Math-Net.Ru

Общероссийский математический портал

E. Clement, Bias correction for the estimation of discretized diffusion processes from an approximated likelihood function, *Теория вероятн. и ее примен.*, 1997, том 42, выпуск 2, 364–370

DOI: <https://doi.org/10.4213/tvp1810>

Использование Общероссийского математического портала Math-Net.Ru подразумевает, что вы прочитали и согласны с пользовательским соглашением
<http://www.mathnet.ru/rus/agreement>

Параметры загрузки:

IP: 54.162.127.20

26 апреля 2023 г., 14:07:54

7. *Drewnowski L.* Boundedness of vector measures with values in the spaces L_0 of Bochner measurable function. — Proc. Amer. Math. Soc., 1984, v. 91, № 4, p. 581–588.
8. *Drewnowski L.* Topological rings of sets, continuous set functions, integration. I. — Bull. Acad. Polon. Sci. Ser. Sci. Math. Astr. Phys., 1972, v. 20, № 4, p. 269–276.

Поступила в редакцию
23.VI.1995

© 1997 г.

CLEMENT E.*

BIAS CORRECTION FOR THE ESTIMATION OF DISCRETIZED DIFFUSION PROCESSES FROM AN APPROXIMATED LIKELIHOOD FUNCTION

Проводится оценивание параметров дискретизированного диффузионного процесса, основанное на аппроксимации функции правдоподобия. Предполагается, что шаг между двумя последовательными наблюдениями процесса фиксирован. Для корректировки смещения оценок предлагается использовать имитации процесса.

Ключевые слова и фразы: диффузионный процесс, схема дискретизации, схема Эйлера, асимптотические свойства оценок.

1. Notation. We consider the univariate diffusion process (X_t) , the solution of the stochastic differential equation:

$$dX_t = b(\theta, X_t) dt + a(\theta, X_t) dW_t, \quad X_0 = x_0, \quad (1)$$

where W_t is a standard brownian motion, θ is a parameter belonging to a compact subset Θ of $\mathbf{R}^p$, a and b are two known functions. We are interested in estimating θ from discrete observations of the process $(X_{t_i})_{0 \leq i \leq n}$, with a step between two consecutive observations $\Delta = t_{i+1} - t_i$ fixed. Since the transition probability of the homogeneous Markov process (X_{t_i}) has an untractable form (see [1]), we will not estimate θ by a maximum likelihood method but we will maximize an approximation of the likelihood function. Since we assume that the observation step is fixed, such estimators are biased (see [4]) and we propose here to correct the bias of these estimators from simulations of the process following the method proposed by Gouriéroux–Monfort–Renault [2].

We denote by θ_0 the true parameter value. We assume that a and b are smooth enough such that equation (1) admits a unique weak solution for the true value θ_0 of parameter (see [5]) and we denote by $P_{x_0}^{\theta_0}$ the distribution of this solution on the space of continuous functions from $\mathbf{R}^+$ into $\mathbf{R}$ equipped with the canonical filtration. P^W stands for the brownian measure. We denote by $p_\Delta^\theta(x, y)$ and $\pi^\theta(x)$ respectively the densities of the transition probability at step Δ and the invariant measure of the process solution of (1) for any parameter value θ .

From the observations $(X_{t_i})_{0 \leq i \leq n}$ of the process, we construct a first step estimator:

$$F_n = \arg \min_{\beta \in \Theta} L_n(\beta), \quad (2)$$

*E.N.S.A.E. 3, Avenue Pierre Larousse, 92245 Malakoff Cedex, France.

with

$$L_n(\beta) = \frac{1}{n} \sum_{i=0}^{n-1} \left(\log a(\beta, X_{t_i}) + \frac{(X_{t_{i+1}} - X_{t_i} - b(\beta, X_{t_i}) \Delta)^2}{2a(\beta, X_{t_i})^2} \right).$$

This first step estimator F_n is obtained by minimizing the opposite of the likelihood function of the Euler approximation at step Δ of the process solution of (1), defined by

$$X_{t_{i+1}} - X_{t_i} = b(\theta, X_{t_i}) \Delta + a(\theta, X_{t_i}) (W_{t_{i+1}} - W_{t_i}).$$

Since this Euler approximation converges in distribution to the process solution of (1) as Δ tends to zero, the contrast $L_n(\beta)$ can be viewed as an approximation of the likelihood function.

We note

$$L(\theta, \beta) = \int \log a(\beta, x) \pi^\theta(x) dx + \int \int \frac{(y - x - b(\beta, x) \Delta)^2}{2a(\beta, x)^2} p_\Delta^\theta(x, y) \pi^\theta(x) dy dx.$$

We assume that for all θ in Θ , the function $L(\theta, \cdot)$ admits $F(\theta)$ as its unique minimum:

$$F(\theta) = \arg \min_{\beta \in \Theta} L(\theta, \beta).$$

Finally, we denote by $l(\beta, x, y)$ the function

$$l(\beta, x, y) = \log a(\beta, x) + \frac{(y - x - b(\beta, x) \Delta)^2}{2a(\beta, x)^2}.$$

We make the following hypotheses:

H₀: Θ is compact and θ_0 is an interior point of Θ .

H₁: $x \mapsto b(\theta, x)$ and $x \mapsto a(\theta, x)$ are C^∞ for each θ in Θ and all their derivatives are bounded; a is bounded; $a(\theta, x) > a > 0$ for each (θ, x) in $\Theta \times \mathbf{R}$; there exists a compact subset K and a strictly positive real γ such that $xb(\theta, x) \leq -\gamma x^2$, for all x not in K .

H₂: $\theta \mapsto b(\theta, x)$ and $\theta \mapsto a(\theta, x)$ are C^3 for each x in $\mathbf{R}$ and their derivatives have at most polynomial growth with respect to x uniformly in θ ; $(\theta, \beta) \mapsto (\partial L / \partial \beta)(\theta, \beta)$ is C^2 ; F is C^2 ; the matrices $DF(\theta_0)$ and $(\partial^2 L / \partial \beta^2)(\theta_0, F(\theta_0))$ are positive definite.

H₃: $F(\theta) = F(\theta_0)$ if and only if $\theta = \theta_0$.

H₄: F is a C^1 -diffeomorphism.

We have the following result for the first step estimator defined by (2).

Proposition 1. Under H_0 , H_1 and H_2 , the first step estimator F_n converges almost surely under $P_{x_0}^{\theta_0}$ to $F(\theta_0)$.

P r o o f. From the ergodicity assumption H_1 , we obtain the almost sure convergence under $P_{x_0}^{\theta_0}$ of $L_n(\beta)$ to $L(\theta_0, \beta)$ for each β in Θ . The Lipschitz conditions on the coefficients a and b of the process (assumption H_2) imply the almost sure convergence of $\sup_\beta |(\partial L_n / \partial \beta)(\beta)|$, and this implies in particular that $\sup_\beta |(\partial L_n / \partial \beta)(\beta)|$ is bounded independently of n , almost surely. We deduce then the almost sure equicontinuity of the family of functions $L_n(\beta)$ and as a consequence of Ascoli's theorem the almost sure uniform convergence of $L_n(\beta)$ for β in Θ . Since $F(\theta_0)$ is the unique minimum point of $L(\theta_0, \cdot)$ we obtain the almost sure convergence of F_n to $F(\theta_0)$.

This first step estimator is then biased and we will see in the next sections how to correct this estimator.

E x a m p l e. Consider the Ornstein-Uhlenbeck process, the solution of

$$dX_t = -\mu X_t dt + \sigma dW_t, \quad X_0 = 0, \quad (3)$$

where $\theta = (\mu, \sigma^2)^t$ is the unknown parameter value. We suppose that μ and σ are strictly positive. The process X_t is then ergodic, π^θ is the density of the normal distribution

$\mathcal{N}(0, \sigma^2/(2\mu))$ and $p_\Delta^\theta(x, \cdot)$ is the density of the normal distribution $\mathcal{N}(\exp(-\mu\Delta)x, \sigma^2(1 - \exp(-2\mu\Delta))/(2\mu))$. For this process, we can estimate θ by a maximum likelihood method but it is no more the case for more complicated processes. Suppose then that we estimate θ by minimizing in $\beta = (\beta_1, \beta_2)^t$ the following contrast:

$$\frac{1}{n} \sum_{i=0}^{n-1} \left[\frac{1}{2} \log(2\pi\beta_2\Delta) + \frac{1}{2\beta_2\Delta} (X_{t_{i+1}} - (1 - \beta_1\Delta)X_{t_i})^2 \right], \quad (4)$$

which is the opposite of the likelihood function of the Euler discretization scheme of the process (X_{t_i}) solution of (3):

$$X_{t_{i+1}} - X_{t_i} = -\mu X_{t_i}\Delta + \sigma(W_{t_{i+1}} - W_{t_i}). \quad (5)$$

We obtain a first step estimator $F_n = (\bar{\mu}_n, \bar{\sigma}_n^2)^t$ which is biased if we assume Δ fixed. We have

$$\lim_{n \rightarrow +\infty} F_n = F(\theta) = \begin{pmatrix} \frac{1 - \exp(-\mu\Delta)}{\Delta} \\ \frac{\sigma^2}{2\mu\Delta} (1 - \exp(-2\mu\Delta)) \end{pmatrix}.$$

2. An auxiliary estimator. We will first consider that the function $F(\theta)$ is known (this is the case for the Ornstein-Uhlenbeck process) and define the estimator

$$\hat{\theta}_n = \arg \min_{\theta \in \Theta} \|F_n - F(\theta)\|, \quad (6)$$

where $\|\cdot\|$ denotes the Euclidean norm. This estimator has just a theoretical interest since the function $F(\theta)$ depends on the transition probability of the process and is generally unknown. We have the result:

Proposition 2. Under H_0 - H_3 , we have

- i) $\lim_{n \rightarrow +\infty} \hat{\theta}_n = \theta_0$ a.s. under $P_{x_0}^{\theta_0}$,
- ii) $\sqrt{n}(\hat{\theta}_n - \theta_0) \Rightarrow \mathcal{N}(0, DF(\theta_0)^{-1} \tilde{\Sigma} DF^t(\theta_0)^{-1})$, $n \rightarrow +\infty$, with

$$\begin{aligned} \tilde{\Sigma} &= \frac{\partial^2 L}{\partial \beta^2}(\theta_0, F(\theta_0))^{-1} \Sigma^0 \left(\frac{\partial l}{\partial \beta} \right) \frac{\partial^2 L}{\partial \beta^2}(\theta_0, F(\theta_0))^{-1}, \\ \Sigma^0(f) &= \iint (f(x, y) - \mathbb{E}_{p_{\Delta}^{\theta_0} \pi^{\theta_0}} f) (f(x, y) - \mathbb{E}_{p_{\Delta}^{\theta_0} \pi^{\theta_0}} f)^t p_{\Delta}^{\theta_0}(x, y) \pi^{\theta_0}(x) dy dx \\ &\quad + 2 \int \left(\int (f(x, y) - \mathbb{E}_{p_{\Delta}^{\theta_0} \pi^{\theta_0}} f) p_{\Delta}^{\theta_0}(x, y) dy \right) G^t(x) \pi^{\theta_0}(x) dx, \\ G(x) &= \sum_{k \geq 1} \int (f(x, y) - \mathbb{E}_{p_{\Delta}^{\theta_0} \pi^{\theta_0}} f) p_{k\Delta}^{\theta_0}(x, y) dy. \end{aligned}$$

P r o o f. i) From Proposition 1, we have

$$\lim_{n \rightarrow +\infty} F_n = F(\theta_0) \quad \text{a.s.}$$

We can deduce that

$$\lim_{n \rightarrow +\infty} \|F_n - F(\theta)\| = \|F(\theta_0) - F(\theta)\| \quad P_{x_0}^{\theta_0}\text{-a.s.}$$

Moreover the family of functions $(\|F_n - F(\theta)\|)$ is equicontinuous since from H_0 and H_2 it follows

$$\left| \|F_n - F(\theta)\| - \|F_n - F(\theta')\| \right| \leq \sup_{\theta \in \Theta} \|DF(\theta)\| \|\theta - \theta'\|.$$

This implies the almost sure convergence under $P_{x_0}^{\theta_0}$ of $\|F_n - F(\theta)\|$ as n tends to infinity. From H_3 , the limit function $\|F(\theta_0) - F(\theta)\|$ has its minimum at the unique point θ_0 , this implies the almost sure convergence of $\hat{\theta}_n$.

ii) We denote by $\varphi_n(\theta)$ the function $(F_n - F(\theta))^t(F_n - F(\theta))$. Since $\varphi_n(\theta)$ is $\mathcal{C}^2(H_2)$ and θ_0 belongs to the interior of $\Theta(H_0)$, we can expand it around θ_0 , on the set $\{\hat{\theta}_n \in \Theta\}$, whose probability tends to 1 since from i) $\hat{\theta}_n$ tends to θ_0 almost surely, we have then

$$D\varphi_n(\hat{\theta}_n) = D\varphi_n(\theta_0) + \int_0^1 D^2\varphi_n(\theta_0 + u(\hat{\theta}_n - \theta_0)) du \cdot (\hat{\theta}_n - \theta_0),$$

with $D\varphi_n(\hat{\theta}_n) = 0$.

This expansion can be expressed in function of F and its differentials of order one and two, we obtain

$$\begin{aligned} DF(\theta_0)^t \cdot (F_n - F(\theta_0)) &= \int_0^1 DF^t(\theta_0 + u(\hat{\theta}_n - \theta_0)) DF(\theta_0 + u(\hat{\theta}_n - \theta_0)) \cdot (\hat{\theta}_n - \theta_0) du \\ &+ \int_0^1 D^2F(\theta_0 + u(\hat{\theta}_n - \theta_0)) \cdot (F_n - F(\theta_0 + u(\hat{\theta}_n - \theta_0)), \hat{\theta}_n - \theta_0) du. \end{aligned}$$

Since F is $\mathcal{C}^2$ and Θ is compact and since from i) $\hat{\theta}_n$ converges to θ_0 almost surely we obtain the convergence by Lebesgue's theorem:

$$\begin{aligned} \lim_{n \rightarrow +\infty} \int_0^1 DF^t(\theta_0 + u(\hat{\theta}_n - \theta_0)) DF(\theta_0 + u(\hat{\theta}_n - \theta_0)) du &= DF^t(\theta_0) DF(\theta_0), \\ \lim_{n \rightarrow +\infty} \int_0^1 D^2F(\theta_0 + u(\hat{\theta}_n - \theta_0)) \cdot (F_n - F(\theta_0 + u(\hat{\theta}_n - \theta_0)), 1) du &= 0. \end{aligned}$$

It suffices to prove the asymptotic normality of $\sqrt{n}(F_n - F(\theta_0))$ with

$$F_n = \arg \min_{\beta} L_n(\beta).$$

An expansion of the derivative with respect to β of the function $L_n(\beta)$ around $F(\theta_0)$ gives

$$\int_0^1 \frac{\partial^2 L_n}{\partial \beta^2} \left(F(\theta_0) + u(F_n - F(\theta_0)) \right) du \cdot (F_n - F(\theta_0)) = -\frac{\partial L_n}{\partial \beta} (F(\theta_0)).$$

From assumption H_2 , the function $(\partial l / \partial \beta)(\beta, x, y)$ belongs to $L^2(p_{\Delta}^{\theta_0} \pi^{\theta_0})$. Applying the result of Dacunha-Castelle and Florens-Zmirou [1], this function satisfies a central limit theorem and we have

$$\sqrt{n} \frac{\partial L_n}{\partial \beta} (\theta_0, F(\theta_0)) \Rightarrow \mathcal{N} \left(0, \Sigma^0 \left(\frac{\partial l}{\partial \beta} \right) \right), \quad n \rightarrow +\infty,$$

where $\Sigma^0(\partial l / \partial \beta)$ is defined in Proposition 2. Moreover we have the almost sure convergence of $\sup_{\beta} |(\partial^2 L_n / \partial \beta^2)(\beta)|$ from the Lipschitz conditions on derivatives of order one and two of coefficients a and b . We deduce then

$$\lim_{n \rightarrow +\infty} \int_0^1 \frac{\partial^2 L_n}{\partial \beta^2} \left(F(\theta_0) + u(F_n - F(\theta_0)) \right) du = \frac{\partial^2 L}{\partial \beta^2} (F(\theta_0)).$$

Finally we obtain

$$\sqrt{n}(F_n - F(\theta_0)) \Rightarrow \mathcal{N}(0, \tilde{\Sigma}), \quad n \rightarrow +\infty,$$

where $\tilde{\Sigma}$ is defined in Proposition 2 and Proposition 2 is proved.

3. Simulated estimator. Since $F(\theta)$ is unknown we will approximate it from simulated observations. For a value of parameter θ , we will consider the process $(X_{t,h}^{\theta,s})$ solution of the Euler discretization scheme with a discretization step equal to Δ/h :

$$X_{t+\Delta/h,h}^{\theta,s} - X_{t,h}^{\theta,s} = b(\theta, X_{t,h}^{\theta,s}) \frac{\Delta}{h} + a(\theta, X_{t,h}^{\theta,s}) (W_{t+\Delta/h}^s - W_t^s), \quad X_{0,h}^{\theta,s} = x, \quad (7)$$

where W^s is a brownian motion independent of W . We can then construct N simulated observations $(X_{t,h}^{\theta,s})_{0 \leq t \leq N}$. From these simulated observations, we define a first step simulated estimator $F_N^h(\theta)$:

$$F_N^h(\theta) = \arg \min_{\beta \in \Theta} L_N^h(\theta, \beta),$$

where

$$L_N^h(\theta, \beta) = \frac{1}{N} \sum_{i=0}^{N-1} \left(\log a(\beta, X_{t_i,h}^{\theta,s}) + \frac{(X_{t_{i+1},h}^{\theta,s} - X_{t_i,h}^{\theta,s} - b(\beta, X_{t_i,h}^{\theta,s})\Delta)^2}{2a(\beta, X_{t_i,h}^{\theta,s})^2} \right).$$

In practice, such trajectories of the process can be simulated for a finite number of parameter values. We consider then a regular grid Θ_m of Θ made of m^p points and we define the simulated estimator as follows:

$$\hat{\theta}_{n,N}^{m,h} = \arg \min_{\theta \in \Theta_m} \|F_n - F_N^h(\theta)\|.$$

We will then determine m , N and h as functions of n such that the simulated estimator $\hat{\theta}_{n,N}^{m,h}$ inherits the same asymptotic properties as the auxiliary estimator $\hat{\theta}_n$.

Proposition 3. Let $m(n)$ be a sequence such that

$$\lim_{n \rightarrow +\infty} \frac{\sqrt{n}}{m(n)} = 0,$$

then, under H_0 , H_1 , H_2 and H_4 there exist sequences $h(n)$ and $N(n)$ such that

$$\sqrt{n} \left(\hat{\theta}_{n,N(n)}^{m(n),h(n)} - \hat{\theta}_n \right) \rightarrow 0 \text{ in probability under } P_{x_0}^{\theta_0} \otimes P^W \otimes \delta_x, \quad n \rightarrow +\infty.$$

From Propositions 3 and 2, we deduce immediately the corollary.

Corollary. Under hypotheses of Proposition 3, there exist $N(n)$ and $h(n)$ such that

$$\lim_{n \rightarrow +\infty} \hat{\theta}_{n,N(n)}^{m(n),h(n)} = \theta_0 \text{ in probability under } P_{x_0}^{\theta_0} \otimes P^W \otimes \delta_x, \\ \sqrt{n} \left(\hat{\theta}_{n,N(n)}^{m(n),h(n)} - \theta_0 \right) \Rightarrow \mathcal{N} \left(0, DF(\theta_0)^{-1} \tilde{\Sigma} DF(\theta_0)^{-1} \right), \quad n \rightarrow +\infty,$$

where $\tilde{\Sigma}$ is defined in Proposition 2.

Proof of Proposition 3. The proof of Proposition 3 is based on the following lemma.

Lemma. If $\|F_{N(n)}^h(\theta) - F(\theta)\| \leq 1/m(n)$ for every $\theta \in \Theta_m$ then, under H_0 - H_4 , we have almost surely for n large enough

$$\|\hat{\theta}_{n,N(n)}^{m(n),h(n)} - \hat{\theta}_n\| \leq \frac{C}{m(n)},$$

where C is a constant independent of n .

P r o o f. From H_4 , we obtain by a Taylor expansion of F^{-1} the following inequality:

$$\|\hat{\theta}_{n,N(n)}^{m(n)} - \hat{\theta}_n\| \leq C \|F(\hat{\theta}_{n,N(n)}^{m(n)}) - F(\hat{\theta}_n)\|,$$

where C is a constant. Since F_n converges almost surely to $F(\theta_0)$ (Proposition 1), there exists n_0 (with $n_0 < \infty$ almost surely) such that $(\forall n \geq n_0)$ F_n is in $F(\Theta)$, from H_4 it yields $F(\hat{\theta}_n) = F_n$, and then

$$\|\hat{\theta}_{n,N(n)}^{m(n)} - \hat{\theta}_n\| \leq C \|F_n - F(\hat{\theta}_{n,N(n)}^{m(n)})\| \leq C \left(\frac{1}{m(n)} + \|F_n - F_{N(n)}(\hat{\theta}_{n,N(n)}^{m(n)})\| \right).$$

Now since Θ_m is a regular grid of Θ made of m^p points, we can choose $\theta_{n,m}$ in Θ_m such that

$$\|\hat{\theta}_n - \theta_{n,m}\| \leq \frac{C}{m(n)},$$

but by definition of $\hat{\theta}_{n,N(n)}^{m(n)}$ and since $\theta_{n,m}$ belongs to Θ_m , we have

$$\|F_n - F_{N(n)}(\hat{\theta}_{n,N(n)}^{m(n)})\| \leq \|F_n - F_{N(n)}(\theta_{n,m})\|,$$

so we obtain

$$\begin{aligned} \|F_n - F_{N(n)}(\hat{\theta}_{n,N(n)}^{m(n)})\| &\leq \|F_n - F_{N(n)}(\theta_{n,m})\| \\ &\leq \|F(\hat{\theta}_n) - F(\theta_{n,m})\| + \|F(\theta_{n,m}) - F_{N(n)}(\theta_{n,m})\| \leq \frac{C}{m(n)}, \end{aligned}$$

and then

$$\|\hat{\theta}_{n,N(n)}^{m(n)} - \hat{\theta}_n\| \leq \frac{C}{m(n)},$$

this is the result of the lemma.

From the lemma, to prove Proposition 3 it suffices to verify that

$$\lim_{n \rightarrow +\infty} P \left\{ (\forall \theta \in \Theta_{m(n)}) \|F_{N(n)}^{h(n)}(\theta) - F(\theta)\| \leq \frac{1}{m(n)} \right\} = 1$$

or equivalently

$$\lim_{n \rightarrow +\infty} P \left\{ (\exists \theta \in \Theta_{m(n)}) \text{ such that } \|F_{N(n)}^{h(n)}(\theta) - F(\theta)\| > \frac{1}{m(n)} \right\} = 0.$$

We note $F^h(\theta)$ the limit as N tends to infinity of $F_N^h(\theta)$ (for the existence of this limit see [6]). We have

$$\begin{aligned} &P \left\{ (\exists \theta \in \Theta_m) \|F_{N(n)}^{h(n)}(\theta) - F(\theta)\| > \frac{1}{m(n)} \right\} \\ &\leq m(n)^p \sup_{\theta \in \Theta_m} P \left\{ \|F_{N(n)}^{h(n)}(\theta) - F^h(\theta)\| > \frac{1}{2m(n)} \right\} \\ &\quad + m(n)^p \sup_{\Theta_m} P \left\{ \|F^h(\theta) - F(\theta)\| > \frac{1}{2m(n)} \right\}. \end{aligned}$$

We note $L^h(\theta, \beta)$ the almost sure limit of $L_N^h(\theta, \beta)$. The expression of $L^h(\theta, \beta)$ is then

$$L^h(\theta, \beta) = \int \log a(\beta, x) \pi^{\theta, h}(x) dx + \int \int \frac{(y - x - b(\beta, x)\Delta)^2}{2a(\beta, x)^2} p_{\Delta}^{\theta, h}(x, y) \pi^{\theta, h}(x) dy dx,$$

where $p_{\Delta}^{\theta,h}(x,y)$ and $\pi^{\theta,h}(x)$ denote respectively the densities of the transition probability and invariant measure of the process solution of (7). From [6], we have the convergence of $L^h(\theta, \beta)$ to $L(\theta, \beta)$ for all (θ, β) and this convergence is uniform in β . We deduce then the convergence of $F^h(\theta)$ to $F(\theta)$ for all θ . This implies the existence of a sequence $h(n)$ such that

$$\mathbf{P}\left\{\left\|F^{h(n)}(\theta) - F(\theta)\right\| > \frac{1}{2m(n)}\right\} = 0.$$

Since $F_N^h(\theta)$ converges almost surely to $F^h(\theta)$ for all θ , we can choose $N(n)$ such that

$$\mathbf{P}\left\{\left\|F_{N(n)}^{h(n)}(\theta) - F^{h(n)}(\theta)\right\| > \frac{1}{2m(n)}\right\} < \frac{1}{m(n)^{p+1}},$$

and we obtain the result of Proposition 3.

Acknowledgement. The author wish to thank Valentine Genon-Catalot and Jean Jacod for helpful discussions.

REFERENCES

1. *Dacunha-Castelle D., Florens-Zmirou D.* Estimation of the coefficient of a diffusion from discrete observations. — *Stochastics*, 1986, v. 19, p. 263–284.
2. *Gourieroux C., Monfort A., Renault E.* Indirect inference. — *J. Appl. Econometrics*, 1993, v. 8, p. S85–S119.
3. *Лунцер П. III., Ширяев А. Н.* Статистика случайных процессов. М.: Наука, 1974.
4. *Pedersen A. R.* A new approach to maximum likelihood estimation for stochastic differential equations based on discrete observations. — *Scand. J. Statist.*, 1995, v. 22, p. 55–71.
5. *Revuz D., Yor M.* Continuous Martingales and Brownian Motion. Berlin: Springer-Verlag, 1991.
6. *Talay D., Tubaro L.* Expansion of the global error for numerical schemes solving stochastic differential equations. — *Stoch. Anal. Appl.*, 1990, v. 8, p. 483–509.

Поступила в редакцию
18.X.1995