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ON MOMENTS OF POLYNOMIALS¹⁾Исследуется эквивалентность L^p -норм многочленов от случайной величины

Ключевые слова и фразы: моменты неравенства, многочлены.

Let ξ be a random variable with nondegenerate distribution F_ξ on $\mathbf{R}$. For ξ normal, Yu. V. Prokhorov showed [3] that, for any polynomial $Q = Q(x)$, $x \in \mathbf{R}$, of degree $n \geq 1$ with $\mathbf{E}Q(\xi) = 0$,

$$(\mathbf{E}|Q(\xi)|^2)^{1/2} \leq c \mathbf{E}|Q(\xi)|, \quad (1)$$

where $c = c_n$ depends on n , only. He proved in [4] a similar inequality for ξ which has $\Gamma(\alpha)$ -distribution with large parameter $\alpha (\geq 400n \log 2)$. In the first case he used an expansion of Q in Hermite polynomials, and in the second paper, he used an expansion of Q in Laguerre polynomials. In this note, we suggest a simple argument to extend (1) to general r.v. ξ under some integrability assumptions. We show that the condition $\mathbf{E}Q(\xi) = 0$ is not necessary. We start with a more general inequality (which is stronger than (1) when $p \geq 2$), namely,

$$\|Q(\xi)\|_p = (\mathbf{E}|Q(\xi)|^p)^{1/p} \leq c \|Q(\xi)\|_0 \quad (2)$$

for the geometric mean $\|Q(\xi)\|_0 = \lim_{p \rightarrow 0+} \|Q(\xi)\|_p = \exp \mathbf{E} \log |Q(\xi)|$ instead of the L^1 -norm $\|Q(\xi)\|_1$ which is difficult to work with (all the constants $c = c_n(p, F_\xi)$ which we deal with may depend on n, p, F_ξ but not on Q). In contrast to the L^1 -norm, one may use a general identity $\|f_1 \cdots f_n\|_0 = \|f_1\|_0 \cdots \|f_n\|_0$ which reduces (2) to polynomials of type $Q(x) = (x - z)^n$. More precisely, we have:

Proposition 1. Let $\mathbf{E}|\xi|^{np} < \infty$, ($p > 0$). The optimal constant c in (2) is given by

$$c^{1/n} = \sup_{z \in \mathbf{C}} \frac{\|\xi - z\|_{np}}{\|\xi - z\|_0}. \quad (3)$$

In the case $c < \infty$ (and only in this case), there exist positive constants a and b such that for any polynomial $Q(x) = \prod_{i=1}^n (x - z_i)$, $z_i \in \mathbf{C}$ ($1 \leq i \leq n$), and for all $r \in [0, p]$,

$$a \prod_{i=1}^n (1 + |z_i|) \leq \|Q(\xi)\|_r \leq b \prod_{i=1}^n (1 + |z_i|). \quad (4)$$

Remark 2. Let $\mathbf{E}|\xi|^{np} < \infty$. The constant c defined by (3) is finite if and only if

$$\sup_{a \in \mathbf{R}} \int_0^1 \frac{\mathbf{P}\{|\xi - a| < t\}}{t} dt < +\infty. \quad (5)$$

Proof. Clearly, for some $c_1, c_2 > 0$, we have

$$c_1 (1 + |z|) \leq \|\xi - z\|_{np} \leq c_2 (1 + |z|),$$

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whenever $z \in \mathbf{C}$. Hence, $c < \infty$ holds if and only if $\inf_z \|\xi - z\|_0 / (1 + |z|) > 0$. This infimum may be taken over all real $z = a$. Indeed, writing $z = a + bi$, $a, b \in \mathbf{R}$ and assuming that $\|\xi - a\|_0 \geq c_3(1 + |a|)$, we have

$$\begin{aligned} \|\xi - z\|_0 &= \|(\xi - a)^2 + b^2\|_0^{1/2} \geq (\|\xi - a\|_0^2 + b^2)^{1/2} \\ &\geq (c_3^2(1 + |a|)^2 + b^2)^{1/2} \\ &\geq c_4(1 + |z|), \end{aligned}$$

where we used the property $\|f + g\|_0 \geq \|f\|_0 + \|g\|_0$ for f and g non-negative with $f = (\xi - a)^2$ and $g = b^2$.

Now, write the relation $\inf_{a \in \mathbf{R}} \|\xi - a\|_0 / (1 + |a|) > 0$ in the form

$$\mathbf{E} \log |\xi - a| \geq D + \log(1 + |a|), \quad a \in \mathbf{R}, \quad (6)$$

for some $D \in \mathbf{R}$. Since $\mathbf{E} \log(1 + |\xi|) < +\infty$, we get, for all $a \in \mathbf{R}$, obvious estimates

$$\begin{aligned} \left| \mathbf{E} \log(1 + |\xi - a|) - \log(1 + |a|) \right| &\leq \mathbf{E} \log(1 + |\xi|), \\ \mathbf{E} \log(1 + |\xi - a|) - \mathbf{E} \log |\xi - a| \mathbf{1}_{\{|\xi - a| \geq 1\}} &\leq \log 2, \end{aligned}$$

from which it follows that

$$\left| \mathbf{E} \log |\xi - a| \mathbf{1}_{\{|\xi - a| \geq 1\}} - \log(1 + |a|) \right| \leq D', \quad a \in \mathbf{R},$$

for some $D' \in \mathbf{R}$. Therefore, (6) may be written as

$$-\mathbf{E} \log |\xi - a| \mathbf{1}_{\{|\xi - a| < 1\}} \leq D'', \quad a \in \mathbf{R},$$

for some $D'' \in \mathbf{R}$, which is exactly (5).

For example, when the distribution ξ is unimodal, with mode a_0 , we have for all a ,

$$\mathbf{P}\{|\xi - a| < t\} \leq \mathbf{P}\{|\xi - a_0| < 2t\},$$

and (5) becomes

$$\int_0^1 \frac{\mathbf{P}\{|\xi - a_0| < t\}}{t} dt < \infty.$$

In particular, this condition is satisfied for $\Gamma(\alpha)$ -distribution with arbitrary $\alpha > -1$. It should be noted however that the method used by Prokhorov yields constants c_n in (1) which increase exponentially (as functions of n), while in the case $r = 0$, we get a worse rate in (4) unless ξ is bounded: in this case, according to (3), the sequence $c^{1/n} = c_n(p, F_\xi)^{1/n}$ tends to infinity.

Proof of Proposition 1. Writing $Q(x) = A \prod_{i=1}^n (x - z_i)$, $x \in \mathbf{C}$, and defining c as in (3), we get, by Hölder's inequality,

$$\|Q(\xi)\|_p \leq |A| \prod_{i=1}^n \|\xi - z_i\|_{np} \leq c |A| \prod_{i=1}^n \|\xi - z_i\|_0 = c \|Q(\xi)\|_0.$$

Thus, (2) is proved. Similarly, it suffices to consider in (4) only polynomials of the form $Q(x) = (x - z)^n$. In addition, the cases $r = 0$ for the left inequality and $r = p$ for the right inequality have to be considered, only. The second case, i.e., the inequality

$\|\xi - z\|_{np} \leq \text{const} (1 + |z|)$ holds due to the assumption $\|\xi\|_{np} < \infty$. The first case, i.e., the inequality $\|\xi - z\|_0 \geq \text{const} (1 + |z|)$ is equivalent to $c < \infty$ (as explained in Remark 2).

R e m a r k 3. It would be very interesting to know, whether or not inequalities between L^p - and L^q -norms like (1) hold for polynomials Q in d random identically distributed random variables $\xi = (\xi_1, \dots, \xi_d)$. One may expect that, under a general assumption such as integrability, such a conjecture is valid, and the constants are independent of the dimension d . This is true for example for random variables ξ_i which are uniformly distributed in an interval, and thus, by the central limit theorem, for normally distributed ξ_i as well (cf. [2] for a more general result). Some related Khinchin-Kahane-type inequalities for Bernoullian random variables may be found in [1] and [5].

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