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© 2007 г. MAKAGON A.*, MIAMEE A. G.*, SALEHI H.*, SOLTANI A. R.***
 ON SPECTRAL DOMAIN OF PERIODICALLY CORRELATED PROCESSES¹⁾

Статью можно разделить на три части. Первая часть имеет дело с теоремой о спектральном представлении для периодически коррелированных последовательностей. Во второй части указывается некоторый класс квадратично-интегрируемых функций, который изоморфен временной области рассматриваемой последовательности. Этот полный класс назван спектральной областью последовательности. Третья часть связана с безуспешными предыдущими попытками других авторов построить полное функциональное пространство как спектральную область. Все это суть естественные обобщения результатов, известных в стационарном случае, на случай периодически коррелированных последовательностей.

Ключевые слова и фразы: спектральная область, периодически коррелированные процессы, полнота.

1. Introduction. The completeness of $L^2(\mu)$, the spectral domain of a stationary process, plays an important role in the prediction theory of such processes; this is because of the isomorphism between this space and the time domain of the process. For non-stationary harmonizable processes the search for identifying a proper space of functions which isometrically corresponds to the time domain has continued for decades by several authors, where various spaces and methods have been proposed for this purpose. It was H. Cramér who first introduced a function space [3], but gave the warning that it may not be complete. It happened that the function space introduced later by M. M. Rao [11] also failed to be complete as was shown in [9]. A general procedure in the direction of identifying a spectral domain of a harmonizable process is proposed by Soltani [13], where spectral domain is viewed as a closed subspace of $L^2(\mu)$. The main goal of this article is to resolve the issue for periodically correlated sequences. Following the path introduced by Soltani, in Section 3 a new spectral characterization is given for a periodically correlated sequence, where we view such a sequence as a process with time dependent spectrum. In Section 4 a complete function space is presented and is proved to be isomorphic to the time domain. Section 5 presents an example which provides more insights on the lack of completeness of the spaces introduced by Cramér and Rao.

For recent achievements on periodically correlated processes see [14], [6], [7].

2. Preliminaries. A sequence $X(n)$ is called periodically correlated (PC) if its correlation function $R(m, n) = \mathbf{E}X(m)\bar{X}(n)$ satisfies

$$R(m, n) = R(m + T, n + T) \quad \forall m, n \in \mathbf{Z},$$

for some integers $T > 0$. The smallest such T is called its period. Any PC process is harmonizable: $X(n) = \int_0^{2\pi} e^{-inx} \Psi(dx)$, $n \in \mathbf{Z}$, where Ψ is a random measure on $[0, 2\pi)$, with possibly dependent increments. For PC processes the restriction of Ψ to

* Department of Mathematics, Hampton University, Hampton, VA 23668, USA.

** Department of Statistics and Probability, Michigan State University, USA.

*** Department of Statistics & Operational Research, Faculty of Science, Kuwait University, Safat, 13060, State of Kuwait; e-mail: soltani@kuc01.kuniv.edu.kw

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each $[2\pi p/T, 2\pi(p+1)/T]$, $p = 0, 1, \dots, T-1$, is a random measure with independent increments; more precisely,

$$\Psi(\Delta) = \sum_{p=0}^{T-1} W_p \left(\Delta \cap \left[\frac{2\pi p}{T}, \frac{2\pi(p+1)}{T} \right) \right),$$

$$\mathbf{E} \left(W_p \left(\Delta + \frac{2\pi p}{T} \right) \overline{W}_q \left(\Delta' + \frac{2\pi q}{T} \right) \right) = 0,$$

for every $\Delta, \Delta' \subseteq [0, 2\pi/T]$ with $\Delta \cap \Delta' = \emptyset$ and every $p, q = 0, 1, \dots, T-1$. This representation in this form is contained in [14]. It is the equivalent form for the spectral random measure corresponding to the spectral distribution of PC sequences characterized first in [5], that can be seen in [6] as well. The term $\mathbf{E}\Psi(\Delta)\overline{\Psi}(\Delta')$ defines a complex measure β on $[0, 2\pi)^2$ which is called the spectral measure of the process, and is concentrated on $2T-1$ parallel equidistant line segments $\lambda = \theta + 2\pi k/T$, $k = -T+1, \dots, T-1$, inside the square $[0, 2\pi)^2$.

For any measure ν , any real number a , and any set $E \subset [0, 2\pi)$, $\nu_a(E)$ stands for $\nu((E+a) \cap [0, 2\pi))$. Let

$$F_k(d\theta) = \mathbf{E}\Psi(d\theta)\overline{\Psi}_{2\pi k/T}(d\theta) = \beta(d\theta, d\theta + 2\pi k T^{-1}), \quad k = -T+1, \dots, T-1, \quad (2.1)$$

then F_k 's are complex measures on $[0, 2\pi)$, $k = 0, 1, \dots, T-1$, which satisfy

$$F_k(A) = 0 \quad \text{for } A \subset [2\pi(T-k)T^{-1}, 2\pi),$$

$$F_{-k}(A) = 0 \quad \text{for } A \subset [0, 2\pi k T^{-1}),$$

and

$$F_k(A) = \overline{F}_{-k}(A + 2\pi k T^{-1}) \quad \text{for } A \subset [0, 2\pi(T-k)T^{-1}).$$

Moreover,

$$\mathbf{F}(d\theta) = \|F_{k-j}(d\theta + 2\pi j T^{-1})\|_{j,k=0,1,\dots,T-1}, \quad \theta \in [0, 2\pi T^{-1}),$$

defines a Hermitian nonnegative definite matrix-valued measure on $[0, 2\pi/T)$ [5]. Let μ be a positive measure on $[0, 2\pi)$ with

$$F_k \ll \mu \quad \text{for all } k = -T+1, \dots, T-1,$$

and

$$\mu_{2\pi k/T}(A) = \mu(A) \quad \text{for } A \subset [0, 2\pi k T^{-1}); \quad k = 0, 1, \dots, T-1. \quad (2.2)$$

Such a measure always exists; for example, one can take

$$\mu(E) = \sum_{j=1}^T \sum_{k=1-T}^{T-1} \nu_{2\pi k/T} \left(E \cap \left[\frac{2\pi(j-1)}{T}, \frac{2\pi j}{T} \right) \right), \quad \text{where } \nu = F_0 + \sum_{p=-T+1}^{T-1} |F_p|.$$

Now let

$$\mathbf{f}(\theta) = \frac{d\mathbf{F}}{d\theta} = \left\| f_{k-j} \left(\theta + \frac{2\pi j}{T} \right) \right\|_{j,k=0,1,\dots,T-1}, \quad \theta \in \left[0, \frac{2\pi}{T} \right), \quad (2.3)$$

where for each k , $f_k = dF_k/d\mu$; we call $\mathbf{f}$ the spectral density of the PC process. Since $\mathbf{f}$ is a nonnegative definite matrix-valued function, by virtue of (2.1), (2.2), and (2.3) we get

$$R(m, n) = \sum_{k=1-T}^{T-1} \int_0^{2\pi} e^{-im\theta + in(\theta + 2\pi k/T)} f_k(\theta) \mu(d\theta), \quad m, n \in \mathbf{Z}. \quad (2.4)$$

3. A structural spectral representation and its consequences. In this section we develop the following important integral representation for PC processes in terms of an orthogonal increment random measure. This representation brings about a two-fold fact property. First, it views a PC process as a process with a time dependent spectrum and gives a precise form for the time dependent kernel. Second, it provides a spectral pair for a PC process, the term introduced for harmonizable processes in [13]. Some other important applications of the representation are also given in the section.

Theorem 3.1 (Structural theorem). *A process $X(n)$, $n \in \mathbf{Z}$, is PC if and only if it admits a spectral integral representation of the form*

$$X(n) = \sum_{k=0}^{T-1} \int_0^{2\pi} e^{-in\theta} a_k(\theta) \Phi_{-2\pi k/T}(d\theta), \quad n \in \mathbf{Z}, \quad (3.1)$$

where

(i) Φ is an orthogonal increment random measure whose spectral measure $\mu = \mathbf{E}\Phi\bar{\Phi}$ is a finite measure on $[0, 2\pi]$ satisfying (2.2),

(ii) a_k is a function on $[0, 2\pi]$ which is zero on $[0, 2\pi k/T]$, $k = 0, 1, \dots, T$. When this is the case, the upper triangular matrix-valued function $\mathbf{A}(\theta)$ defined on $[0, 2\pi/T]$ by $\mathbf{A}(\theta) = \|a_{j-k}(\theta + 2\pi j T^{-1})\|_{j \geq k, k, j=0,1,\dots,T-1}$ is the Cholesky factor [1] in the decomposition of the spectral matrix $\mathbf{f}(\theta)$,

$$\mathbf{f}(\theta) = \mathbf{A}(\theta) \mathbf{A}(\theta)^*, \quad x \in [0, 2\pi T^{-1}]. \quad (3.2)$$

P r o o f. For the «if» part we note that the correlation function of the process in (3.1) is given by

$$R(m, n) = \sum_{k=0}^{T-1} \sum_{l=0}^{T-1} \int_0^{2\pi} \int_0^{2\pi} e^{-im\theta + in\lambda} a_k(\theta) \bar{a}_l(\lambda) \mathbf{E}\Psi_{-2\pi k/T}(d\theta) \bar{\Psi}_{-2\pi l/T}(d\lambda).$$

However,

$$\mathbf{E}\Phi_{-2\pi k/T}(d\theta) \bar{\Phi}_{-2\pi l/T}(d\lambda) = \begin{cases} \mu_{-2\pi k/T}(d\theta) & \text{if } \theta - 2\pi k T^{-1} = \lambda - 2\pi l T^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Letting $p = 1 - k$ and noting that $\mu_{-2\pi k/T}(A) = \mu(A)$, $A \subset [2\pi k/T, 2\pi)$, $k = 0, 1, \dots, T-1$, and that a_k is zero on $[0, 2\pi k/T]$, we obtain

$$\begin{aligned} R(m, n) &= \sum_{p=-(T-1)}^{-1} \int_0^{2\pi} e^{-im\theta + in(\theta + 2\pi p/T)} \sum_{k=-p}^{T-1} a_k(\theta) \bar{a}_{k+p} \left(\theta + \frac{2\pi p}{T} \right) \mu(d\theta) \\ &\quad + \int_0^{2\pi} e^{-im\theta + in\theta} \sum_{k=0}^{T-1} |a_k(\theta)|^2 \mu(d\theta) \\ &\quad + \sum_{p=1}^{T-1} \int_0^{2\pi} e^{-im\theta + in(\theta + 2\pi p/T)} \sum_{k=0}^{T-1-p} a_k(\theta) \bar{a}_{k+p} 2\pi \left(\theta + \frac{2\pi p}{T} \right) \mu(d\theta), \quad (3.3) \end{aligned}$$

which shows that the process is PC.

For the «only if» part assume that the process $X(n)$ is PC. Hence $X(n)$ is harmonizable and its covariance function is given by (2.4). Now let $\mathbf{A}(\theta) = [A_{jk}(\theta)]_{j,k=0,1,\dots,T-1}$ be the upper triangular Cholesky factor of the corresponding density function $\mathbf{f}$ with $A_{kk}(\theta) \geq 0$, for all $k = 0, 1, 2, \dots, T-1$ and μ -a.a. θ . So we can write

$$\mathbf{f} = \mathbf{A}(\theta) \mathbf{A}^*(\theta), \quad \theta \in [0, 2\pi).$$

For each $k = 0, 1, \dots, T-1$ define a_k on $[0, 2\pi)$ by

$$a_k(\theta) = \begin{cases} 0, & 0 \leq \theta < 2\pi k T^{-1}, \\ A_{j,j+k}(\theta - 2\pi(k+j) T^{-1}), & 2\pi(k+j) T^{-1} \leq \theta < 2\pi(k+j+1) T^{-1}. \end{cases}$$

So the entries of $\mathbf{f}$ can be identified by the $a_0, 1, \dots, a_{T-1}$ as follows:

$$\begin{aligned} \sum_{k=-p}^{T-1} a_k(\theta) \bar{a}_{p+k} \left(\theta + \frac{2\pi p}{T} \right) &= f_{-p}(\theta), \quad p = -T+1, \dots, -1, \\ \sum_{k=0}^{T-1} |a_k(\theta)|^2 &= f_0(\theta), \end{aligned} \quad (3.4)$$

$$\sum_{k=0}^{T-1-p} a_k(\theta) \bar{a}_{k+p} \left(\theta + \frac{2\pi p}{T} \right) = f_{-p}(\theta), \quad p = 1, \dots, T-1.$$

By replacing the corresponding terms for the entries of $\mathbf{f}$ from (3.4) in (2.4), it follows immediately that the covariance function in (2.4) takes the form as in (3.3) which is indeed the covariance function of the $X(n)$ given by (3.1). This completes the proof.

Remark 3.1. Theorem 3.1, indeed, expresses that a PC process can be identified with a positive measure μ and T functions $a_0, \dots, a_{T-1}$.

Remark 3.2. For a given μ , the conditions in (ii) that are imposed on the $a_0, \dots, a_{T-1}$ make them unique. However any square root of $\mathbf{f}$, i.e., a $T \times T$ matrix valued function $\mathbf{B}$ for which $\mathbf{f}(\theta) = \mathbf{B}(\theta)\mathbf{B}^*(\theta)$ for μ -a.a. $\theta \in [0, 2\pi/T)$, gives rise to a number of T functions on $[0, 2\pi)$, as constructed in the proof, to replace $a_0, \dots, a_{T-1}$. The Cholesky factor, the positive square root, and analytic factors are those that are of frequent use in applications.

Theorem 3.1 has several important consequences that are stated below as theorems. Proofs are direct.

Theorem 3.2. Every PC process $X(t)$ with period T is the sum of T stationary processes. In particular,

$$X(n) = \sum_{k=0}^{T-1} X_k(n) \quad \text{with} \quad X_k(n) = \int_0^{2\pi} e^{-in\theta} a_k(\theta) \Phi_{-2\pi k/T}(d\theta), \quad (3.5)$$

$$\mathbf{E}X_k(m) \overline{X_l}(n) = \int_0^{2\pi} e^{-im\theta + in(\theta + 2\pi/T(l-k))} a_k(\theta) \overline{a_l}\left(\theta + \frac{2\pi(l-k)}{T}\right) \mu(d\theta), \quad m, n \in \mathbf{Z},$$

for $k, l = 0, 1, \dots, T-1$, where a_k and μ are as in Theorem 3.1.

Theorem 3.3. A PC process is a process with time dependent spectrum. Indeed it admits an integral representation

$$X(n) = \int_0^{2\pi} e^{-in\theta} V_n(\theta) \Phi(d\theta) \quad \text{with} \quad V_n(\theta) = \sum_{k=0}^{T-1} e^{-i2\pi kn/T} a_k\left(\theta + \frac{2\pi k}{T}\right), \quad (3.6)$$

where random measure Φ and functions a_k are as in Theorem 3.1.

Therefore $|V_n(\theta)|^2 \mu(d\theta)$ is a dependent spectrum for the process.

A spectral pair $(\mu, \mathcal{F})$ can be assigned to a harmonizable process, where μ is a positive measure and $\mathcal{F}$ is a $L^2(\mu)$ -valued measure for which

- (i) Ψ and $\mathcal{F}$ possesses the same variation and the same semi-variation,
- (ii) a complex function g is integrable with respect to Ψ , in the sense of Dunford and Schwartz [4], if and only if it is integrable with respect to $\mathcal{F}$, and in this case

$$\mathbf{E}\left(\int g d\Psi\right)^2 = \left\|\int g d\mathcal{F}\right\|_{L^2(\mu)}^2,$$

- (iii) if g is integrable with respect to Ψ , then the second order random variable $\int g d\Psi$ is given by $\int f d\Psi = \int_I (\int f d\mathcal{F})(x) d\Phi(x)$, see [13].

A PC process is harmonizable and Theorem 3.1 provides a spectral pair for it, as follows.

Theorem 3.4. A PC process has a spectral pair $(\mu, \mathcal{F})$, where

$$\mathcal{F}(A)(x) = \sum_{k=0}^{T-1} a_k\left(x + \frac{2\pi k}{T}\right) \mathbf{1}_{A-2\pi k/T}(x), \quad A \in \mathcal{B}_{[0, 2\pi)},$$

i.e., for each x , $\mathcal{F}(\cdot)(x)$ is concentrated on the set $\{x + 2\pi k/T, k = 0, 1, \dots, T-1\}$, the measure μ and the functions $a_0, \dots, a_{T-1}$ are as in Theorem 3.1.

The final application of Theorem 3.1 in this section (below) provides a necessary and sufficient condition for complete regularity of a PC process. A PC process is called completely regular if

$$\bigcap_n \overline{\text{sp}}\{X(m), m \leq n\} = 0,$$

see [5].

Theorem 3.5. A PC process $X(n)$ is completely regular if and only if

$$\int_0^{2\pi} \ln a_0(\theta) d\theta > -\infty.$$

P r o o f. It was proved by Gladyshev in [5] that a PC process $X(n)$ is completely regular if and only if $\int_0^{2\pi/T} \ln \det \mathbf{f}(\theta) d\theta > -\infty$, where $\mathbf{f}$ is the spectral density given in (2.3), the measure μ can be taken to be the Lebesgue measure. It follows from (3.2) that

$$\det \mathbf{f}(\theta) = \prod_{j=0}^{T-1} a_0\left(\theta + \frac{2\pi j}{T}\right)^2, \quad \theta \in \left[0, \frac{2\pi}{T}\right).$$

Therefore

$$\int_0^{2\pi/T} \ln \det \mathbf{f}(\theta) d\theta = 2 \int_0^{2\pi} \ln a_0(\theta) d\theta,$$

giving the result.

Theorem 3.1 in [8] easily follows from this observation, as the function f_0 in [8] dominates the function a_0 in Theorem 3.1 given here.

4. A complete spectral domain. In this section we present a spectral domain for a PC process that is complete and is isometrically isomorphic to the time domain of the process. The integral representation in Theorem 3.3 suggests a natural approach to build an appropriate spectral domain. It obviously leads us to a Kolmogorov isomorphism

$$X(n) \leftrightarrow e^{-in\theta} V_n(\theta), \quad (4.1)$$

between the set of the process values $\{X(n), n \in \mathbf{Z}\}$ in $L^2(\Omega, \mathbf{F}, \mathbf{P})$ and the set $\{e^{-in\theta} V_n(\theta), n \in \mathbf{Z}\}$ in $L^2(\mu)$. Therefore under the mapping in (4.1), the time domain is isomorphic to

$$\mathcal{S} = \overline{\text{sp}}\{e^{-in\theta} V_n(\theta), n \in \mathbf{Z}\} \quad \text{in } L^2(\mu). \quad (4.2)$$

Obviously in this settings the completeness here is not the issue, since $\mathcal{S}$ is clearly complete. So any function space isomorphic to $\mathcal{S}$ can be considered as a spectral domain. Let us present such a complete function space. This will be accomplished by establishing an isomorphism between $\mathcal{S}$ and a suitable complete scalar valued function space associated with the Hilbert space

$$M(\mathbf{f}) = \left\{ G: \left[0, \frac{2\pi}{T}\right) \rightarrow C^T, \int_0^{2\pi/T} G(x) \mathbf{f}(x) G(x)^* \mu(dx) < \infty \right\}, \quad (4.3)$$

with the inner product

$$(G, K)_M = \int_0^{2\pi/T} G(x) \mathbf{f} K(x)^* \mu(dx), \quad (4.4)$$

note that $M(\mathbf{f})$ is the $L^2(\mathbf{f})$ in [12], [15]. Let us define the following mapping

$$e^{-in\theta} V_n(\theta) \longleftrightarrow (e^{-inx}, e^{-in(x+2\pi/T)}, \dots, e^{-in(x+2\pi(T-1)/T)}) \quad (4.5)$$

Lemma 4.1. The mapping (4.5) is an isometry between the set $\{e^{-in\theta} V_n(\theta), n \in \mathbf{Z}\}$ in $L^2(\mu)$ and the set $\{(e^{-inx}, e^{-in(x+2\pi/T)}, \dots, e^{-in(x+2\pi(T-1)/T)}), n \in \mathbf{Z}\}$ in $M(\mathbf{f})$.

P r o o f. Note that it easily follows that

$$\begin{aligned} & \left(e^{-in\theta} V_n(\theta), e^{-im\theta} V_m(\theta) \right)_{L^2(\mu)} \\ &= \int_0^{2\pi} e^{-i(n-m)\theta} \sum_{k=0}^{T-1} \sum_{j=0}^{T-1} e^{-i2\pi kn/T} a_k \left(\theta + \frac{2\pi k}{T} \right) \bar{a}_j \left(\theta + \frac{2\pi j}{T} \right) e^{i2\pi jm/T} \mu(d\theta) \\ &= \sum_{k=0}^{T-1} \sum_{j=0}^{T-1} \int_0^{2\pi} a_k(\theta) \bar{a}_j \left(\theta + \frac{2\pi(j-k)}{T} \right) e^{-i[(n-m)\theta + 2\pi(j-k)m/T]} \mu(dx) \end{aligned}$$

$$= \int_0^{2\pi} \left\{ \sum_{p=-T+1}^{-1} f_{-p}(\theta) e^{-i[(n-m)\theta + 2\pi pm/T]} + f_0(\theta) e^{-i(n-m)\theta} \right. \\ \left. + \sum_{p=1}^{T-1} f_p(\theta) e^{-i[(n-m)\theta + 2\pi pm/T]} \right\} \mu(d\theta),$$

where the last equality follows from (3.4). On the other hand, it follows from (2.3) without much difficulty that for $g_n(x) = (e^{-inx}, e^{-in(x+2\pi/T)}, \dots, e^{-in(x+2\pi(T-1)/T)})$, we have

$$\begin{aligned} (g_n(x), g_m(x))_{M(\mathbf{f})} &= \int_0^{2\pi/T} g_n(x) \mathbf{f}(x) \bar{g}_m(x)' \mu(dx) \\ &= \int_0^{2\pi/T} \sum_{j=0}^{T-1} \sum_{k=0}^{T-1} e^{-in(x+2\pi j/T)} f_{jk} e^{im(x+2\pi k/T)} \mu(dx) \\ &= \sum_{p=-T+1}^{-1} \int_0^{2\pi} f_p(\theta) e^{-i[(n-m)\theta - 2\pi mp/T]} \mu(d\theta) \\ &\quad + \int_0^{2\pi} f_0(\theta) e^{-i(n-m)\theta} \mu(d\theta) \\ &\quad + \sum_{p=1}^{T-1} \int_0^{2\pi} f_p(\theta) e^{-i[(n-m)\theta - 2\pi mp/T]} \mu(d\theta). \end{aligned}$$

Therefore

$$(e^{-in\theta} V_n(\theta), e^{-im\theta} V_m(\theta))_{L^2(\mu)} = (g_n(x), g_m(x))_{M(\mathbf{f})}.$$

The proof of the lemma is complete.

It follows from Lemma 4.1 that $\mathcal{S}$ is isomorphic to

$$m(\mathbf{f}) = \overline{\text{sp}} \left\{ (e^{-inx}, e^{-in(x+2\pi/T)}, \dots, e^{-in(x+2\pi(T-1)/T)}) \right\}, \quad n \in \mathbf{Z} \quad \text{in } M(\mathbf{f}).$$

Let us now show that $m(\mathbf{f}) = M(\mathbf{f})$. With $y = Tx$,

$$m(\mathbf{f}) = \overline{\text{sp}} \{ e^{-iny} e^{-iny/T} \mathbf{j}_r, \quad r = 0, 1, \dots, T-1, \quad n \in \mathbf{Z} \}$$

is in $M(\mathbf{f})$. Since $\{\mathbf{j}_r, r = 0, 1, \dots, T-1\}$ is an orthonormal basis in C^T and $e^{-iny/T} \neq 0$, it readily follows from the analysis of Fourier coefficients that the annihilator of $m(\mathbf{f})$ is $\{0\}$ in $M(\mathbf{f})$.

A further analysis shows that $\mathcal{S}$ could indeed be identified with a space of scalar-valued functions. To establish this, let for each $G: [0, 2\pi/T] \rightarrow C^T$ define $g_G: [0, 2\pi] \rightarrow C$ by

$$g_G(x) = G_p \left(x - \frac{2\pi p}{T} \right), \quad \text{if } x \in \left[\frac{2\pi p}{T}, \frac{2\pi(p+1)}{T} \right), \quad p = 0, 1, \dots, T-1, \quad (4.6)$$

and for any $g: [0, 2\pi] \rightarrow C$, we define $G_g: [0, 2\pi/T] \rightarrow C^T$ by $G_g(x) = (g(x), g(x+2\pi T^{-1}), \dots, g(x+2\pi(T-1)T^{-1}))$. Also let $\Upsilon = \{g: [0, 2\pi] \rightarrow C, G_g \in M(\mathbf{f})\}$ be equipped with the inner product $(g_1, g_2)_\Upsilon = (G_{g_1}, G_{g_2})_{M(\mathbf{f})}$, then the function space Υ equipped with the inner product $(\cdot, \cdot)_\Upsilon$ is isomorphic to the closed subspace $\mathcal{S}$ in $L^2(\mu)$, and is our candidate for a spectral domain.

5. Earlier attempts on constructing a spectral domain. In this section we provide a brief background on the earlier attempts in constructing spectral domains for harmonizable processes. Since it is now apparent that harmonizable processes form a large class of nonstationary processes, it is conceivable that certain models of function spaces cannot exhibit spectral domains properties for harmonizable processes in the sense of identifying complete spectral domain for such processes. The approach presented in [13], which was applied for PC processes in the preceding section, is more promising. The early work on the issue goes back to Cramér [3]. A harmonizable process $X(n)$ with random measure

Ψ for which its corresponding bimeasure $\beta(\Delta, \Delta') = \mathbf{E}\Psi(\Delta)\overline{\Psi}(\Delta')$ constitutes a measure on $[0, 2\pi]^2$ was considered by Cramér in [3], where he defined an inner product space $\mathcal{C} = \{h: [0, 2\pi] \rightarrow C: h(\theta)\overline{h(\lambda)} \in L^2(\beta)\}$, $(h, g)_{\mathcal{C}} = \int_0^{2\pi} h(\lambda)\overline{g(\theta)}\beta(d\lambda, d\theta)$, together with an equivalence relation on $\mathcal{C}$, that makes h equivalent to g if and only if

$$\|h - g\|_{\mathcal{C}} = (h - g, h - g)_{\mathcal{C}} = 0.$$

Cramér referred to the corresponding quotient space together with $(\cdot, \cdot)_{\mathcal{C}}$ as a spectral domain of $X(n)$. However Cramér raised the question of the completeness of this function space. This settings was extended by Rao [11], to the case that β is merely a bimeasure. However in [9] an example is given showing that these spectral domains are not in general complete. Pursuing the issue, the example in [9] is modified below exhibiting that even for PC processes these models of function spaces are not complete. Indeed, the idea of an example is simple. It is well known that if $X^i(n)$, $i = 1, 2$, are two jointly stationary discrete processes and $\mathcal{E}(\cdot)$ is the spectral resolution of the shift of the process, then the measure $\Psi(\Delta) = \mathcal{E}(2(\Delta \cap [0, \pi))X^1(0) + \mathcal{E}(2(\Delta \cap [\pi, 2\pi) - \pi))X^2(0)$ is the random measure of a PC process of period 2. Let $\beta(\Delta, \Delta') = \mathbf{E}\Psi(\Delta)\overline{\Psi}(\Delta')$. In view of a one-to-one correspondence between elements of $\mathcal{C}$ and Ψ -integrable functions [2], the completeness of $\mathcal{C}$ would imply that every $Y \in M_X = \overline{\text{sp}}\{X^i(n): n \in \mathbf{Z}, i = 1, 2\}$ has the form $Y = Y^1 + Y^2$, where $Y^i \in M_i = \overline{\text{sp}}\{X^i(n): n \in \mathbf{Z}\}$, $i = 1, 2$. But it is known that the latter property may be violated for jointly stationary sequences [15, p. 108].

A counter example is only formulated here. For details interested readers are referred to [7].

Example 5.1. Let $G_1(\theta) = \langle f_{11}(\theta), f_{12}(\theta) \rangle$ and $G_2(\theta) = \langle f_{21}(\theta), f_{22}(\theta) \rangle$ be two $L^2(d\theta, C^2)$ functions defined on $[0, 2\pi]$ such that:

- (A1) $f_{11}(\theta) > 0$ and $f_{21}(\theta) = \overline{f_{12}(\theta)}$ $d\theta$ -a.e.,
- (A2) $f_{11}(\theta)f_{22}(\theta) - 2f_{21}(\theta)f_{12}(\theta) > 0$ $d\theta$ -a.e.

Let $X(n)$ be the sequence in $\mathbf{H} = L^2(d\theta, C^2)$, $\theta \in [0, 2\pi]$, defined by

$$X(n)(\theta) = \exp(inu/2)G_1(\theta) + \exp(in(\theta + 2\pi)/2)G_2(\theta), \quad \theta \in [0, 2\pi].$$

Then $X(n)$ is periodically correlated with period $T = 2$, with time domain $M(X) =: \overline{\text{sp}}\{X(n): n \in \mathbf{Z}\} = L^2(dt, C^2)$; the spectral measure β of $X(n)$ is only supported by the lines $y = x - \pi$, $y = x$, $y = x + \pi$ with respective density masses

$$\begin{aligned} & 2(G_1(2\theta - 2\pi), G_2(2\theta - 2\pi))d\theta, \quad \theta \in [\pi, 2\pi], \\ & 2[|G_1(2\theta)|^2 1_{[0, \pi)}(\theta) + |G_2(2\theta - 2\pi)|^2 1_{[\pi, 2\pi)}(\theta)]d\theta, \\ & 2(G_1(2\theta), G_2(2\theta))d\theta, \quad \theta \in [0, \pi]. \end{aligned}$$

The corresponding function space $\mathcal{C}$ can be embedded into $L^2(dt, C^2)$ by the isometry mapping $(Ig)(\theta) = g(\theta/2)G_1(\theta) + g((\theta + 2)/2)G_2(\theta)$, $g \in \mathcal{C}$. Finally it can be observed that the constant function $\phi(t) = (1, 1)$ in $L^2(dt, C^2)$ is not the image of any g in $\mathcal{C}$.

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PAULAUSKAS V.*, SURGAILIS D.*

ON THE RATE OF APPROXIMATION IN LIMIT THEOREMS FOR SUMS OF MOVING AVERAGES¹⁾

Рассматривается линейный процесс $X_t = \sum_{j=0}^{\infty} a_j \varepsilon_{t-j}$, $t \geq 1$, где ε_i , $i \in \mathbf{Z}$, — независимые одинаково распределенные случайные величины, принадлежащие области притяжения устойчивого закона с индексом α , $0 < \alpha \leq 2$, $\alpha \neq 1$. Накладывая условия на величины ε_i и коэффициенты a_j , мы оцениваем точность аппроксимации распределений сумм $S_n = B_n^{-1} \sum_{t=1}^n X_t$ соответствующими устойчивыми законами. Полученные оценки имеют правильный порядок относительно n .

Ключевые слова и фразы: линейные процессы, устойчивые законы, точность аппроксимации.

* Vilnius University, Department of Mathematics and Informatics, Naugarduko, 24, Vilnius 03225, Lithuania; Institute of Mathematics and Informatics, Akademijos, 4, Vilnius, Lithuania; e-mail: vygantas.paulauskas@maf.vu.lt; sdonatas@ktl.mii.lt

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