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CLOSEDNESS OF SUM SPACES AND
THE GENERALIZED SCHRÖDINGER PROBLEM

Установлены некоторые общие свойства замкнутости сумм пространств измеримых функций. В качестве применения доказаны существование и единственность решений обобщенной задачи Шредингера при некотором условии интегрируемости, но без каких-либо предположений о топологии или ограниченности. Полученные свойства позволяют также доказать интересный результат о структуре законов с многомерными частными распределениями, установить существование оптимальных аппроксимаций в аддитивных статистических моделях и обобщить представление Колмогорова для непрерывных функций нескольких переменных. Из последнего результата вытекает, что любая локально ограниченная измеримая функция имеет точное представление нейронной сетью с одним скрытым слоем.

Ключевые слова и фразы: уравнение Шредингера, сумма пространств, аддитивные модели, многомерные частные распределения, представление Колмогорова.

1. Introduction

The main motivation of this paper is to derive existence and uniqueness results for some generalized versions of the Schrödinger equations. It turns out that these results are based on some crucial closedness properties of sum spaces of measurable functions, which are derived in Section 2. These closedness properties also allow to prove some interesting existence results for projections in additive statistical models, furthermore, to develop an interesting structural property of distributions with multivariate marginals, and to derive an extension of Kolmogorov's famous representation theorem for continuous functions of n -variables.

Let μ be a probability measure on a product space $(E, \mathcal{A}) = (E_1, \mathcal{A}_1) \otimes (E_2, \mathcal{A}_2)$ with marginals μ_i on E_i . Let $L^0(E_i, \mathcal{A}_i, \mu_i) = L^0(\mu_i)$ denote the class of $\mathcal{A}_i$ -measurable real functions. It is clear from the context, if $f \in L^0(\mu_i)$ stands for some version or for the equivalence-class mod μ_i -null-functions. Furthermore, let $\nu_i \in M^1(E_i, \mathcal{A}_i)$ be probability measures on

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$(E_i, \mathcal{A}_i)$ continuous with respect to μ_i with densities $r_i = d\nu_i/d\mu_i$. Consider the following system of nonlinear integral equations which we denote as (generalized) *Schrödinger-problem*:

$$\begin{aligned} E_\mu(a \otimes b \mid \pi_1 = x) &= r_1(x) & \mu_1\text{-a.e.}, \\ E_\mu(a \otimes b \mid \pi_2 = y) &= r_2(y) & \mu_2\text{-a.e.}, \end{aligned} \quad (1.1)$$

where $a \otimes b \in L_+^0(\mu)$, $a \otimes b(x, y) := a(x)b(y)$, and $\pi_i: E_1 \times E_2 \rightarrow E_i$ are the projections on E_i .

If μ has a density h with respect to $\mu_1 \otimes \mu_2$, i.e., $\mu = h \mu_1 \otimes \mu_2$, then (1.1) is equivalent to

$$\begin{aligned} \int a(x)b(y)h(x, y)\mu_1(dy) &= r_1(x) & \mu_1\text{-a.e.}, \\ \int a(x)b(y)h(x, y)\mu_2(dx) &= r_2(y) & \mu_2\text{-a.e.}, \end{aligned} \quad (1.2)$$

Equation (1.1) is equivalent to the following marginal problem. Let $M(\nu_1, \nu_2)$ denote the set of all probability measures on $(E, \mathcal{A})$ with marginals ν_1, ν_2 . Then (1.1) is equivalent to:

Find $a \otimes b \in L_+^0(\mu)$ such that

$$\nu = a \otimes b \mu \in M(\nu_1, \nu_2). \quad (1.3)$$

Problem (1.1), (1.3) is of interest for the structure of (generalized) Schrödinger bridges (cf. [2], [3], [4], [6], [7], [11], [15], [21]). In our previous paper [18] we proved the existence and uniqueness of solutions of (1.2) with $a \in L_+^0(\mu_1)$, $b \in L_+^0(\mu_2)$ under an integrability condition but without any assumptions on the boundedness of h or topological restrictions extending related results in [1], [8], [10], [11] and [20]. In this paper we investigate the general problem (1.1), (1.3). The progress is established by some crucial closedness properties of sum spaces of measurable functions in Section 2 which need the development of some new tools, and which also find some further applications in Sections 3 and 4. In particular we consider applications to optimal approximations in additive statistical models, some structure results for distributions with given marginals and a «generalization» of Kolmogorov's representation theorem to measurable functions. A part of the problems is caused by the possibly complicated structure of the support of μ .

2. Closedness Properties of Sum Spaces

Consider the sum space of measurable functions

$$F = L^0(\mu_1) \oplus L^0(\mu_2) = \{\varphi = f \oplus g; f \in L^0(\mu_1), g \in L^0(\mu_2)\}, \quad (2.1)$$

where $f \oplus g(x, y) = f(x) + g(y)$. F is a subspace of

$$G = (M(E_1) \oplus M(E_2)) \cap L^0(\mu), \quad (2.2)$$

where $M(E_i)$ is the class of all real valued functions (maps) on E_i and $L^0(\mu) = L^0(E_1 \times E_2, \mathcal{A}_1 \otimes \mathcal{A}_2, \mu)$ is supplied with the complete metric of μ -stochastic convergence. In this section we are concerned with a characterization of the closure of F in $L^0(\mu)$ with respect to μ -stochastic convergence (and related types of convergence). The following result is obtained by using essentially methods of the proofs in [18].

Proposition 2.1. *G is closed in $L^0(\mu)$ with respect to μ -stochastic convergence. In particular, the closure of F in $L^0(\mu)$ is a subset of G .*

Proof. 1. For any limit point $\varphi = \varphi(x, y)$ of G with respect to μ -stochastic convergence, there exists a sequence $f_n \oplus g_n$ in G such that

$$f_n(x) + g_n(y) \longrightarrow \varphi(x, y), \quad \mu\text{-a.e.}$$

Then $C := \{(x, y) \in E; \lim_n (f_n(x) + g_n(y)) \text{ exists}\} \in \mathcal{A}_1 \otimes \mathcal{A}_2$, $\mu(C) = 1$ and without loss of generality

$$\varphi(x, y) = \lim_n (f_n(x) + g_n(y)) 1_C(x, y).$$

2. Choose $(x_1, y_1) \in C$ and define

$$\begin{aligned} f_n^1(x) &:= f_n(x) - f_n(x_1), & g_n^1(y) &:= g_n(y) + f_n(x_1), \\ C_1^1 &:= \{x \in E_1; f^1(x) := \lim_n f_n^1(x) \text{ exists}\}, \\ C_2^1 &:= \{y \in E_2; g^1(y) := \lim_n g_n^1(y) \text{ exists}\}. \end{aligned} \quad (2.3)$$

Then $(x_1, y_1) \in C_1^1 \times C_2^1$ and for $(x, y) \in C_1^1 \times C_2^1$ it holds

$$f_n(x) + g_n(y) = (f_n(x) - f_n(x_1)) + (g_n(y) + f_n(x_1)) \longrightarrow f^1(x) + g^1(y), \quad (2.4)$$

i.e., $(x, y) \in C$. Therefore, $C_1^1 \times C_2^1 \subset C$.

3. Choose $(x_2, y_2) \in C \setminus C_1^1 \times C_2^1$ and define $f_n^2, g_n^2, f^2, g^2, C_1^2, C_2^2$ as in 2. with (x_1, y_1) replaced by (x_2, y_2) . Then

$$C_1^1 \cap C_1^2 = \emptyset, \quad C_2^1 \cap C_2^2 = \emptyset. \quad (2.5)$$

To prove (2.5) let $x \in C_1^1 \cap C_1^2$. Then $f_n(x) - f_n(x_1) \longrightarrow f^1(x)$ and $f_n(x) - f_n(x_2) \longrightarrow f^2(x)$ and, therefore,

$$f_n(x_2) - f_n(x_1) \longrightarrow f^1(x) - f^2(x) = f^1(x_2).$$

This implies that $x_2 \in C_1^1$. From

$$\begin{aligned} g_n(y_2) + f_n(x_1) &= (g_n(y_2) + f_n(x_2)) - (f_n(x_2) - f_n(x_1)) \\ &\longrightarrow g^2(y_2) - f^1(x_2) \end{aligned} \quad (2.6)$$

it follows by (2.3) that $y_2 \in C_2^1$. This contradicts $(x_2, y_2) \notin C_1^1 \times C_2^1$. By (transfinite) induction we obtain a (possibly uncountable) family $(C_1^i \times C_2^i, i \in I)$, such that $(x_i, y_i) \in C_1^i \times C_2^i$, $C_1^i \cap C_1^j = \emptyset$, $C_2^i \cap C_2^j = \emptyset$ for all $i, j \in I$, $i \neq j$, and $C = \sum_{i \in I} C_1^i \times C_2^i$. Furthermore, as in point 2 we obtain functions $f^i 1_{C_1^i}$ and $g^i 1_{C_2^i}$ such that

$$\varphi(x, y) 1_C(x, y) = (f(x) + g(y)) 1_C(x, y), \quad (2.7)$$

where $f(x) := \sum_{i \in I} f^i(x) 1_{C_1^i}(x)$ and $g(y) := \sum_{i \in I} g^i(y) 1_{C_2^i}(y)$.

R e m a r k s. a) Additionally, we point out that the components $C_1^i \times C_2^i$ of the convergence set $C = \sum_{i \in I} C_1^i \times C_2^i$ are in $\mathcal{A}_1 \otimes \mathcal{A}_2$. Indeed, $C_{x_i} = C_2^i$, where C_{x_i} denotes the x_i -section of $C \in \mathcal{A}_1 \otimes \mathcal{A}_2$, hence

$$C_2^i \in \mathcal{A}_2 \quad \text{and} \quad C_1^i \in \mathcal{A}_1 \quad \text{for all } i \in I \quad (2.8)$$

as well. Furthermore, the corresponding components $f 1_{C_1^i}$ and $g 1_{C_2^i}$ of f and g are measurable. Clearly, the x_i -section of φ , $\varphi(x_i, \cdot) = f(x_i) 1_{C_2^i} + g 1_{C_2^i}$ is $\mathcal{A}_2$ -measurable, hence

$$g 1_{C_2^i} \text{ is } \mathcal{A}_2\text{-measurable and } f 1_{C_1^i} \text{ is } \mathcal{A}_1\text{-measurable} \quad (2.9)$$

as well.

b) H. Kellerer pointed out to us a very simple and nice argument which yields the stated representation $C = \sum_{i \in I} C_1^i \times C_2^i$, $C_\nu^i \cap C_\nu^j = \emptyset$ for $i, j \in I$, $i \neq j$, and $\nu = 1, 2$ in the proof of Proposition 2.1: The two-dimensional differences of φ_n disappear. Therefore, given $x_1, x_2 \in E_1$ and $y_1, y_2 \in E_2$, it follows that $(x_2, y_2) \in C$ if $(x_1, y_1), (x_1, y_2), (x_2, y_1) \in C$. This implies that the sections C_{x_1} and C_{x_2} are either disjoint or coincide, which yields the assertion.

c) F is closed, if I is countable, or if all components $C_1^i \times C_2^i$ are one-point sets. But F is not closed in general. The following beautiful counterexample due to N. Gantert (private communication) has surprised us and in fact it pointed out a mistake in a first version of this paper.

E x a m p l e (N. Gantert). Consider $E_1 = E_2 = \{0, 1\}^{\mathbb{N}}$, $\mathcal{A}_1 = \mathcal{A}_2 = (\mathcal{P}(\{0, 1\}))^{\mathbb{N}}$ where $\mathcal{P}(A)$ is the set of all subsets of A and $\mu_1 = (B(1, \frac{1}{2}))^{\mathbb{N}}$; define the Markov-kernel K from E_1 to E_2 by $K(x, \cdot) = \sum_{k=1}^{\infty} \frac{1}{2^k} \varepsilon_{x(k)}$, where $x(k) = (x_1, \dots, x_{k-1}, x_{k+1}, x_{k+2}, \dots)$, and let $\mu = \mu_1 \otimes K$ be the product on $E_1 \times E_2$. Furthermore, consider the measurable functions

$$f_n(x) = \sum_{i=1}^n x_i, \quad g_n(y) = - \sum_{i=1}^{n-1} y_i, \quad n \geq 2.$$

Obviously, $f_n \rightarrow \infty$ (μ -a.e.), $g_n \rightarrow -\infty$ (μ -a.e.) and

$$f_n \oplus g_n \rightarrow \varphi = \varphi 1_C \in L^0(\mu) \quad \mu\text{-a.e.},$$

where $C = \{(x, y) \in E; \lim_{n \rightarrow \infty} (f_n(x) + g_n(y)) \text{ exists}\}$. Clearly, $(x, y) \in C$ if and only if $x_{i+1} = y_i$ eventually, and x, x' are in the same component if and only if $x_i = x'_i$ eventually, i.e., the components are the atoms of the tail σ -field on E_1 .

Suppose that $\varphi = f \oplus g$ (μ -a.s.) for some $f \oplus g \in L^0(\mu_1) \oplus L^0(\mu_2)$. Let $X = (X_i, i \in \mathbf{N}) = \text{id}_{E_1}$ and $X^k = (X_i^k, k \in \mathbf{N})$, $X_i^k = X_i$ for $i > k$, $X_i^k = 0$ for $i \leq k$. X and X^k are in the same component, hence $f \circ X - f \circ X^k = \lim_{n \rightarrow \infty} (f_n \circ X - f_n \circ X^k) = \sum_{i=1}^k X_i$, i.e., $f \circ X = \sum_{i=1}^k X_i + f \circ X^k$. Note that $\sum_{i=1}^k X_i$, $f \circ X^k$ are independent and that $\sum_{i=1}^k X_i$ is $B(k, \frac{1}{2})$ -distributed.

Given $0 < \varepsilon \leq \frac{1}{20}$, choose $a \in \mathbf{R}$ with $\mu\{|f \circ X| > a\} < \varepsilon$ and choose $b < 0$ (by the central limit theorem) such that $\lim_{n \rightarrow \infty} \mu\{\sum_{i=1}^n X_i < \frac{1}{2}(n + b\sqrt{n})\} = \frac{1}{8}$, hence $\mu\{\sum_{i=1}^n X_i \geq b_n\} > \frac{3}{4}$ ($\forall n \geq n_b$), $b_n := \frac{1}{2}(n + b\sqrt{n})$. Then, on one hand,

$$\begin{aligned} \varepsilon > \mu\{f \circ X > a\} &\geq \mu\{f \circ X^n > a - b_n\} \mu\left\{\sum_{i=1}^n X_i \geq b_n\right\} \\ &> \frac{3}{4} \mu\{f \circ X^n > a - b_n\} \end{aligned}$$

yields $\mu\{f \circ X^n > a - b_n\} < \frac{4}{3}\varepsilon$ ($\forall n \geq n_b$). On the other hand,

$$\begin{aligned} \varepsilon > \mu\{f \circ X \leq -a\} &\geq \mu\{f \circ X^n \leq a - b_n\} \mu\left\{\sum_{i=1}^n X_i < b_n - 2a\right\} \\ &\geq \left(1 - \frac{4}{3}\varepsilon\right) \frac{1}{16} \quad (\forall n \geq m_b \geq n_b) \quad \text{for some } m_b \in \mathbf{N} \end{aligned}$$

yields $\varepsilon > \frac{3}{52}$, which contradicts $\varepsilon \leq \frac{1}{20}$. Therefore, the limit φ having a decomposition in G can not have a decomposition in F .

We next state some sufficient conditions which ensure the existence of a measurable decomposition of a limit point $\varphi \in F$, i.e.,

$$\varphi = f \oplus g \in L^0(\mu_1) \oplus L^0(\mu_2).$$

If $\varphi = f \oplus g 1_C = \tilde{f} \oplus \tilde{g} 1_C \in G$ are two different representations in G , then $f 1_{C_1^i} = \tilde{f} 1_{C_1^i} + \alpha_i 1_{C_1^i}$ and $g 1_{C_2^i} = \tilde{g} 1_{C_2^i} - \alpha_i 1_{C_2^i}$ for some $\alpha_i \in \mathbf{R}$ and all $i \in I$, i.e., the additive representation of φ is unique up to constants on the components $C_1^i \times C_2^i$ and we call $\tilde{f}, \tilde{g}$ a modification of f, g .

Define the μ -completions of $\mathcal{A}$ and $\mathcal{A}_i$,

$$\begin{aligned} \mathcal{A}' &:= (\mathcal{A}_1 \otimes \mathcal{A}_2)_\mu, \\ \mathcal{A}'_1 &:= \{A_1 \subset E_1; A_1 \times E_2 \in \mathcal{A}'\}, \\ \mathcal{A}'_2 &:= \{A_2 \subset E_2; E_1 \times A_2 \in \mathcal{A}'\}. \end{aligned} \tag{2.10}$$

Then $\mathcal{A}'_1 \otimes \mathcal{A}'_2$ is the largest product- σ -algebra in $\mathcal{A}'$, and $\mathcal{A}'_i$ depend on μ , not merely on the marginal measures. μ extends uniquely to a measure μ'

on $\mathcal{A}'$. Let μ'_i denote the marginals of μ' on $\mathcal{A}'_i$ and

$$F' := L^0(\mu'_1) \oplus L^0(\mu'_2). \quad (2.11)$$

The proof of the following closedness theorem and proposition shows that centering at the conditional median is the appropriate tool to obtain closedness. We formulate however the theorem with an easy to verify sufficient condition. For the appropriate and nearly necessary but not in general easy to check condition for closedness see the following remarks.

Theorem 2.1. *Suppose $\varphi_n = f_n \oplus g_n \in F'$ is μ' -stochastically convergent to φ . If*

$$\sup_n |\varphi_n| \text{ is real-valued,} \quad (2.12)$$

then φ is in F' .

Proof. We adhere to the notation in the proof of Proposition 2.1. Without loss of generality. $f_n \oplus g_n \rightarrow \varphi$ (μ' -a.e.).

1. We can assume that $\sum_{i \in I} C_1^i = E_1$ and $\sum_{i \in I} C_2^i = E_2$. Indeed, by $C \subset \pi_1(C) \times E_2$, and $\mu'(C) = 1$ it follows that $\pi_1(C) \times E_2 \in \mathcal{A}'$, hence $\pi_1(C) \in \mathcal{A}'_1$ and $\pi_2(C) \in \mathcal{A}'_2$ as well. Thus $\varphi(x, y) = f(x) 1_{\pi_1(C)}(x) + g(y) 1_{\pi_2(C)}(y)$ (μ' -a.e.) and $E_1 = \sum_{i \in I} C_1^i + (\pi_1(C))^c$, $E_2 = \sum_{i \in I} C_2^i + (\pi_2(C))^c$.

2. Define $T_1: (E_1, \mathcal{A}'_1) \rightarrow (I, J)$, $T_1(x) = i$ if $x \in C_1^i$, and J the σ -algebra generated by the one-point sets $\{i\}$, $i \in I$. Let $F_n(i, \cdot)$ denote the (right continuous) distribution function of the conditional distribution $\mu^{f_n|T_1=i}$, and let $F_n^{-1}(i, \cdot)$ denote the (left continuous) inverse distribution function (the quantile function) of $F_n(i, \cdot)$. Then $m_n(i) := F_n^{-1}(i, \frac{1}{2})$ is the smallest conditional median of f_n given $T_1 = i$. Since $F_n^{-1}(i, \cdot)$ is left continuous $\forall i \in I$, and $F_n^{-1}(\cdot, u)$ is J -measurable $\forall u \in]0, 1[$, we obtain that F_n^{-1} is $J \otimes (\mathcal{B} \cap [0, 1])$ -measurable, where $\mathcal{B} \cap [0, 1]$ is the Borel σ -algebra on $[0, 1]$. This implies that m_n is J -measurable and, therefore, $m_n \circ T_1$ is $\sigma(T_1)$ -measurable.

3. Let $(x_i, y_i) \in C_1^i \times C_2^i$, $i \in I$, be as in the proof of Proposition 2.1. Then

$$m(i) := \liminf (m_n(i) - f_n(x_i)) \text{ is finite } \forall i \in I. \quad (2.13)$$

Indeed, if $\psi := \sup |\varphi_n|$ and $(x_0, y_0) \in E$, then by assumption $|f_n(x) + g_n(y_0)| \leq \psi(x, y_0)$ and $|f_n(x_0) + g_n(y)| \leq \psi(x_0, y) \quad \forall x \in E_1, y \in E_2$. Consider the modification $\tilde{f}_n(x) = f_n(x) - \frac{1}{2}(f_n(x_0) - g_n(y_0))$, $\tilde{g}_n(y) = g_n(y) + \frac{1}{2}(f_n(x_0) - g_n(y_0))$. Then $|\tilde{f}_n| \leq F := \psi(\cdot, y_0) + \frac{1}{2}\psi(x_0, y_0) \in L^0(\mu'_1)$, $|\tilde{g}_n| \leq G := \psi(x_0, \cdot) + \frac{1}{2}\psi(x_0, y_0) \in L^0(\mu'_2)$ and $\varphi_n = \tilde{f}_n \oplus \tilde{g}_n$. Hence we assume without loss of generality that $f_n = \tilde{f}_n$ and $g_n = \tilde{g}_n$. Then $|m_n(i)| = |\text{med}(f_n | T_1 = i)| \leq \text{med}(F | T_1 = i) < \infty$ and $|f_n(x_i)| \leq F(x_i) < \infty \quad \forall n \in \mathbb{N}, \forall i \in I$. This yields that $m(i) = \liminf (m_n(i) - f_n(x_i))$ is finite.

4. Define

$$\hat{f}(x) := \overline{\lim} (f_n(x) - m_n \circ T_1(x)) = f(x) - m \circ T_1(x). \quad (2.14)$$

By (2.7) and (2.13) the function $\hat{f}$ is real valued and $\hat{f}$ is $\mathcal{A}'_1$ -measurable, since $f_n, m_n \circ T_1$ are $\mathcal{A}'_1$ -measurable. Define $T_2: (E_2, \mathcal{A}'_2) \rightarrow (I, J)$, $T_2(y) = i$ if $y \in C_2^i$. Then $m_n \circ T_1(x) = m_n \circ T_2(y) \forall (x, y) \in C$. Since

$$\begin{aligned}\hat{g}(y) &:= \underline{\lim}(g_n(y) + m_n(i)) \\ &= \underline{\lim}(g_n(y) + f_n(x) - f_n(x) + m_n(i)) \\ &= \varphi(x, y) - \hat{f}(x) \quad \text{for all } (x, y) \in C_1^i \times C_2^i, \\ &\quad \text{and all } i \in I\end{aligned}\tag{2.15}$$

we obtain the $\mathcal{A}'_2$ -measurable function

$$\hat{g}(y) = \underline{\lim}(g_n(y) + m_n \circ T_2(y))\tag{2.16}$$

and, by (2.14), (2.15), the desired decomposition

$$\varphi 1_C = (\hat{f} + \hat{g}) 1_C, \quad \text{i.e., } \varphi \in L^0(\mu'_1) \oplus L^0(\mu'_2).$$

Let us point out that if a measurable decomposition of the limit point φ in Proposition 2.1 exists at all, then the centering on the conditional median (as in the proof of Theorem 2.1) yields a measurable decomposition. To be precise:

Proposition 2.2. *Let $\varphi(x, y) 1_C(x, y) = (f(x) + g(y)) 1_C(x, y)$ be a limit point of $L^0(\mu_1) \oplus L^0(\mu_2)$ in $L^0(\mu)$ as in Proposition 2.1. If there exists a measurable decomposition $\varphi 1_C = \hat{f} \oplus \hat{g} 1_C \in L^0(\mu'_1) \oplus L^0(\mu'_2)$, then*

$$\begin{aligned}\tilde{f} &:= f - \sum_{i \in I} \text{med}(f 1_{C_1^i} | T_1 = i) 1_{C_1^i} \quad \text{is } \mathcal{A}'_1\text{-measurable,} \\ \tilde{g} &:= g + \sum_{i \in I} \text{med}(f 1_{C_1^i} | T_1 = i) 1_{C_2^i} \quad \text{is } \mathcal{A}'_2\text{-measurable, and}\end{aligned}$$

$$\varphi \cdot 1_C = \tilde{f} \oplus \tilde{g} 1_C,$$

where $T_1: (E_1, \mathcal{A}_1) \rightarrow (I, J)$, $T_1(C_1^i) = i$, and J is generated by the one-point sets.

P r o o f. The additive representation of $\varphi 1_C$ is unique up to constants on the components $C_1^i \times C_2^i \in \mathcal{A}_1 \otimes \mathcal{A}_2$, hence $\tilde{f} = f + \sum_{i \in I} \alpha_i 1_{C_1^i}$ for certain $\alpha_i \in \mathbb{R}$, $i \in I$. Since $f 1_{C_1^i}$ is $\mathcal{A}_1$ -measurable, we obtain

$$\begin{aligned}\tilde{f} 1_{C_1^i} - \text{med}(\tilde{f} 1_{C_1^i} | T_1 = i) 1_{C_1^i} \\ = f 1_{C_1^i} + \alpha_i 1_{C_1^i} - \text{med}(f 1_{C_1^i} + \alpha_i 1_{C_1^i} | T_1 = i) 1_{C_1^i} \\ = f 1_{C_1^i} - \text{med}(f 1_{C_1^i} | T_1 = i) 1_{C_1^i},\end{aligned}$$

and, therefore,

$$\tilde{f} = f - \text{med}(\hat{f} | T_1) = f - \sum_{i \in I} \text{med}(f 1_{C_1^i} | T_1 = i) 1_{C_1^i}$$

is $\mathcal{A}_1$ -measurable. Define

$$\tilde{g}(y) := g(y) + \sum_{i \in I} \text{med}(f 1_{C_i} | T_1 = i) 1_{C_i}.$$

Then $\tilde{g}$ is $\mathcal{A}_2$ -measurable. Indeed, $\tilde{g}(y) 1_C(x, y) = (\varphi(x, y) - \tilde{f}(x)) 1_C(x, y)$ is $\mathcal{A}'_1 \otimes \mathcal{A}'_2$ -measurable and for any $\alpha \in \mathbf{R}$ it holds $\{\tilde{g} 1_C \leq \alpha\} = E_1 \times \{\tilde{g} \leq \alpha\} \cap C \subset E_1 \times \{\tilde{g} \leq \alpha\} \subset E_1 \times \{\tilde{g} \leq \alpha\} \cap C + C^c$ with $\mu(C^c) = 0$. We obtain by definition of $\mathcal{A}'_1$ that $\{\tilde{g} \leq \alpha\} \in \mathcal{A}'_1$. Hence $\tilde{g}$ is $\mathcal{A}'_2$ -measurable and $\varphi 1_C = \tilde{f} \oplus \tilde{g} 1_C$.

Let ψ be a real valued function on E , i.e., $\psi \in M(E)$. Then denote by F'_ψ the subset of F' bounded by ψ , i.e.,

$$F'_\psi = \{h \in L^0(\mu'_1) \oplus L^0(\mu'_2); |h| \leq \psi\}. \quad (2.17)$$

Corollary 2.1. F'_ψ is closed in $L^0(\mu')$ with respect to μ' -stochastic convergence.

P r o o f. For any limit point φ of F'_ψ there exists a sequence φ_n in F'_ψ such that $\varphi_n \rightarrow \varphi$ (μ' -a.e.). Since $\sup_n |\varphi_n| \leq \psi$, the assertion follows from Theorem 2.1.

Corollary 2.2. Suppose (φ_n) is a sequence in $L^1(\mu'_1) \oplus L^1(\mu'_2) \subset L^1(\mu')$ and $|\varphi_n| \leq \psi \in M(E)$.

a) If $\varphi_n \rightarrow \varphi$ in $L^1(\mu)$, then $\varphi \in L^0(\mu'_1) \oplus L^0(\mu'_2)$. Moreover, $\varphi \in L^1(\mu'_1) \oplus L^1(\mu'_2)$ if $\psi(\cdot, y_0) \in L^1(\mu'_1)$ and $\psi(x_0, \cdot) \in L^1(\mu'_2)$ for some $(x_0, y_0) \in E$.

b) If $\varphi_n \rightarrow \varphi$ with respect to the weak topology $\sigma(L^1(\mu'), L^\infty(\mu'))$, then $\varphi \in L^0(\mu'_1) \oplus L^0(\mu'_2)$.

P r o o f. a) follows from Theorem 2.1. The integrability follows from majorized convergence.

b) follows from a) observing that $(L^1(\mu'_1) \oplus L^1(\mu'_2)) \cap \{h \in L^1(\mu'); |h| \leq \psi\}$ is convex and, therefore, the weak and the norm-closure of this set coincide.

R e m a r k s. a) The limit φ in Corollary 2.2 a) is not in $L^1(\mu'_1) \oplus L^1(\mu'_2)$ in general (cf. [18]).

b) An inspection of the proof of Theorem 2.1 shows that the boundedness condition (2.12) is only used to prove the finiteness condition (2.13), which is violated in the counterexample. So we can reformulate Theorem 2.1 in a more general form substituting (2.12) by (2.13), but in general it seems to be difficult to check the latter condition directly. By Proposition 2.2 it is seen that condition (2.13) is «nearly» necessary, too.

c) Let $L^0_{++}(\mu'_i) := \{f \in L^0(\mu'_i); f > 0 \text{ } (\mu'_i\text{-a.e.})\}$ and define the product $f \otimes g(s, t) := f(s)g(t)$. Then the limit points of sequences in $L^0_{++}(\mu'_1) \otimes L^0_{++}(\mu'_2)$ bounded in $L^0_{++}(\mu')$ are in $L^0_{++}(\mu'_1) \otimes L^0_{++}(\mu'_2)$ again. The proof is immediate from Theorem 2.1 by using logarithms.

d) One might think that it is possible to get a more direct construction of a measurable representation of a limit point $\varphi(x, y) \in L^1(\mu)$ in the form $f(x) + g(y)$ by means of the optimal L^1 -approximation operators S_i with respect to $L^1(\mu_i)$, i.e., the conditional medians

$$S_1\varphi(x) = \text{med}_\mu(\varphi \mid \pi_1 = x), \quad S_2\varphi(y) = \text{med}_\mu(\varphi \mid \pi_2 = y) \quad (2.18)$$

(cf. [19]). Consider the following example:

$$E_1 = E_2 = \{0, 1\}, \quad 0 < \varepsilon < \frac{1}{4},$$

$$\mu := \left(\frac{1}{2} - \varepsilon\right) (\delta_{\{(0,0)\}} + \delta_{\{(1,1)\}}) + \varepsilon (\delta_{\{(0,1)\}} + \delta_{\{(1,0)\}}) \quad (2.19)$$

and $\varphi(x, y) := f(x) + g(y)$ with $f(0) = g(0) = 0$, $f(1) = -g(1) = 1$. Then

$$S_1\varphi(x) = \text{med}_\mu(\varphi \mid \pi_1 = x) = 0 \quad \text{and}$$

$$S_2\varphi(y) = \text{med}_\mu(\varphi \mid \pi_2 = y) = 0. \quad (2.20)$$

In particular, the alternating projection algorithm does not converge to an optimal approximation and it seems impossible to identify f, g in terms of S_1, S_2 . The optimal approximation is given by the projection equations

$$S_1(\varphi - g) = f, \quad S_2(\varphi - f) = g. \quad (2.21)$$

But it is not clear how to solve them.

Theorem 2.1 shows that the limit point in $L^0(\mu')$ of a sequence $f_n \oplus g_n$ in F' satisfying (2.12) or (2.13) is of the form $f \oplus g$, where f, g are measurable only with respect to the μ -completions $\mathcal{A}'_i$. Therefore, let us point out that even in fairly general situations equality $\mathcal{A}'_i = \mathcal{A}_i$ holds if $\mathcal{A}_i$ are assumed to be complete in the usual sense. Concerning the notion of perfect measure spaces we refer to [16, Definition 2.1].

Theorem 2.2. *If $(E_i, \mathcal{A}_i, \mu_i)$ are complete perfect measure spaces, then $\mathcal{A}_i = \mathcal{A}'_i$. Therefore, given $\psi \in M(E)$, the set $F_\psi \subset (L^0(\mu_1) \oplus L^0(\mu_2))$ is closed in $L^0(\mu)$.*

Proof. A basic result in perfect measure spaces (cf. [16, Theorem 12.2.1]) implies that $\mu_*(A_1 \times E_2) = (\mu_1)_*(A_1)$ for any $A_1 \subset E_1$, where $\mu_*, (\mu_1)_*$ are the inner measures of μ, μ_1 . Given $A_1 \in \mathcal{A}'_1$, $(\mu_1)_*(A_1) + (\mu_1)_*(A_1^c) = \mu_*(A_1 \times E_2) + \mu_*(A_1^c \times E_2) = \mu'(A_1 \times E_2) + \mu'(A_1^c \times E_2) = 1$. Hence, there are $B_1 \in \mathcal{A}_1$, $B_1 \subset A_1$, $B_2 \subset A_1^c$ with $\mu_1(B_1) + \mu_1(B_2) = 1$. Therefore, $B_1 \subset A_1 \subset B_2^c$ and $\mu_1(B_1) = \mu(B_2^c)$. This implies $A_1 \in \mathcal{A}_1$, since $\mathcal{A}_1$ is μ_1 -complete.

Remark. Note that perfectness of a measure space is preserved by completion (cf. [16, 2.2, p. 5]). In particular, Theorem 2.2 implies the closedness property of F_ψ for polish spaces and, more general, for analytic spaces. The necessity to enlarge $\mathcal{A}_i$ to $\mathcal{A}'_i$ in Theorem 2.1 arises from the fact that

projections of product measurable sets onto the marginal spaces are known to be in the completions of the marginal spaces only under additional assumptions like analyticity. Therefore, we conjecture that the enlargement of A_i to A_i^c in Theorem 2.1 can not be omitted in general.

3. On the Generalized Schrödinger Problem.

Let $M(\mu_1, \mu_2)$ be the set of all probability measures on $(E_1 \times E_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$ with marginals μ_i . Let $I(\nu | \mu) = \int \ln(d\nu/d\mu) d\nu$ denote the Kullback-Leibler distance and define $\nu^* \in M(\nu_1, \nu_2)$ to be the I -projection of μ on $M(\nu_1, \nu_2)$ if

$$I(\nu^* | \mu) = \inf \{ I(\nu | \mu); \nu \in M(\nu_1, \nu_2), \nu \ll \mu \} =: m. \quad (3.1)$$

From the closedness results in Section 2 we obtain the existence of a solution of the generalized Schrödinger equation (1.1), (1.3), if m is finite.

Theorem 3.1. Assume that $\nu_i \ll \mu_i$, $i = 1, 2$, and $m < \infty$.

- a) There exists a uniquely determined I -projection ν^* on $M(\nu_1, \nu_2)$.
- b) There exists $a \otimes b \in L_+^0(\nu^*)$ such that $(d\nu^*/d\mu)(x, y) = a(x)b(y)$, (ν^* -a.e.).

P r o o f. a) $M(\nu_1, \nu_2)$ is closed in variation norm and so the assertion follows from [5, Theorem 2.1].

b) By Theorem 3.1 of [5], $\ln(d\nu^*/d\mu)$ belongs to the closure of $L^1(\nu_1) \oplus L^1(\nu_2)$ in $L^1(\nu^*)$. By Proposition 2.1, therefore, $\ln(d\nu^*/d\mu)(x, y) = f(x) + g(y)$ (ν^* -a.e.) for some $f \oplus g \in G$ and the result follows with $a = \exp f$, $b = \exp g$.

R e m a r k s. a) The functions a, b are not unique in general, but only the product $a \otimes b$ is unique ν^* -a.e. It is easily seen that a and b are unique up to constants if and only if any two points (x', y') and (x'', y'') in the support A of ν^* are finitely connected in A , i.e., there are finitely many points $(x_1, y_1) := (x', y'), (x_2, y_2), \dots, (x_n, y_n) := (x'', y'')$ in A , such that $x_i = x_{i+1}$ or $y_i = y_{i+1}$ for all $i = 1, \dots, n-1$. For example, this is true if $\mu = h\mu_1 \otimes \mu_2$, $h > 0$, $\mu_1 \otimes \mu_2$ -a.e.

b) Theorem 3.1 is of importance for the derivation of characterizations and properties of (generalized) Schrödinger bridges. To be precise, let P_0 be the Wiener measure on $S := C([0, T], \mathbf{R}^d)$, $0 < T < \infty$. Furthermore, with $\Omega = S^{\mathbf{N}}$ and with $P = P_0^{\mathbf{N}}$ let X_i be the projections. Föllmer [7] noted that Schrödinger bridges have a large deviation interpretation. If $\mu_n := \frac{1}{n} \sum_{i=1}^n \delta_{X_i}$ is the empirical measure of $X_1, \dots, X_n$, then, by Sanov's large deviation theorem, (μ_n) satisfies a LD-principle with rate function $I(Q_0 | P_0)$.

For a subset Γ of probability measures on S it holds:

$$P\{\mu_n \in \Gamma\} \sim \exp \left\{ -n \inf_{Q_0 \in \Gamma} I(Q_0 | P_0) \right\}. \quad (3.2)$$

If $\mu = P_0^{(X(0), X(T))}$ is the joint distribution of $(X(0), X(T))$ with respect to Wiener measure and

$$\Gamma := \{Q_0 \in M^1(S); Q_0^{X_0} = \nu_1, Q_0^{X(T)} = \nu_2\} \quad (3.3)$$

with $\nu_i \ll \mu_i$, then Γ describes large deviations at the initial and final time points.

For any $Q_0 \in \Gamma$ it holds

$$I(Q_0 | P_0) = I(\nu | \mu) + \int I(Q_x^y | P_x^y) \nu(dx, dy), \quad (3.4)$$

where $\nu = Q_0^{(X(0), X(T))}$ and Q_x^y, P_x^y are the conditional distributions of Q_0, P_0 given $X(0) = x, X(T) = y$. This implies

$$I(Q_0 | P_0) \geq I(\nu | \mu) \geq I(\nu^* | \mu) = I(P^* | P_0), \quad (3.5)$$

where $P^* := \int P_x^y \nu^*(dx, dy)$ is the Schrödinger bridge. This minimum entropy interpretation of P^* yields via Sanov's theorem (3.2) that P^* is «favoured» in the sense of large deviations by the empirical distribution in Γ .

Related results can be derived for modifications of Γ (like Prokhorov-type neighbourhoods) and also for general P_0 , replacing the Wiener measure, and thus leading to generalized Schrödinger bridges P^* . For the ample literature on these kind of problems we refer to [7], [14], [6], [4], [2], [20], [3], [11].

The factorization property $(d\nu^* d\mu)(x, y) = a(x)b(y)$, (ν^* -a.e.) is crucial for the derivation of characteristic properties of the Schrödinger bridges (like the (reciprocal) Markov property, diffusion property, h -path process). Theorem 3.1 gives a product form of the density but in general with nonmeasurable factors a, b only. It is however not clear whether this factorization is enough to derive the above mentioned properties of Schrödinger bridges.

4. Some Extensions and Applications

Clearly, the statements in Section 2 carry over to product spaces of $k \in \mathbb{N}$ factors. In particular, let $(E, \mathcal{A}) = \otimes_{i=1}^k (E_i, \mathcal{A}_i)$, let μ be a probability measure on $(E, \mathcal{A})$ with marginals μ_i and $\psi \in M(E)$.

Proposition 4.1. a) $(\oplus_{i=1}^k M(E_i)) \cap L^0(\mu')$ is closed in the metric space $L^0(\mu')$ with respect to μ' -stochastic convergence.

b) $(\oplus_{i=1}^k L^0(\mu'_i))_\psi$ is closed in $L^0(\mu')$.

For another aspect let $(\Omega, \mathcal{B}, P)$ be a complete probability space. Let $\mathcal{B}_1, \dots, \mathcal{B}_k \subset \mathcal{B}$ be sub- σ -algebras of $\mathcal{B}$ containing the P -null sets. $P_i := P | \mathcal{B}_i$ denote the restriction of P to $\mathcal{B}_i$ and $L^0(P_i) = L^0(\Omega, \mathcal{B}_i, P_i)$ are the real-valued $\mathcal{B}_i$ -measurable functions. Define μ on $\otimes_{i=1}^k (\Omega, \mathcal{B}_i)$ by $\mu(B_1 \times \dots \times B_k) = P(B_1 \cap \dots \cap B_k)$ and let $T_i : (\Omega, \mathcal{B}) \rightarrow (\Omega, \mathcal{B}_i)$, $T_i = \text{id}_\Omega$. Note that

$P(T_1, \dots, T_k) = \mu$, $P^{T_i} = P_i = \mu_i$, and that $\mathcal{B}_i$ coincides with its μ -completion. This leads to the following

Proposition 4.2. $(\oplus_{i=1}^k L^0(P_i)_\psi)$ is closed in $L^0(P)$ for any $\psi \in M(\Omega)$.

P r o o f. Let φ be a limit point and $\varphi_n = \sum_{i=1}^k h_{ni}$ be a sequence in $(\oplus_{i=1}^k L^0(P_i)_\psi)$ such that $\varphi_n \rightarrow \varphi$ (P -a.e.). Clearly, h_{ni} are $\sigma(T_i) = \mathcal{B}_i$ -measurable and φ is $\sigma(T_1, \dots, T_k) = \sigma(\cup_{i=1}^k \mathcal{B}_i)$ -measurable. By the factorization theorem, there are measurable functions $k_{ni} : (\Omega, \mathcal{B}_i) \rightarrow (\mathbf{R}, \mathbf{B})$ and $k : (\Omega^k, \otimes_{i=1}^k \mathcal{B}_i) \rightarrow (\mathbf{R}, \mathbf{B})$ such that $k_{ni} \circ T_i = h_{ni}$ and $k \circ (T_1, \dots, T_k) = \varphi$. This yields $1 = P\{\lim_{n \rightarrow \infty} \sum_{i=1}^k h_{ni} = \varphi\} = \mu\{\lim_{n \rightarrow \infty} \sum_{i=1}^k k_{ni} = k\}$, since $P(T_1, \dots, T_k) = \mu$. By assumption, $|\sum_{i=1}^k k_{ni}(\omega_i)| \leq \sum_{i=1}^k \psi(\omega_i) < \infty$ for all $(\omega_1, \dots, \omega_k) \in \Omega^k$, hence Theorem 2.1 or Proposition 4.1 yields $k(\omega_1, \dots, \omega_k) = \sum_{i=1}^k k_i(\omega_i)$ for some $k_i \in L^0(\mu_i) = L^0(P_i)$. This implies $\varphi = \sum_{i=1}^k h_i$, (P -a.e.), where $h_i = k_i \circ T_i$.

R e m a r k. In particular, Proposition 4.2 applies to functions of several variables like $\sum_{i=1}^{k-1} f(x_i, x_{i+1})$ or as in ridge regression theory to functions of the type $\sum_{i=1}^k f_i(A_i x)$, where A_i are some fixed functions on E .

From Theorem 3.1 and these propositions we obtain an interesting structure theorem for distributions with multivariate marginals. Let $(E, \mathcal{A}) = \otimes_{i=1}^k (E_i, \mathcal{A}_i)$, let $\mathcal{H} \subset \mathcal{P}(\{1, \dots, k\})$ be a system of subsets of $\{1, \dots, k\}$ and let $\mu \in M^1(E, \mathcal{A})$ have H -marginals $\mu_H := \mu^{\pi_H}$, $H \in \mathcal{H}$, where $\pi_H : E \rightarrow \prod_{j \in H} E_j =: E_H$ is the projection on the H -component. Let $\nu_H \in M^1(E_H, \mathcal{A}_H)$, $\mathcal{A}_H := \otimes_{j \in H} \mathcal{A}_j$, be continuous with respect to μ_H , $H \in \mathcal{H}$, and let $M_{\mathcal{H}} := M((\nu_H), H \in \mathcal{H})$ be the set of all probabilities on $(E, \mathcal{A})$ with H -marginals ν_H , $H \in \mathcal{H}$. The following proposition gives an interesting relation between $M(\mu_H, H \in \mathcal{H})$ and $M(\nu_H, H \in \mathcal{H})$.

Proposition 4.3. If $M_{\mathcal{H}} := M(\nu_H, H \in \mathcal{H}) \neq \emptyset$ and $\inf\{I(\nu | \mu); \nu \in M_{\mathcal{H}}\} < \infty$, then there exists $\otimes_{H \in \mathcal{H}} a_H \in L_+^0(\mu)$ such that $\nu^* = (\otimes_{H \in \mathcal{H}} a_H)\mu \in M_{\mathcal{H}}$, where $\otimes_{H \in \mathcal{H}} a_H(x) = \prod_{H \in \mathcal{H}} a_H(x_H)$, $x_H := \pi_H(x)$.

P r o o f. The proof is similar to that of Theorem 3.1 applying Proposition 4.1 with sub- σ -algebras $\sigma(\pi_H) \subset \mathcal{A} = \otimes_{j=1}^n \mathcal{A}_j$, $H \in \mathcal{H}$.

The following proposition gives a basic existence result for additive statistical models. Consider $(E, \mathcal{A}) = \otimes_{i=1}^k (E_i, \mathcal{A}_i)$ and a probability measure μ on $(E, \mathcal{A})$ with marginals μ_i . In additive statistical models the systematic part of the observations is assumed to be in $F^\alpha := \otimes_{i=1}^k L^\alpha(\mu_i) \subset L^\alpha(\mu)$, where $0 < \alpha \leq \infty$.

Proposition 4.4. a) For any $0 < \alpha \leq \infty$ there exists a probability space $(E, \mathcal{A}, \mu)$ such that F^α is not closed in $L^\alpha(\mu)$.

b) For $1 < \alpha < \infty$ and any $\varphi \in L^\alpha(\mu)$ there exists a unique optimal approximation $\bar{\varphi}$ of φ in the closure of F^α in $L^\alpha(\mu)$ and $\bar{\varphi} \in (\oplus_{i=1}^k M(E_i)) \cap L^0(\mu')$.

P r o o f. a) If $0 < \alpha < \infty$, then consider $E_1 = E_2 = \mathbf{N}$ and probabilities $\mu_1 = \mu_2$ with density $p(2n-1) = p(2n) = \frac{1}{2}2^{-n}$ and the probability measure $\mu \ll \mu_1 \otimes \mu_2$ with density $q(2n, 2n-1) = q(2n-1, 2n) = \frac{1}{2}2^{-2n}$,

$q(2n, 2n) = q(2n-1, 2n-1) = \frac{1}{2}(2^{-n} - 2^{-2n})$. μ has marginals μ_1, μ_2 . Define $f, g: \mathbf{N} \rightarrow \mathbf{R}^1$, $f(2n-1) = 2^{n/\alpha}$, $f(2n) = 0$, and $g(2n-1) = -2^{n/\alpha}$, $g(2n) = 0$. Then $f \notin L^\alpha(\mu_1)$, $g \notin L^\alpha(\mu_2)$ but $\varphi = f \oplus g \in L^\alpha(\mu)$ and $\int |\varphi|^\alpha d\mu = 1$. With $f_n := f \cdot 1_{\{1, \dots, n\}}$, $g_n := g \cdot 1_{\{1, \dots, n\}}$, $f_n \in L^\alpha(\mu_1)$, $g_n \in L^\alpha(\mu_2)$ and $\varphi_n := f_n \oplus g_n \rightarrow \varphi$ in $L^\alpha(\mu)$. Therefore, φ is in the closure of F^α in $L^\alpha(\mu)$. For any representation $\varphi = f' \oplus g'$ (μ -a.e.), it holds $f'(i) = f(i) + c_i$, $g'(j) = g(j) - c_j$ for some constants c_i and (i, j) with $q(i, j) > 0$. With $d_n := c_{2n-1} = -c_{2n}$ and using the inequality $|a + b|^\alpha \leq 2^\alpha(|a|^\alpha + |b|^\alpha)$ we obtain $\int |f'|^\alpha d\mu_1 = \sum_{n=1}^\infty [|2^{n/\alpha} + d_n|^\alpha \cdot \frac{1}{2} \cdot 2^{-n} + |-d_n|^\alpha \frac{1}{2} 2^{-n}] = \infty$. This implies that φ is not in F^α .

If $\alpha = \infty$, then consider $E_1 = E_2 = \mathbf{R}^1$, $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{B}^1$. Let $Q_1 = Q_2 = N(0, 1)$, $A = \{(x, y); x-1 \leq y \leq x+1\} \in \mathcal{B}^2$ and let μ be the product $Q_1 \otimes Q_2$ conditional on A . Furthermore, let $f = \text{id}_{\mathbf{R}^1}$, $g = -\text{id}_{\mathbf{R}^1}$ and $\varphi := f \oplus g$. $\varphi = f \oplus g$ is the unique measurable additive decomposition of φ but $f \notin L^\infty(\mu_1)$, $g \notin L^\infty(\mu_2)$, $\varphi \in L^\infty(\mu)$, since $|\varphi| \leq 1$ (μ -a.e.).

b) Let $(\varphi_n) \subset F^\alpha$ be a sequence such that $\|\varphi - \varphi_n\|_\alpha \rightarrow \inf\{\|\varphi - h\|_\alpha; h \in F^\alpha\} =: a$. Then (φ_n) is norm-bounded and, therefore, relatively weakly sequentially compact. Since F^α is convex the norm-closure and the weak closure coincide and, therefore, $\|\varphi - \bar{\varphi}\| = a$ for any accumulation point $\bar{\varphi}$ of (φ_n) . Obviously, $\bar{\varphi}$ is uniquely determined and by Proposition 4.1, $\bar{\varphi} \in (\oplus_{i=1}^k M(E_i)) \cap L^0(\mu)$ and $\bar{\varphi}$ is the optimal approximation of φ .

The final application will be an extension of a famous representation theorem of Kolmogorov (1957) for continuous functions $f: [0, 1]^n \rightarrow \mathbf{R}^1$ of n variables, disproving conjecture no. 13 of Hilbert's list of 23 problems. Kolmogorov's theorem states the existence of real continuous functions $\alpha_{ij}: [0, 1] \rightarrow \mathbf{R}^1$, $1 \leq i \leq 2n+1$, $1 \leq j \leq n$, such that any continuous function $f = f(x_1, \dots, x_n)$ has a representation of the form

$$f(x_1, \dots, x_n) = \sum_{i=1}^{2n+1} g\left(\sum_{j=1}^n \alpha_{ij}(x_j)\right) \quad (4.1)$$

with some continuous functions g depending on f . The functions α_{ij} even may be chosen to be Lipschitz-continuous, and $(\sum_j \alpha_{ij})$ to be one-to-one. As reference for this result and several modifications we refer to Chapter 11 of [13]. (4.1) is considered to be a theoretical justification of neural networks. We obtain a «generalization» to measurable functions as follows.

Proposition 4.5. *Let $\mu \in M^1(\mathbf{R}^n, \mathcal{B}^n)$ and let $f \in L^0(\mu)$ be locally bounded. Then*

$$f(x_1, \dots, x_n) = \sum_{i=1}^{2n+1} g\left(\sum_{j=1}^n \alpha_{ij}(x_i)\right) \quad (\mu\text{-a.e.})$$

for some measurable functions $g: \mathbf{R}^1 \rightarrow \mathbf{R}^1$ and $\alpha_{ij}: \mathbf{R}^1 \rightarrow \mathbf{R}^1$ piecewise Lipschitz-continuous.

P r o o f. Since f is assumed to be locally bounded on $\mathbf{R}^n$, it is majorized by a continuous function h on $\mathbf{R}^n$, $|f| \leq h$. Let without loss of generality $f \geq 0$, then decomposing $\mathbf{R}^n$ into disjoint unions of cubes we can approximate f by a sequence $0 \leq f_m \leq h$ of continuous functions with respect to μ -stochastic convergence. By (4.1) f_m has a representation of the form

$$f_m(x) = \sum_{i=1}^{2n+1} g_m \left(\sum_{j=1}^n \alpha_{ij}(X_j) \right),$$

where g_m are continuous on the cubes and, in particular, measurable on $\mathbf{R}^n$ and the (α_{ij}) are as in (4.1). From the proof of (4.1) in [14] it is seen that a representing function $g = g_f$ in (4.1) can be chosen such that $f_1 \leq f_2$ implies $g_{f_1} \leq g_{f_2}$. This yields $0 \leq g_m \leq g_h$ for all $m \in \mathbf{N}$. Then, by Proposition 4.2 with $\mathcal{B}_i = \sigma(T_i)$, Remark b) implies the existence of a measurable function g such that $f(x) = \sum_{i=1}^{2n+1} g(\sum_{j=1}^n \alpha_{ij}(x_j))$ (μ -a.e.).

R e m a r k s. a) Kolmogorov's theorem is considered as a theoretical foundation of neural networks (cf. [9, p. 122–137], [17]). There are many papers on generalizations of this theorem to non continuous functions and to variants of the theorem with special kind of functions α_{ij} in the literature on neural networks. Proposition 4.5 improves the known representation and approximation results for neural networks like the back propagation theorem (cf. [9, p. 132] and references given there). For the first time it states an exact representation result for standard feed forward networks with one hidden layer. It was known up to now that two hidden layers are sufficient (cf. the discussion of this point in [17] and [9]).

b) By the monotonicity property of the representing function g in (4.1) it is clear that the representation (4.1) holds for any continuous function f on $\mathbf{R}^n$. Therefore, it is possible to extend the representation property to measurable functions majorized on open subsets U_j by continuous functions f_j such that $\mu(\partial U_j) = 0$, (U_j) pairwise disjoint and $\cup \overline{U_j} = \mathbf{R}^n$.

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REFERENCES

1. *Beurling A.* An automorphism of product measures. — *Ann. Math.*, 1960, v. 72, p. 189–200.
2. *Blaquiere A., Sigal-Pouchard M.* Schrödinger processes with unbounded or singular potentials, conditional Sanov property. — *Proceedings of 5th Workshop of the Bellman Continuum*, Hawai, 1993.
3. *Bolthausen E.* Markov process large deviations in τ -topology. — *Stoch. Process. Appl.*, 1987, v. 25, p. 95–108.
4. *Brunaud M.* Finite Kullback information diffusion laws with fixed marginals and

- associated large deviations functionals. — Stoch. Process. Appl., 1993, v. 44, p. 329–345.
5. *Csiszar I.* I-divergence geometry of probability distributions and minimization problems. — Ann. Probab., 1975, v. 3, p. 146–158.
 6. *Dawson D., Gorostiza L., Wakolbinger A.* Schrödinger processes and large deviations. — J. Math. Phys., 1990, v. 31, p. 2385–2388.
 7. *Föllmer H.* Random fields and diffusion processes. — Lect. Notes Math., 1988, v. 1362, p. 101–203.
 8. *Fortet P. R.* Résolution d'un système d'équations de M. Schrödinger. — J. Math. Pures Appl., 1940, v. 9, p. 83–105.
 9. *Hecht-Nielsen R.* Neurocomputing. Reading, MA: Addison-Wesley, 1989.
 10. *Hobby C., Pyke R.* Doubly stochastic operators obtained from positive operators. — Pac. J. Math., 1965, v. 15, p. 153–157.
 11. *Jamison B.* Reciprocal processes. — Z. Wahrscheinlichkeitstheor. verw. Geb., 1974, B. 30, S. 65–86.
 12. *Колмогоров А. Н.* О представлении непрерывных функций нескольких переменных суперпозициями непрерывных функций меньшего числа переменных. — Докл. АН СССР, т. 108, с. 179–182.
 13. *Light W. A., Cheney E. W.* Approximation Theory in Tensor Product Spaces. (Lect. Notes Math., v. 1169). Berlin: Springer-Verlag, 1985.
 14. *Lorentz G. G.* Approximation of Functions. New York: Holt, Rinehart and Winston, 1966.
 15. *Nagasawa M.* Can the Schrödinger equation be a Boltzmann equation? — In: Diffusion Processes and Related Problems in Analysis/Ed. by M. Pinsky. Basel: Birkhäuser, 1990.
 16. *Ramachandran D.* Perfect Measures. New York: Macmillan, 1979.
 17. *Ripley B. D.* Statistical Aspects of Neural Networks. (Monographs on Statist. Appl. Probab. 50.) London: Chapman & Hall, 1993, p. 40–123.
 18. *Rüschendorf L., Thomsen W.* Note on the Schrödinger equation and I -projections. — Statist. Probab. Lett., 1993, v. 17, p. 369–375.
 19. *Shintani T., Ando T.* Best approximation in L^1 -space. — Z. Wahrscheinlichkeitstheor. verw. Geb., 1975, B. 33, S. 33–39.
 20. *Schrödinger E.* Über die Umkehrung der Naturgesetze. Sitzungsberichte Preuss. — Akad. Wiss. Berlin, Phys. Math., 1931, B. 144, S. 144–153.
 21. *Wakolbinger A.* Schrödinger bridges from 1931 to 1991. — In: Proc. 4th Latin America Congress in Probab. Math. Statist./Ed. by E. Cabana, 1992, p. 61–79.

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