta}^2 + \|\delta z\|_{T,\beta}^2).
\end{aligned}$$

This implies that ψ is contractive if we can choose the constants K and μ such that

$$\mu^2 < \beta, \quad K = \max\{1, (\beta - \mu^2)^{-1}\}, \quad 2 \frac{K}{\mu^2} L^2 < 1.$$

This is equivalent to looking for K and μ (given L and β) such that $\mu^2 > 2L^2$ and $\beta > \mu^2 + 2L^2/\mu^2$. Elementary calculations show that it is possible if and only if $\beta > \beta_0(L)$. Hence, in this case, there exists a unique fixed point for the map ψ in the space $\mathcal{H}_{T,\beta}^2 \times \mathcal{H}_{T,\beta}^2 \times \mathcal{L}_\beta^2$, which is the solution to (1). Theorem 2.1 is proved.

The linear case. As a particular Lipschitz case, we deal with the linear case. To state our result, it is worth reminding that the Doléans exponential $\mathcal{E}(X)$ of a local martingale X is a unique solution of the Doléans equation

$$\mathcal{E}_t(X) = 1 + \int_0^t \mathcal{E}_{s-}(X) dX_s;$$

this is a local martingale, given by $\mathcal{E}_t(X) = \exp(X_t - X_0 - \frac{1}{2} \langle X^c \rangle_t) \prod_{0 < s \leq t} (1 + \Delta X_s) e^{-\Delta X_s}$, where $\Delta X_s = X_s - X_{s-}$. We will use the Doléans exponential in order to apply a form of Girsanov's theorem; so we will need $\mathcal{E}(X)$ to be a positive uniformly integrable martingale. If $\Delta X_s > -1$ almost surely for all s , then the process $\mathcal{E}(X)$ is positive. Moreover, [11] provides conditions in order $\mathcal{E}(X)$ to be uniformly integrable (generalizing the Novikov condition, which holds when X is continuous).

We have the following representation for the first component of the solution.

Lemma 2.2 (Linear BSDE's). *Let a, b, c be predictable bounded processes, let $\mathcal{E}$ be the Doléans exponential of the martingale $(\int_0^t b_s dM_s)_{t \geq 0}$ and define $\gamma_t = \exp(\int_0^t a_s d\langle M \rangle_s)$, $\Gamma_t = \gamma_t \mathcal{E}_t$. Suppose that*

(i) $\mathcal{E}$ is a positive uniformly integrable martingale,

(ii) $\mathbf{E}[(\sup_t \gamma_t)^2 \mathcal{E}_T^2] < \infty$.

If the linear backward equation

$$dY_t = -(a_t Y_t + b_t Z_t + c_t) d\langle M \rangle_t + Z_t dM_t + dN_t, \quad Y_T = \xi, \quad (9)$$

has a solution (Y, Z, N) in $\mathcal{H}_T^2 \times \mathcal{H}_T^2 \times \mathcal{L}^2$, then Y is given by

$$Y_t = \mathbf{E} \left[\xi \frac{\Gamma_T}{\Gamma_t} + \int_t^T \frac{\Gamma_s}{\Gamma_t} c_s d\langle M \rangle_s \mid \mathcal{F}_t \right], \quad 0 \leq t \leq T. \quad (10)$$

P r o o f. We introduce the probability measure $\mathbf{Q}$ (equivalent to $\mathbf{P}$) which has density $\mathcal{E}_T$ with respect to $\mathbf{P}$ on $\mathcal{F}_T$. By Girsanov's theorem (see, e.g., [9, Theorem 13.19]), the $(\mathcal{F}_t)$ -adapted process $\widetilde{M}$ defined by $\widetilde{M}_t := M_t - \langle M, \int_0^t b_s dM_s \rangle_t = M_t - \int_0^t b_s d\langle M \rangle_s$ is a $\mathbf{Q}$ -martingale and $\langle \widetilde{M} \rangle = \langle M \rangle$. Notice that also N is a $\mathbf{Q}$ -martingale and it is orthogonal to $\widetilde{M}$ since $\langle N, M \rangle = 0$. Equation (9), under the measure $\mathbf{Q}$, becomes $dY_t = -(a_t Y_t + c_t) d\langle M \rangle_t + Z_t d\widetilde{M}_t + dN_t$, $Y_T = \xi$. So, we apply Itô's formula to the process γY and obtain

$$(\mathbf{Q}) \quad \gamma_t Y_t - \gamma_0 Y_0 + \int_0^t c_s \gamma_s d\langle M \rangle_s = \int_0^t \gamma_s Z_s d\widetilde{M}_s + \int_0^t \gamma_s dN_s. \quad (11)$$

By using condition (ii), one can easily see that the right-hand side of (11) is a $\mathbf{Q}$ -martingale, so the left-hand side too. We obtain, as usual, that $\gamma_t Y_t = \mathbf{E}^{\mathbf{Q}}[\xi \gamma_T + \int_t^T c_s \gamma_s d\langle M \rangle_s \mid \mathcal{F}_t]$, and consequently

$$\Gamma_t Y_t = \mathbf{E} \left[\xi \Gamma_T + \int_t^T c_s \Gamma_s d\langle M \rangle_s \mid \mathcal{F}_t \right].$$

This concludes the proof of Lemma 2.2.

Going back in the proof of the previous lemma, we can also see that, whenever the expression (10) makes sense, it is a solution of the linear equation (9) if the left-hand side of (11) is a $L^2(\mathbf{Q})$ martingale.

The linear case allows us to obtain a comparison result. Just consider two processes (Y^i, Z^i, N^i) in $\mathcal{H}_T^2 \times \mathcal{H}_T^2 \times \mathcal{L}^2$, $i = 1, 2$, verifying

$$Y_t^i = \xi_i + \int_t^T f_i(s, Y_s^i, Z_s^i) d\langle M \rangle_s - \int_t^T Z_s^i dM_s + N_t^i - N_T^i.$$

We define $\delta Y_t = Y_t^1 - Y_t^2$ and $\delta Z_t = Z_t^1 - Z_t^2$, and introduce the processes

$$\begin{aligned}\Delta_Y f_t^1 &= \frac{f_1(t, Y_t^1, Z_t^1) - f_1(t, Y_t^2, Z_t^1)}{\delta Y_t} \mathbf{1}_{\delta Y_t \neq 0}, \\ \Delta_Z f_t^1 &= \frac{f_1(t, Y_t^2, Z_t^1) - f_1(t, Y_t^2, Z_t^2)}{\delta Z_t} \mathbf{1}_{\delta Z_t \neq 0}.\end{aligned}$$

Then it is easy to verify that the process δY solves the following linear BSDE:

$$-d\delta Y_t = (\Delta_Y f_t^1 \delta Y_t + \Delta_Z f_t^1 \delta Z_t + \delta_2 f_t) d\langle M \rangle_t - \delta Z_t dM_t - d\delta N_t, \quad \delta Y_T = \xi_1 - \xi_2.$$

According to Lemma 2.2, if the coefficients $\Delta_Y f^1$ and $\Delta_Z f^1$ verify conditions (i) and (ii), we have that

$$\Gamma_t \delta Y_t = \mathbf{E} \left[\Gamma_T (\xi_1 - \xi_2) + \int_t^T \delta_2 f_s \Gamma_s d\langle M \rangle_s \mid \mathcal{F}_t \right]. \quad (12)$$

Assume now that $\xi_1 \geq \xi_2$ and, for any t , $\delta_2 f_t \geq 0$ $\mathbf{P}$ -a.s. Then, for any t , $Y_t^1 \geq Y_t^2$ $\mathbf{P}$ -a.s.

We have proved the following comparison result.

Theorem 2.2. *Let (Y^1, Z^1, N^1) , (Y^2, Z^2, N^2) be as before and suppose the coefficients $\Delta_Y f^1$ and $\Delta_Z f^1$ verify conditions (i) and (ii) of Lemma 2.2. Assume that $\xi_1 \geq \xi_2$ and, for any t , $\delta_2 f_t(\omega) \geq 0$ $\mathbf{P}$ -a.s. Then, for any t , $Y_t^1 \geq Y_t^2$ $\mathbf{P}$ -a.s.*

E x a m p l e. Finally, we would like to consider an easy linear equation

$$dY_t = -aY_t d\langle M \rangle_t + Z_t dM_t + dN_t, \quad Y_T = \xi, \quad (13)$$

with a constant and ξ in L^2 . We present a toy model, to understand how the conditions in Theorem 2.1 naturally arise in this case. We refer to the literature for examples of linear equations more related to applications (see [7] for the equation driven by a Brownian motion and [3], [14] for the equation driven by a semimartingale).

Here, it is quite easy to understand which are the natural links between the integrability conditions on the parameter ξ and on the solution Y .

We have $\Gamma_t = \gamma_t = e^{a\langle M \rangle_t}$, so the proof above works if $\mathbf{E} e^{2a\langle M \rangle_T} < \infty$ (condition (ii)). Then it is immediate to verify that $Y_t = \mathbf{E}[\xi e^{a(\langle M \rangle_T - \langle M \rangle_t)} \mid \mathcal{F}_t]$ is a unique process in $\mathcal{H}_T^2$ verifying the equation, when ξ is in L_{2a}^2 ; this is required since

$$\|Y\|_T^2 \leq \mathbf{E} \left[\xi^2 \int_0^T e^{2a(\langle M \rangle_T - \langle M \rangle_t)} d\langle M \rangle_t \right] \leq \frac{1}{2a} \mathbf{E} \left[\xi^2 e^{2a\langle M \rangle_T} \right].$$

Moreover,

$$\begin{aligned}\|Y\|_{T,\beta}^2 &\leq \mathbf{E} \left[\xi^2 e^{2a\langle M \rangle_T} \int_0^T e^{(\beta-2a)\langle M \rangle_t} d\langle M \rangle_t \right] \\ &\begin{cases} \leq (\beta - 2a)^{-1} \mathbf{E}[\xi^2 e^{\beta\langle M \rangle_T}] & \text{if } \beta > 2a; \\ \leq (2a - \beta)^{-1} \mathbf{E}[\xi^2 e^{2a\langle M \rangle_T}] & \text{if } \beta < 2a; \\ = \mathbf{E}[\xi^2 e^{2a\langle M \rangle_T} \langle M \rangle_T] & \text{if } \beta = 2a. \end{cases}\end{aligned}$$

This implies that, if ξ is in L_β^2 for some $\beta > 2a$, then Y is in $\mathcal{H}_{T,\beta}^2$, while, if ξ is in L_{2a}^2 , then Y is in $\mathcal{H}_{T,\beta}^2$ at least for all $\beta < 2a$.

3. Nonuniform Lipschitz generators and multidimensional case. We consider now the case of a nonuniform Lipschitz generator in the multidimensional context, with $M = (M_t)_{t \in [0,T]}$ a square integrable $\mathbf{R}^n$ -valued martingale; we will recover some of the results obtained in the previous section and explain why the same proofs work also in these more general conditions.

We adopt the same formulation of the problem used in [6]. We consider the same equation (14) and the same spaces, and we will see that our proofs guarantee the existence and uniqueness of the solution under weaker integrability conditions of the parameters.

$[M]$ and $\langle M \rangle$ are processes taking values in $\mathbf{R}^{n \times n}$. Since the filtration is quasi-left continuous, we can consider a continuous increasing process C , the one appearing in the

BSDE (14), such that $\langle M \rangle_t = \int_0^t m_s m_s^* dC_s$, where m is $\mathbf{R}^{n \times n}$ -valued predictable process. By $*$ we denote the transposition. The BSDE can now be written as

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) dC_s - \int_t^T Z_s^* dM_s - N_T + N_t, \quad t \in [0, T], \quad (14)$$

with an $\mathbf{R}^d$ -valued terminal condition ξ (which is $\mathcal{F}_T$ -measurable) and generator $f: \Omega \times [0, T] \times \mathbf{R}^d \times \mathbf{R}^{n \times d} \rightarrow \mathbf{R}^d$, which is an $\mathcal{F} \otimes \mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbf{R}^d) \otimes \mathcal{B}(\mathbf{R}^{n \times d})$ -measurable map. We use $|\cdot|$ to denote the Euclidean norm of vectors and matrices; if z is in $\mathbf{R}^{n \times d}$, its norm is defined by $|z| = \sqrt{\text{trace}(z^* z)}$.

We suppose f satisfies a Lipschitz condition of the type

$$|f(t, y_1, z_1) - f(t, y_2, z_2)| \leq r_t |y_1 - y_2| + \theta_t |m_t^* (z_1 - z_2)| \quad dC_t \otimes d\mathbf{P}\text{-a.s.}, \quad (15)$$

where r_t and θ_t are two positive predictable processes. We denote $\alpha_t^2 = r_t + \theta_t^2 > 0$ and $A_t = \int_0^t \alpha_s^2 dC_s$.

The natural spaces for the solutions of the BSDE (14) will be now (we use the hat $\hat{\cdot}$ to distinguish from the ones of the previous section):

$\hat{\mathcal{H}}_{T,\beta}^2$, $\beta > 0$, the subspace of the $\mathbf{R}^d$ -valued adapted processes ϕ with $|\phi|_{T,\beta}^2 = \mathbf{E} \int_0^T e^{\beta A_t} |\phi_t|^2 dC_t < \infty$;

$\hat{\mathcal{L}}_\beta^2$ the space of L^2 bounded $\mathbf{R}^d$ -valued martingales N such that $|N|_\beta^2 := \mathbf{E} \int_0^T e^{\beta A_t} d\text{trace}[N]_t < +\infty$.

We will denote by $|Z|_{\beta,T}^2$ the norm of the martingale $Z \cdot M$, i.e., $|Z|_{\beta,T}^2 := \mathbf{E} \int_0^T e^{\beta A_t} |m_t^* Z_t|^2 dC_t$.

The parameters of the equation are β -standard if (i) ξ is in the space $\hat{L}_\beta^2$ of the $\mathcal{F}_T$ -measurable $\mathbf{R}^d$ -valued random vectors such that $|\xi|_\beta^2 = \mathbf{E}[e^{\beta A_T} |\xi|^2] < \infty$; (ii) f satisfies (15) and is such that $|\alpha^{-1} f(\cdot, 0, 0)|_{T,\beta}$ is finite.

The solution (Y, Z, N) will be a process in the space

$$\begin{aligned} \mathcal{S}_\beta = \Big\{ (Y, Z, N): (\alpha Y), m^* Z \in \hat{\mathcal{H}}_{T,\beta}^2, Z \text{ is predictable,} \\ N \in \hat{\mathcal{L}}_\beta^2, \langle N, M \rangle = 0, N_0 = 0 \Big\}, \end{aligned}$$

$$|(Y, Z, N)|_\beta^2 = |\alpha Y|_{T,\beta}^2 + |Z|_{\beta,T}^2 + |N|_\beta^2.$$

We underline that the only differences with respect to the environment in [6] is that the nonuniform Lipschitz condition (15) is slightly different, but, as we said in the Introduction, it is necessary to prove the main result in the multidimensional case.

With respect to the previous section, the main difference is that relaxing the Lipschitz condition requires to use norms (for the parameters and the solution) which depend also on the «Lipschitz» processes, while before the Lipschitz constant simply influenced the choice of β . In this new context, the main result can be written in the following form.

Theorem 3.1. *Let (f, ξ) be β -standard parameters with $\beta > 3$. Then equation (14) has a unique solution (Y, Z, N) in the space $\mathcal{S}_\beta$. Moreover, the random variable $\sup_t (e^{\beta A_t/2} |Y_t|)$ is in $L^2(\Omega, \mathcal{F}, P)$.*

R e m a r k. This assures better results than the corresponding existence and uniqueness theorem in [6] since there the same result was proved only for $\beta > 90$ (see Theorem 6.1 in [6] and write explicitly the condition «for β large enough»).

For the proof, we simply have to revisit the results of the previous section:

— Proposition 2.1 does not change and so also Lemma 2.1, which is a quite direct consequence of the previous proposition;

— For the a priori estimates, and so the existence and uniqueness theorem, the previous proof works; only we need to rewrite these results in the new notation: the norms have changed, and so the choice of β will change. We sketch here briefly the same steps in the new context.

Proposition 3.1 (A priori estimates). *Let (f_1, ξ_1) and (f_2, ξ_2) be two β -standard parameters with $\beta > 3$ and suppose that there exist the corresponding solutions (Y^1, Z^1, N^1) and (Y^2, Z^2, N^2) of (14), respectively, in the space $\mathcal{S}_\beta$. Then, for some positive constants K and μ , $|(\delta Y, \delta Z, \delta N)|_\beta^2 \leq K[|\xi_1 - \xi_2|_\beta^2 + \mu^{-2}|\alpha^{-1}\delta_2 f|_{T,\beta}^2]$.*

P r o o f. Now we have to write Itô's formula for $e^{\beta A_s}|\delta Y_s|^2$; integrating in time and taking expectation we find

$$\begin{aligned} & \mathbf{E} \left(e^{\beta A_T} |\delta Y_T|^2 \right) - \mathbf{E} |\delta Y_0|^2 \\ &= \mathbf{E} \int_0^T e^{\beta A_s} \left\{ \beta \alpha_s^2 |\delta Y_s|^2 dC_s - 2\delta Y_s^* (f_1(s, Y_s^1, Z_s^1) - f_2(s, Y_s^2, Z_s^2)) dC_s \right. \\ & \quad \left. + |m_s \delta Z_s^*|^2 dC_s + d \operatorname{trace}[\delta N]_s + 2\delta Y_s^* \delta Z_s^* dM_s + 2\delta Y_s^* d\delta N_s \right\}. \end{aligned}$$

The last two terms vanish, since the time integrals are martingales, so we have

$$\begin{aligned} & \beta |\alpha \delta Y|_{T,\beta}^2 + |\delta Z^*|_{\beta,T}^2 + |\delta N|_\beta^2 \\ & \leq \mathbf{E} [e^{\beta A_T} |\delta Y_T|^2] + 2\mathbf{E} \left[\int_0^T e^{\beta A_s} \delta Y_s^* (f_1(s, Y_s^1, Z_s^1) - f_2(s, Y_s^2, Z_s^2)) dC_s \right] \\ & \leq |\xi_1 - \xi_2|_\beta^2 + 2\mathbf{E} \left[\int_0^T e^{\beta A_s} |\delta Y_s| \left(r_s |\delta Y_s| + \theta_s |m_s^* \delta Z_s| + |\delta_2 f_s| \right) dC_s \right] \\ & \leq |\xi_1 - \xi_2|_\beta^2 + \mathbf{E} \left[\int_0^T e^{\beta A_s} \left(|\delta Y_s|^2 (2r_s + \theta_s^2 \lambda^2 + \alpha_s^2 \mu^2) \right. \right. \\ & \quad \left. \left. + |m_s^* \delta Z_s|^2 \lambda^{-2} + \mu^{-2} |\alpha_s^{-1} \delta_2 f_s|^2 \right) dC_s \right], \quad (16) \end{aligned}$$

for any $\lambda, \mu > 0$. So $(\beta - 2 - \lambda^2 - \mu^2) |\alpha \delta Y|_{T,\beta}^2 + (1 - \lambda^{-2}) |\delta Z|_{\beta,T}^2 + |\delta N|_\beta^2 \leq |\xi_1 - \xi_2|_\beta^2 + \mu^{-2} |\alpha^{-1} \delta_2 f|_{T,\beta}^2$.

For $\beta > 3$, we can choose λ, μ , and K such that

$$1 - \lambda^{-2} > 0, \quad \beta - 2 - \lambda^2 - \mu^2 > 0,$$

$$K = \max\{(1 - \lambda^{-2})^{-1}, (\beta - 2 - \lambda^2 - \mu^2)^{-1}\}. \quad (17)$$

P r o o f o f T h e o r e m 3.1. Following the scheme of Theorem 2.1 in the previous section, we define a function ψ in the same way. In order to prove that ψ is a contraction, we just take the first inequality in (16) and remember that, in this case, it becomes

$$\begin{aligned} & \beta |\alpha \delta Y|_{T,\beta}^2 + |\delta Z^*|_{\beta,T}^2 + |\delta N|_\beta^2 \leq 2\mathbf{E} \int_0^T e^{\beta A_s} \delta Y_s^* (f(s, y_s^1, z_s^1) - f(s, y_s^2, z_s^2)) dC_s \\ & \leq 2\mathbf{E} \int_0^T e^{\beta A_s} \delta Y_s^* (r_s |\delta y_s| + \theta_s |m_s^* \delta z_s|) dC_s \\ & \leq \mathbf{E} \int_0^T e^{\beta A_s} (2\mu^2 \alpha_s^2 |\delta Y_s|^2 + \alpha_s^2 \mu^{-2} |\delta y_s|^2 + \mu^{-2} |m_s^* \delta z_s|^2) dC_s, \end{aligned}$$

that is $(\beta - 2\mu^2) |\alpha \delta Y|_{\beta,T}^2 + |\delta Z^*|_{\beta,T}^2 + |\delta N|_\beta^2 \leq \mu^{-2} (|\alpha \delta y|_{\beta,T}^2 + |\delta z|_{\beta,T}^2)$.

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CONTRACTIVE APPROXIMATIONS FOR THE VARADHAN'S FUNCTION ON A FINITE MARKOV CHAIN¹⁾

В работе рассматриваются марковские цепи с конечным пространством состояний. Для заданной на этом пространстве состояний вещественнозначной функции соответствующая функция Варадана, характеризующая экспоненциальную скорость роста суммарной цены, является

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