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GENERAL ASYMPTOTIC BAYESIAN THEORY OF QUICKEST CHANGE DETECTION¹⁾

Оптимальное правило обнаружения изменений свойств независимых и одинаково распределенных (н.о.р.) последовательностей в байесовской постановке задачи получено А. Н. Ширяевым в 1960-е годы. Однако задача анализа характеристик этого правила — средней задержки обнаружения и вероятности ложной тревоги — оставалась открытой. В настоящей статье разрабатывается общая асимптотическая теория обнаружения изменений (разладки), которая не ограничена жестким н.о.р.-допущением. Характеристики правила Ширяева исследуются для общих статистических моделей в дискретном времени в асимптотической постановке задачи, когда вероятность ложной тревоги стремится к нулю. Показано, что правило Ширяева асимптотически оптимально в случае зависимых и неодинаково распределенных наблюдений при весьма слабых условиях. Показано также, что две популярные небайесовские процедуры обнаружения — процедура Пейджа и процедура Ширяева-Робертса-Поллака — вообще говоря, неоптимальны (даже асимптотически) для байесовского критерия. Результаты исследования являются особенно важными для изучения асимптотик в нецентрализованных распределенных системах обнаружения.

Ключевые слова и фразы: обнаружение изменений (разладки), последовательное обнаружение, асимптотическая оптимальность, нелинейная теория восстановления, правило Ширяева, процедура кумулятивных сумм.

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1. Introduction

The problem of detecting abrupt changes in stochastic processes arises in a variety of applications including biomedical signal processing, quality control engineering, finance, link failure detection in communication networks, intrusion detection in computer systems, and target detection in surveillance systems [1], [4], [12], [19], [30]. A typical such problem is one of target detection in multisensor systems (radar, infrared, sonar, etc.) [30], [35], [39], where the target appears randomly at an unknown time. The goal is to detect the target as quickly as possible, while maintaining the false alarm rate at a given level. Another application area is intrusion detection in distributed computer networks [4], [12], [39]. Large scale attacks, such as denial-of-service attacks, occur at unknown points in time and need to be detected in the early stages by observing abrupt changes in the network traffic.

The design of the quickest change detection procedures usually involves optimizing the tradeoff between two kinds of performance measures, one being a measure of detection delay and the other being a measure of the frequency of false alarms. There are two standard mathematical formulations for the optimum tradeoff problem. The first of these is a minimax formulation proposed by Lorden [17] and Pollak [21], in which the goal is to minimize the worst-case delay subject to a lower bound on the mean time between false alarms. The second is a Bayesian formulation, proposed by Shiryaev [25]–[27], in which the change point is assumed to have a geometric prior distribution, and the goal is to minimize the expected delay subject to an upper bound on false alarm probability. The asymptotic performance of various change-point detection procedures is well understood in the minimax context for both the discrete- and continuous-time cases (see [1], [3], [7], [9], [17], [18], [20]–[23], [28], [30]–[32], [34], [39], [43], [44]). However, there has been little previous work on the asymptotics of Bayesian procedures. The exception is the work by Lai [16] in which the asymptotic properties of Page's cumulative sum (CUSUM) procedure were studied in a Bayesian (as well as minimax) context for general stochastic models. (See also Beibel [3] for continuous-time Brownian motion and Borovkov [5, Lemma 5] for i.i.d. data models.) Our goal is to provide a general Bayesian asymptotic theory for change point detection.

The paper is organized as follows. In Section 2, we formulate the problem. In Section 3, we study the behavior of the Shiryaev detection procedure for general, non-i.i.d. data models, and prove that it is asymptotically optimal under mild conditions when the false alarm probability goes to zero. We show that this procedure is asymptotically optimal not only with respect to the average detection delay, but that it is also uniformly asymptotically optimal in the sense of minimizing the conditional expected delay for every

change point. Moreover, we study the behavior of higher moments of the detection delay and show that under certain general conditions the Shiryaev procedure minimizes moments of the detection delay up to the given order. In Section 4, we find the asymptotic operating characteristics of the Shiryaev change detection procedure in the i.i.d. case when the false alarm probability goes to zero. The use of nonlinear renewal theory allows us to obtain sharp asymptotic approximations for the false alarm probability and the average detection delay up to vanishing terms. In Section 5, we analyze the asymptotic performance of other well-known change detection procedures (Page's procedure and the Shiryaev–Roberts–Pollak procedure) in the Bayesian framework. The results of this section allow us to conclude that, while being optimal in the minimax context, these procedures may lose their optimality property (even asymptotically) with respect to the Bayesian criterion depending on the structure of the prior distribution. In Section 6, we consider an example of detecting a change in the mean value of an autoregressive process that illustrates general results. In Section 7, we consider two additional examples related to detecting changes in distributed multi-sensor systems. We study the implications of the asymptotic results in decentralized quickest change detection problems assuming that sensors send quantized versions of their observations to a *fusion* center (central processor), where the change detection is performed based on all the sensor messages. Finally, in Section 8, we conclude the paper by giving several remarks.

The results of this paper have been presented in part at the International Symposium on Information Theory [36] and the Information Theory Workshop [37] in 2002, and at the Sixth International Conference on Information Fusion [38] in 2003.

2. Problem formulation

In the conventional setting of the change-point detection problem one assumes that the observed random variables $X_1, X_2, \dots$ are i.i.d., until a change occurs at an unknown point in time λ , $\lambda \in \{1, 2, \dots\}$. After the change occurs, the observations are again i.i.d. but with another distribution. In other words, conditioned on $\lambda = k$, the observations $X_1, X_2, \dots$ are independent with $X_n \sim f_0$ for $n < k$ and $X_n \sim f_1$ for $n \geq k$, where $f_0(x)$ and $f_1(x)$ are, respectively, the pre-change and post-change probability density functions (PDF) with respect to a sigma-finite measure μ . For the sake of brevity, in what follows, this case will be referred to as the «i.i.d. case.»

In a Bayesian setting, the change point λ is assumed to be random with prior probability distribution $\pi_k = \mathbf{P}\{\lambda = k\}$, $k = 0, 1, 2, \dots$. The goal is to detect the change as soon as possible after it occurs, subject to the false alarm probability constraints.

In mathematical terms, a sequential detection procedure is identified with a stopping time τ for an observed sequence $\{X_n\}_{n \geq 1}$, i.e., τ is an extended integer-valued random variable, such that the event $\{\tau \leq n\}$ belongs to the sigma-algebra $\mathcal{F}_n^X = \sigma(X_1, \dots, X_n)$. A false alarm is raised whenever the detection is declared before the change occurs, i.e., when $\tau < \lambda$. A good detection procedure should guarantee a «stochastically small» detection lag $\tau - \lambda$ provided that there is no false alarm (i.e., $\tau \geq \lambda$), while the rate of false alarms should be low.

Let $\mathbf{P}_k$ and $\mathbf{E}_k$ denote the probability measure and the corresponding expectation when the change occurs at time $\lambda = k$. In what follows, $\mathbf{P}^\pi$ stands for the «average» probability measure which is defined as $\mathbf{P}^\pi(\Omega) = \sum_{k=0}^{\infty} \pi_k \mathbf{P}_k(\Omega)$, and $\mathbf{E}^\pi$ denotes the expectation with respect to $\mathbf{P}^\pi$.

In the Bayesian setting, a reasonable measure of the detection lag is the average detection delay (ADD)

$$\begin{aligned} \text{ADD}(\tau) &= \mathbf{E}^\pi(\tau - \lambda \mid \tau \geq \lambda) = \frac{\mathbf{E}^\pi(\tau - \lambda)^+}{\mathbf{P}^\pi\{\tau \geq \lambda\}} \\ &= \frac{1}{\mathbf{P}^\pi\{\tau \geq \lambda\}} \sum_{k=0}^{\infty} \pi_k \mathbf{P}_k\{\tau \geq k\} \mathbf{E}_k\{\tau - k \mid \tau \geq k\}, \end{aligned} \quad (2.1)$$

and the false alarm rate can be measured by the probability of false alarm (PFA)

$$\text{PFA}(\tau) = \mathbf{P}^\pi\{\tau < \lambda\} = \sum_{k=1}^{\infty} \pi_k \mathbf{P}_k\{\tau < k\}. \quad (2.2)$$

Here and henceforth, we use the traditional notation $x^+ = \max(0, x)$ for the positive part of x .

An optimal Bayesian detection procedure is a procedure for which ADD is minimized, while PFA is constrained to be below a given (usually small) level α , $\alpha \in (0, 1)$. Specifically, define the class of change-point detection procedures $\Delta(\alpha) = \{\tau: \text{PFA}(\tau) \leq \alpha\}$ for which the false alarm probability does not exceed the predefined number α . The optimal change-point detection procedure is described by the stopping time

$$\nu = \arg \inf_{\tau \in \Delta(\alpha)} \text{ADD}(\tau).$$

Obviously, if $\text{PFA}(\nu) = \alpha$, then ν also minimizes $\mathbf{E}^\pi(\tau - \lambda)^+$.

Let $\mathbf{X}^n = (X_1, \dots, X_n)$ denote the concatenation of the first n observations, let $\mathcal{F}_n^X = \sigma(\mathbf{X}^n)$ be a sigma-algebra generated by $\mathbf{X}^n$, and let $p_n = \mathbf{P}\{\lambda \leq n \mid \mathcal{F}_n^X\}$ be the posterior probability that the change occurred before time n . For the i.i.d. case, Shiryaev [25]–[27] proved that if the distribution of the change point is geometric, i.e., $\mathbf{P}\{\lambda = 0\} = \pi_0$, $\pi_k = (1 - \pi_0) \rho(1 - \rho)^{k-1}$, $k \geq 1$ ($0 < \rho < 1$, $0 \leq \pi_0 < 1$), then the optimal detection procedure is the one that raises an alarm at the first time such

that the posterior probability p_n exceeds a threshold A , i.e.,²⁾

$$\nu(A) = \inf\{n \geq 1: p_n \geq A\}, \quad (2.3)$$

where the threshold $A = A_\alpha$ should be chosen so that $\text{PFA}(\nu(A)) = \alpha$. A generalization of this result for an arbitrary prior distribution has been stated in [5], albeit without proof (see [5, Theorem 8]). However, except for the case of detecting the change in the drift of the Wiener process observed in continuous time, it is difficult to find a threshold that provides an exact match to the given PFA. Also, there are no results related to the ADD evaluation of this optimal procedure, again, except for the continuous-time Wiener process [27] with exponential prior distribution, and for i.i.d. data models with a geometric prior distribution when $\rho \rightarrow 0$ (see [5, Lemma 5]).

While the exact match of the false alarm probability is related to the estimation of the overshoot in the stopping rule (2.3), and for this reason is problematic, a simple upper bound, which ignores overshoot, can be obtained. Indeed, since $\mathbf{P}^\pi\{\nu(A) < \lambda\} = \mathbf{E}^\pi\{1 - p_{\nu(A)}\}$ and $1 - p_{\nu(A)} \leq 1 - A$ on $\{\nu(A) < \infty\}$, we obtain that the PFA defined in (2.2) obeys the inequality

$$\text{PFA}(\nu(A)) \leq 1 - A. \quad (2.4)$$

It follows that setting $A = A_\alpha = 1 - \alpha$ guarantees the inequality $\text{PFA}(\nu(A_\alpha)) \leq \alpha$. Note that inequality (2.4) holds true for arbitrary (proper), not necessarily geometric, prior distributions.

More generally, assume that observations are non-i.i.d. in the pre-change and post-change modes. Specifically, let $\mathbf{P}_\infty$ stand for the probability measure under which the conditional density of X_n given $\mathbf{X}^{n-1} = (X_1, \dots, X_{n-1})$ is $f_{0,n}(X_n | \mathbf{X}^{n-1})$ for every $n \geq 1$ (i.e., $\lambda = \infty$). Furthermore, for any $\lambda = k$, $1 \leq k < \infty$, let $\mathbf{P}_k$ stand for the probability measure under which the conditional density of X_n is $f_{0,n}(X_n | \mathbf{X}^{n-1})$ if $n \leq k - 1$ and is $f_{1,n}(X_n | \mathbf{X}^{n-1})$ if $n \geq k$. Therefore, if the change occurs at time $\lambda = k$, then the conditional density of the k -th observation changes from $f_{0,k}(X_k | \mathbf{X}^{k-1})$ to $f_{1,k}(X_k | \mathbf{X}^{k-1})$.

The log-likelihood ratio (LLR) for the hypotheses that the change occurred at the point $\lambda = k$ and at $\lambda = \infty$ (no change at all) is

$$Z_n^k := \ln \frac{d\mathbf{P}_k}{d\mathbf{P}_\infty}(\mathbf{X}^n) = \sum_{t=k}^n \ln \frac{f_{1,t}(X_t | \mathbf{X}^{t-1})}{f_{0,t}(X_t | \mathbf{X}^{t-1})}, \quad k \leq n. \quad (2.5)$$

In what follows, we will use the convention that before the observations become available (i.e., for $n = 0$), $Z_0^0 = \ln[f_{1,0}(X_0)/f_{0,0}(X_0)] = 0$ almost everywhere.

In the rest of the paper, we will consider prior distributions $\pi_k = \mathbf{P}\{\lambda = k\}$ concentrated on nonnegative integers. The following two classes

²⁾ Hereafter we use the convention that $\inf \emptyset = \infty$, i.e., $\nu(A) = \infty$ if no such n exists.

of prior distributions will be covered: distributions with exponential tail for which

$$\lim_{k \rightarrow \infty} \frac{\ln \mathbf{P}\{\lambda \geq k+1\}}{k} = -d, \quad d > 0; \quad (2.6)$$

and prior distributions with «heavy» tails for which

$$\lim_{k \rightarrow \infty} \frac{\ln \mathbf{P}\{\lambda \geq k+1\}}{k} = 0. \quad (2.7)$$

The first class will be denoted by $\mathcal{E}(d)$, and the second class by $\mathcal{H}$.

Clearly, the geometric prior distribution with the parameter ρ ,

$$\pi_k = \pi_0 \mathbb{1}_{\{k=0\}} + (1 - \pi_0) \rho (1 - \rho)^{k-1} \mathbb{1}_{\{k \geq 1\}}, \quad 0 < \rho < 1, \quad 0 \leq \pi_0 < 1, \quad (2.8)$$

belongs to the class $\mathcal{E}(d)$ with $d = |\ln(1 - \rho)|$. Here and henceforth, $\mathbb{1}_{\{\mathcal{A}\}}$ denotes an indicator of the set $\mathcal{A}$.

Note that a more general case where a fixed d is replaced with the value of d_α that depends on α and vanishes when $\alpha \rightarrow 0$ can also be handled in a similar way. This more general case has been considered by Lai [16]. However, this slight generalization does not have substantial impact on practical applications.

For $n \geq 0$, define the likelihood ratio of the hypotheses « H_1 : $\lambda \leq n$ » and « H_0 : $\lambda > n$ »

$$\Lambda_n := \frac{d\mathbf{P}\{\mathbf{X}^n | H_1\}}{d\mathbf{P}\{\mathbf{X}^n | H_0\}} = \frac{\sum_{k=0}^n \pi_k \prod_{i=k}^n f_{1,i}(X_i | \mathbf{X}^{i-1}) \prod_{i=1}^{k-1} f_{0,i}(X_i | \mathbf{X}^{i-1})}{\sum_{k=n+1}^{\infty} \pi_k \prod_{i=1}^n f_{0,i}(X_i | \mathbf{X}^{i-1})},$$

where $f_{1,0}(X_0)/f_{0,0}(X_0) = 1$ almost everywhere by the above convention.

Recall that $p_n = \mathbf{P}\{\lambda \leq n | \mathcal{F}_n^X\}$ denotes the posterior probability of the fact that the change occurred before time n . Write $\Pi_n = \mathbf{P}\{\lambda > n\}$. It is easily verified that

$$\Lambda_n = \Lambda_0 + \Pi_n^{-1} \sum_{k=1}^n \pi_k e^{Z_n^k} \quad \text{and} \quad \Lambda_n = \frac{p_n}{1 - p_n}, \quad n \geq 1, \quad (2.9)$$

where $\Lambda_0 = \pi_0/(1 - \pi_0)$. Therefore, the Shiryaev stopping rule given in (2.3) can be written in the following form that is more convenient for asymptotic study:

$$\nu_B = \inf\{n \geq 1: \Lambda_n \geq B\}, \quad B = \frac{A}{1 - A}, \quad (2.10)$$

where $B > \pi_0/(1 - \pi_0)$.

Obviously, inequality (2.4) holds in the general, non-i.i.d. case too. Consequently,

$$B_\alpha = \frac{1 - \alpha}{\alpha} \quad \text{implies} \quad \nu_{B_\alpha} \in \Delta(\alpha) \quad (2.11)$$

provided that $\alpha < 1 - \pi_0$.

It is worth noting that while the Shiryaev procedure (2.10) is optimal in the i.i.d. case (if B is chosen so that $\text{PFA}(\nu_B) = \alpha$), it may not be optimal in the non-i.i.d. scenario even if we can set the threshold to meet the PFA constraint exactly. In fact, the properties of the Shiryaev procedure in the non-i.i.d. scenario have not been investigated previously.

In addition to the Bayesian ADD defined in (2.1), we will also be interested in the behavior of the conditional ADD (CADD) for the fixed change point $\lambda = k$, which is defined by $\text{CADD}_k(\tau) = \mathbf{E}_k(\tau - k | \tau \geq k)$, $k = 1, 2, \dots$, as well as higher moments of the detection delay $\mathbf{E}_k\{(\tau - k)^m | \tau \geq k\}$ and $\mathbf{E}^\pi\{(\tau - \lambda)^m | \tau \geq \lambda\}$ for $m > 1$.

In the next section, we study the operating characteristics of the Bayesian procedure (2.10) for small PFA ($\alpha \rightarrow 0$) in the general, non-i.i.d. case. In Section 4, these results will be specialized to the i.i.d. scenario.

3. Asymptotic operating characteristics of the detection procedure ν_B in a non-i.i.d. case

As we mentioned above, in general, the Shiryaev procedure ν_B is not optimal even if one is able to choose the threshold B in such a way that $\text{PFA}(B) = \alpha$. However, below we show that this procedure with $B = B_\alpha = (1 - \alpha)/\alpha$ is asymptotically optimal for small PFA under some mild conditions. We will show that in the asymptotic setting, ν_{B_α} minimizes not only the ADD, but also CADD_k for all $k \geq 1$. Furthermore, under certain general conditions this procedure minimizes higher moments of the detection delay up to the given order.

3.1. Asymptotic lower bounds for moments of the detection delay. We begin by establishing asymptotic lower bounds for moments of the detection delay, in particular for ADD and CADD of any procedure in the class $\Delta(\alpha)$. Later on these bounds will be used to obtain asymptotic optimality results.

As we will see, the derivation of the lower bounds is based on the application of the Chebyshev inequality that involves certain probabilities, which are shown to go to 0 when $\alpha \rightarrow 0$. We start with the study of the behavior of these probabilities.

Let q be a positive finite number and define $L_\alpha = L_\alpha(q_d) = |\ln \alpha|/q_d$,

$$\begin{aligned}\gamma_{\varepsilon, \alpha}(\tau) &= \mathbf{P}^\pi\{\lambda \leq \tau < \lambda + (1 - \varepsilon)L_\alpha\}, \\ \gamma_{\varepsilon, \alpha}^{(k)}(\tau) &= \mathbf{P}_k\{k \leq \tau < k + (1 - \varepsilon)L_\alpha\}, \quad 0 < \varepsilon < 1,\end{aligned}$$

where $q_d = q + d$ in the case of prior distributions with exponential right tail, $\pi \in \mathcal{E}(d)$, and $q_d = q_0 = q$ in the case of heavy-tailed prior distributions $\pi \in \mathcal{H}$.

The significance of the number q should be explained in more detail. We do not assume any particular model for the observations, and as a result,

there is no «structure» of the LLR process. We hence have to impose some conditions on the behavior of the LLR process at least for large n . It is natural to assume that there exists a positive finite number $q = q(f_1, f_0)$ such that $Z_n^k/(n - k)$ converges almost surely to q , i.e.,

$$\frac{1}{n} Z_{k+n}^k \xrightarrow[n \rightarrow \infty]{\mathbf{P}_k - \text{a.s.}} q \quad \text{for every } k < \infty. \quad (3.1)$$

This is always true for i.i.d. data models, in which case $q = D(f_1, f_0) = \mathbf{E}_1 Z_1^1$ is the Kullback–Leibler information number. It turns out that the a.s. convergence condition (3.1) is sufficient for obtaining lower bounds for all positive moments of the detection delay (but not necessary). In fact, the key condition (3.2) in Lemma 1 and Theorem 1 holds whenever $Z_n^k/(n - k)$ converges almost surely to the number q . Therefore, this number is of paramount importance in the general change-point detection theory.

Note that $\mathbf{P}_k\{\tau < k\} = \mathbf{P}_\infty\{\tau < k\}$, since $\{\tau < k\}$ belongs to the sigma-field $\mathcal{F}_{k-1}^X$, and hence, $\text{PFA}(\tau) = \sum_{k=1}^\infty \pi_k \mathbf{P}_\infty\{\tau < k\}$.

The following «fundamental» lemma will be repeatedly used to derive lower bounds for the performance indices.

Lemma 1. *Let Z_n^k be defined as in (2.5) and assume that for some $q > 0$*

$$\mathbf{P}_k \left\{ \frac{1}{M} \max_{0 \leq n < M} Z_{k+n}^k \geq (1 + \varepsilon) q \right\} \xrightarrow[M \rightarrow \infty]{} 0 \quad \text{for all } \varepsilon > 0 \text{ and } k \geq 1. \quad (3.2)$$

Then, for all $0 < \varepsilon < 1$ and $k \geq 1$,

$$\lim_{\alpha \rightarrow 0} \sup_{\tau \in \Delta(\alpha)} \gamma_{\varepsilon, \alpha}^{(k)}(\tau) = 0, \quad (3.3)$$

and for all $0 < \varepsilon < 1$,

$$\lim_{\alpha \rightarrow 0} \sup_{\tau \in \Delta(\alpha)} \gamma_{\varepsilon, \alpha}(\tau) = 0. \quad (3.4)$$

Proof. Changing the measure $\mathbf{P}_\infty \rightarrow \mathbf{P}_k$, we obtain that for any $C > 0$ and $\varepsilon \in (0, 1)$

$$\begin{aligned} \mathbf{P}_\infty\{k \leq \tau < k + (1 - \varepsilon) L_\alpha\} &= \mathbf{E}_k \left\{ \mathbb{1}_{\{k \leq \tau < k + (1 - \varepsilon) L_\alpha\}} e^{-Z_\tau^k} \right\} \\ &\geq \mathbf{E}_k \left\{ \mathbb{1}_{\{k \leq \tau < k + (1 - \varepsilon) L_\alpha, Z_\tau^k < C\}} e^{-Z_\tau^k} \right\} \\ &\geq e^{-C} \mathbf{P}_k \left\{ k \leq \tau < k + (1 - \varepsilon) L_\alpha, \max_{k \leq n < k + (1 - \varepsilon) L_\alpha} Z_n^k < C \right\} \\ &\geq e^{-C} \left[\mathbf{P}_k \{k \leq \tau < k + (1 - \varepsilon) L_\alpha\} - \mathbf{P}_k \left\{ \max_{0 \leq n < (1 - \varepsilon) L_\alpha} Z_{k+n}^k \geq C \right\} \right], \end{aligned} \quad (3.5)$$

where the last inequality follows trivially from the fact that for any events A and B (with B^c being a complement to B), $\mathbf{P}(A \cap B) \geq \mathbf{P}(A) - \mathbf{P}(B^c)$.

Setting $C = (1 - \varepsilon^2) q L_\alpha$, we obtain

$$\begin{aligned} \gamma_{\varepsilon, \alpha}^{(k)}(\tau) &\leq e^{(1 - \varepsilon^2) q L_\alpha} \mathbf{P}_\infty\{k \leq \tau < k + (1 - \varepsilon) L_\alpha\} \\ &\quad + \mathbf{P}_k \left\{ \max_{0 \leq n < (1 - \varepsilon) L_\alpha} Z_{k+n}^k \geq (1 - \varepsilon^2) q L_\alpha \right\}. \end{aligned} \quad (3.6)$$

Write $p_k(\alpha, \varepsilon) = e^{(1-\varepsilon^2)qL_\alpha} \mathbf{P}_\infty\{k \leq \tau < k + (1-\varepsilon)L_\alpha\}$ and $\beta_k(\alpha, \varepsilon) = \mathbf{P}_k\{\max_{0 \leq n < (1-\varepsilon)L_\alpha} Z_{k+n}^k \geq (1-\varepsilon^2)qL_\alpha\}$.

By condition (3.2), for every $0 < \varepsilon < 1$ and all $k \geq 1$,

$$\beta_k(\alpha, \varepsilon) = \mathbf{P}_k\left\{\frac{1}{(1-\varepsilon)L_\alpha} \max_{0 \leq n < (1-\varepsilon)L_\alpha} Z_{k+n}^k \geq (1+\varepsilon)q\right\} \xrightarrow{\alpha \rightarrow 0} 0. \quad (3.7)$$

Next, for any $\tau \in \Delta(\alpha)$ and $n \geq 1$, $\alpha \geq \text{PFA}(\tau) \geq \mathbf{P}\{\tau < n\} \cap \{\lambda > n\} = \mathbf{P}\{\tau < n \mid \lambda > n\} \mathbf{P}\{\lambda > n\} = \mathbf{P}_\infty\{\tau < n\} \Pi_n$, and, therefore,

$$\mathbf{P}_\infty\{\tau < n\} \leq \alpha(\Pi_n)^{-1}, \quad n \geq 1. \quad (3.8)$$

It follows that the first term in inequality (3.6)

$$p_k(\alpha, \varepsilon) \leq \alpha e^{(1-\varepsilon^2)qL_\alpha} (\Pi_{m_\alpha(k)})^{-1}, \quad (3.9)$$

where $m_\alpha(k) = \lfloor k + (1-\varepsilon)L_\alpha \rfloor$ is the greatest integer number $\leq k + (1-\varepsilon)L_\alpha$. Since $\alpha = e^{-qdL_\alpha}$ and $(m_\alpha(k) - k - 1)/(1-\varepsilon) \leq L_\alpha \leq (m_\alpha(k) - k)/(1-\varepsilon)$, we obtain

$$p_k(\alpha, \varepsilon) \leq (\Pi_{m_\alpha(k)})^{-1} \exp\left\{-\frac{d + \varepsilon^2 q}{1-\varepsilon} (m_\alpha(k) - k - 1)\right\}$$

and

$$\frac{\ln p_k(\alpha, \varepsilon)}{m_\alpha(k)} \leq -\frac{\ln \Pi_{m_\alpha(k)}}{m_\alpha(k)} - \frac{d + \varepsilon^2 q}{1-\varepsilon} \frac{m_\alpha(k) - k - 1}{m_\alpha(k)}.$$

By conditions (2.6) and (2.7),

$$\lim_{\alpha \rightarrow 0} \frac{\ln p_k(\alpha, \varepsilon)}{m_\alpha(k)} \leq d - \frac{d + \varepsilon^2 q}{1-\varepsilon} = -\frac{\varepsilon}{1-\varepsilon} (d + \varepsilon q),$$

where $d > 0$ for $\pi \in \mathcal{E}(d)$ and $d = 0$ for $\pi \in \mathcal{H}$. It follows that, for any $\pi \in \mathcal{H} \cup \mathcal{E}(d)$,

$$p_k(\alpha, \varepsilon) \rightarrow 0 \quad \text{as } \alpha \rightarrow 0 \quad \text{for all } k \geq 1 \text{ and } 0 < \varepsilon < 1. \quad (3.10)$$

Therefore, we obtain that for every $\tau \in \Delta(\alpha)$ and $\varepsilon > 0$,

$$\gamma_{\varepsilon, \alpha}^{(k)}(\tau) \leq p_k(\alpha, \varepsilon) + \beta_k(\alpha, \varepsilon), \quad (3.11)$$

where by (3.7) and (3.10), $\beta_k(\alpha, \varepsilon)$ and $p_k(\alpha, \varepsilon)$ go to zero as $\alpha \rightarrow 0$. Since both $p_k(\alpha, \varepsilon)$ and $\beta_k(\alpha, \varepsilon)$ do not depend on a particular stopping time τ , (3.3) holds.

Let $N_\alpha = \lfloor \varepsilon L_\alpha \rfloor$ be the greatest integer number less or equal to εL_α . Obviously,

$$\gamma_{\varepsilon, \alpha}(\tau) = \sum_{k=1}^{\infty} \pi_k \gamma_{\varepsilon, \alpha}^{(k)}(\tau) \leq \sum_{k=1}^{N_\alpha} \pi_k \gamma_{\varepsilon, \alpha}^{(k)}(\tau) + \Pi_{N_\alpha}. \quad (3.12)$$

Using (3.11) and (3.12), we obtain

$$\gamma_{\varepsilon, \alpha}(\tau) \leq \Pi_{N_\alpha} + \sum_{k=1}^{N_\alpha} \pi_k \beta_k(\alpha, \varepsilon) + \sup_{k \leq N_\alpha} p_k(\alpha, \varepsilon). \quad (3.13)$$

The term $\Pi_{N_\alpha} \rightarrow 0$ as $\alpha \rightarrow 0$. The second term goes to zero as $\alpha \rightarrow 0$ by condition (3.2) (see (3.7)) and Lebesgue's dominated convergence theorem. It suffices to show that the third term vanishes as $\alpha \rightarrow 0$.

By (3.9),

$$\sup_{k \leq N_\alpha} p_k(\alpha, \varepsilon) \leq \alpha e^{(1-\varepsilon^2)qL_\alpha} (\Pi_{m_\alpha(N_\alpha)})^{-1} = (\Pi_{m_\alpha(N_\alpha)})^{-1} \exp \{ - (d + \varepsilon^2 q) L_\alpha \},$$

where $m_\alpha(N_\alpha) = \lfloor N_\alpha + (1 - \varepsilon) L_\alpha \rfloor$. Obviously, $L_\alpha \leq m_\alpha(N_\alpha) \leq L_\alpha + 2$, and hence,

$$\frac{\ln \sup_{k \leq N_\alpha} p_k(\alpha, \varepsilon)}{m_\alpha(N_\alpha)} \leq - \frac{\ln \Pi_{m_\alpha(N_\alpha)}}{m_\alpha(N_\alpha)} - (d + \varepsilon^2 q) \frac{L_\alpha}{L_\alpha + 2}.$$

Since $-(\ln \Pi_{m_\alpha(N_\alpha)})/m_\alpha(N_\alpha) \rightarrow d$ as $\alpha \rightarrow 0$, we obtain that for any $\pi \in \mathcal{H} \cup \mathcal{E}(d)$,

$$\lim_{\alpha \rightarrow 0} \frac{\ln \sup_{k \leq N_\alpha} p_k(\alpha, \varepsilon)}{m_\alpha(N_\alpha)} \leq -\varepsilon^2 q,$$

showing that $\sup_{k \leq N_\alpha} p_k(\alpha, \varepsilon) \rightarrow 0$ as $\alpha \rightarrow 0$.

Since the right-hand side in (3.13) does not depend on τ , (3.4) follows, and the proof is complete.

In the next theorem, we derive the lower bounds for the positive moments of the detection delay of any procedure from the class $\Delta(\alpha)$ under the conditions postulated in Lemma 1. A similar bound has been obtained by Lai [16] for $\mathbf{E}^\pi(\tau - \lambda)^+$ and heavy-tailed prior distributions under a slightly stronger conditions with $\sup_k$ in (3.2). Recall that $q_d = q + d$ when $\pi \in \mathcal{E}(d)$ and $q_d = q$ when $\pi \in \mathcal{H}$.

Theorem 1. *Let condition (3.2) hold for some positive finite number q . Then, for every $k \geq 1$ and $m > 0$,*

$$\inf_{\tau \in \Delta(\alpha)} \mathbf{E}_k[(\tau - k)^m \mid \tau \geq k] \geq \left\{ \frac{|\ln \alpha|}{q_d} \right\}^m (1 + o(1)) \quad \text{as } \alpha \rightarrow 0 \quad (3.14)$$

and for all $m > 0$

$$\inf_{\tau \in \Delta(\alpha)} \mathbf{E}^\pi[(\tau - \lambda)^m \mid \tau \geq \lambda] \geq \left\{ \frac{|\ln \alpha|}{q_d} \right\}^m (1 + o(1)) \quad \text{as } \alpha \rightarrow 0, \quad (3.15)$$

where $o(1) \rightarrow 0$ as $\alpha \rightarrow 0$.

P r o o f. By the Chebyshev inequality, for any $0 < \varepsilon < 1$ and $m > 0$

$$\mathbf{E}_k[(\tau - k)^+]^m \geq [(1 - \varepsilon) L_\alpha]^m \mathbf{P}_k\{\tau - k \geq (1 - \varepsilon) L_\alpha\}.$$

Obviously, $\mathbf{P}_k\{\tau - k \geq (1 - \varepsilon) L_\alpha\} = \mathbf{P}_k\{\tau \geq k\} - \gamma_{\varepsilon, \alpha}^{(k)}(\tau)$. Therefore,

$$\begin{aligned} \mathbf{E}_k[(\tau - k)^m \mid \tau \geq k] &= \frac{\mathbf{E}_k[(\tau - k)^+]^m}{\mathbf{P}_k\{\tau \geq k\}} \\ &\geq \frac{[(1 - \varepsilon) L_\alpha]^m}{\mathbf{P}_k\{\tau \geq k\}} \left[\mathbf{P}_k\{\tau \geq k\} - \gamma_{\varepsilon, \alpha}^{(k)}(\tau) \right]. \end{aligned} \quad (3.16)$$

Next, by (3.8), for any $\tau \in \Delta(\alpha)$ and $\alpha < \Pi_k$

$$\mathbf{P}_k\{\tau \geq k\} = 1 - \mathbf{P}_\infty\{\tau < k\} \geq 1 - \alpha(\Pi_k)^{-1}. \quad (3.17)$$

Using (3.16) and (3.17) yields the inequality

$$\mathbf{E}_k[(\tau - k)^m \mid \tau \geq k] \geq [(1 - \varepsilon) L_\alpha]^m \left[1 - \frac{\gamma_{\varepsilon, \alpha}^{(k)}(\tau)}{1 - \alpha(\Pi_k)^{-1}} \right], \quad (3.18)$$

which holds for any $\tau \in \Delta(\alpha)$, $0 < \varepsilon < 1$, and $\{k: \Pi_k < \alpha\}$. By Lemma 1, $\gamma_{\varepsilon, \alpha}^{(k)}(\tau) \rightarrow 0$ as $\alpha \rightarrow 0$ uniformly over all $\tau \in \Delta(\alpha)$, which implies

$$\inf_{\tau \in \Delta(\alpha)} \mathbf{E}_k[(\tau - k)^m \mid \tau \geq k] \geq [(1 - \varepsilon) L_\alpha]^m (1 + o(1)) \quad \text{as } \alpha \rightarrow 0.$$

Since ε is arbitrary, the asymptotic lower bound (3.14) follows.

To prove the asymptotic lower bound (3.15) we again use the Chebyshev inequality, according to which for any $0 < \varepsilon < 1$ and any $\tau \in \Delta(\alpha)$

$$\mathbf{E}^\pi[(\tau - \lambda)^+]^m \geq [(1 - \varepsilon) L_\alpha]^m \mathbf{P}^\pi\{\tau - \lambda \geq (1 - \varepsilon) L_\alpha\},$$

where $\mathbf{P}^\pi\{\tau - \lambda \geq (1 - \varepsilon) L_\alpha\} = \mathbf{P}^\pi\{\tau \geq \lambda\} - \gamma_{\varepsilon, \alpha}(\tau)$. Since $\mathbf{P}^\pi\{\tau \geq \lambda\} \geq 1 - \alpha$ for any $\tau \in \Delta(\alpha)$, it follows that

$$\begin{aligned} \mathbf{E}^\pi[(\tau - \lambda)^m \mid \tau \geq \lambda] &= \frac{\mathbf{E}^\pi[(\tau - \lambda)^+]^m}{\mathbf{P}^\pi\{\tau \geq \lambda\}} \geq [(1 - \varepsilon) L_\alpha]^m \left[1 - \frac{\gamma_{\varepsilon, \alpha}(\tau)}{\mathbf{P}^\pi\{\tau \geq \lambda\}} \right] \\ &\geq [(1 - \varepsilon) L_\alpha]^m \left[1 - \frac{\gamma_{\varepsilon, \alpha}(\tau)}{1 - \alpha} \right]. \end{aligned} \quad (3.19)$$

Since ε can be arbitrarily small and, by Lemma 1, $\sup_{\tau \in \Delta(\alpha)} \gamma_{\varepsilon, \alpha}(\tau) \rightarrow 0$ as $\alpha \rightarrow 0$, the asymptotic lower bound (3.15) follows, and the proof is complete.

Remark 1. It is important to emphasize that the vanishing term $o(1)$ in (3.14) depends on k . For this reason, the lower bound (3.15) does not follow directly from the inequality (3.14), and an additional effort was needed to prove it.

3.2. Asymptotic performance of the procedure ν_B for large B .

We begin with the evaluation of the performance of the Shiryaev detection procedure ν_B for large values of B regardless of the false alarm constraint.

Write

$$Y_t = \ln \frac{f_{1,t}(X_t \mid \mathbf{X}^{t-1})}{f_{0,t}(X_t \mid \mathbf{X}^{t-1})},$$

and, for every $k = 1, 2, \dots$ and $\varepsilon > 0$, define the random variable

$$T_\varepsilon^{(k)} = \sup \left\{ n \geq 1: \left| \frac{1}{n} \sum_{t=k}^{k+n-1} Y_t - q \right| > \varepsilon \right\}, \quad (3.20)$$

where $\sup \emptyset = 0$. Clearly, in terms of $T_\varepsilon^{(k)}$, the almost sure convergence of (3.1) may be written as $\mathbf{P}_k\{T_\varepsilon^{(k)} < \infty\} = 1$ for all $\varepsilon > 0$ and $k \geq 1$, which implies condition (3.2).

While these conditions are sufficient for obtaining lower bounds for moments of the detection delay (in particular, for the average detection delay), they need to be strengthened in order to establish asymptotic optimality properties of the detection procedure ν_B , and to obtain asymptotic expansions for moments of the detection delay. Indeed, in general these conditions do not even guarantee finiteness of $\text{CADD}_k(\nu_B)$ and $\text{ADD}(\nu_B)$. In order to study asymptotics for the average detection delay one may impose the following constraints on the rate of convergence in the strong law for Z_{k+n}^k/n :

$$\mathbf{E}_k T_\varepsilon^{(k)} < \infty \quad \text{for all } \varepsilon > 0 \text{ and } k \geq 1, \quad \text{and} \quad (3.21)$$

$$\sum_{k=1}^{\infty} \pi_k \mathbf{E}_k T_\varepsilon^{(k)} < \infty \quad \text{for all } \varepsilon > 0. \quad (3.22)$$

Note that (3.21) is closely related to the condition

$$\sum_{n=1}^{\infty} \mathbf{P}_k \left\{ \left| \sum_{t=k}^{k+n-1} Y_t - qn \right| > \varepsilon n \right\} < \infty \quad \text{for all } \varepsilon > 0 \text{ and } k \geq 1,$$

which is nothing but the complete convergence of Z_{k+n}^k/n to q under $\mathbf{P}_k$ (cf. [11]). We write this compactly as

$$\frac{1}{n} Z_{k+n}^k \xrightarrow[n \rightarrow \infty]{\mathbf{P}_k\text{-completely}} q \quad \text{for every } k \geq 1. \quad (3.23)$$

The convergence condition (3.22) is a joint condition on the rates of convergence of Z_{k+n}^k/n for each $\lambda = k$ and the prior distribution. We write this condition compactly as

$$\frac{1}{n} Z_{\lambda+n}^\lambda \xrightarrow[n \rightarrow \infty]{\mathbf{P}^\pi\text{-average-completely}} q. \quad (3.24)$$

To study asymptotics for higher moments of the detection delay, the complete convergence conditions (3.23) and (3.24) should be further strengthened. A natural generalization is to require, for some $r \geq 1$,

$$\mathbf{E}_k [T_\varepsilon^{(k)}]^r < \infty \quad \text{for all } \varepsilon > 0 \text{ and } k \geq 1 \quad (3.25)$$

and

$$\sum_{k=1}^{\infty} \pi_k \mathbf{E}_k [T_\varepsilon^{(k)}]^r < \infty \quad \text{for all } \varepsilon > 0. \quad (3.26)$$

If (3.25) holds, it is said that Z_{k+n}^k/n converges r -quickly to q (cf. Lai [13]). If (3.26) holds, we will say that $Z_{\lambda+n}^\lambda/n$ converges average- r -quickly to q . We will write these modes of convergence compactly as

$$\frac{1}{n} Z_{k+n}^k \xrightarrow[n \rightarrow \infty]{\mathbf{P}_k\text{-}r\text{-quickly}} q \quad \text{for every } k \geq 1 \quad (3.27)$$

and

$$\frac{1}{n} Z_{\lambda+n}^\lambda \xrightarrow[n \rightarrow \infty]{\mathbf{P}^\pi\text{-average-}r\text{-quickly}} q. \quad (3.28)$$

Complete and r -quick convergence conditions have been previously used by Lai [14], Tartakovsky [33], and Dragalin, Tartakovsky, and Veeravalli [8] to establish the asymptotic optimality of sequential hypothesis tests for general statistical models. Below we take advantage of these results and prove that the conditions (3.23), (3.24), (3.27), and (3.28) are sufficient for asymptotic optimality of the Shiryaev change-point detection procedure.

In the following theorem, we establish the operating characteristics of the detection procedure ν_B for large values of the threshold B regardless of the false alarm rate constraints for general statistical models when the r -quick convergence conditions (3.27) and (3.28) hold. Hereafter $X_B \sim Y_B$ as $B \rightarrow \infty$ means that $\lim_{B \rightarrow \infty} (X_B/Y_B) = 1$. For the sake of compactness, in the rest of the paper we will write $\mathbf{ED}_m^\pi(\tau)$ for $\mathbf{E}^\pi[(\tau - \lambda)^m | \tau \geq \lambda]$ and $\mathbf{ED}_m^{(k)}(\tau)$ for $\mathbf{E}_k[(\tau - k)^m | \tau \geq k]$, respectively.

Theorem 2. (i) *Let condition (3.27) hold for some positive q and $r \geq 1$. Then for all $m \leq r$*

$$\mathbf{ED}_m^{(k)}(\nu_B) \sim \left(\frac{\ln B}{q_d} \right)^m \quad \text{as } B \rightarrow \infty \text{ for all } k \geq 1. \quad (3.29)$$

(ii) *Let condition (3.28) hold for some positive q and $r \geq 1$. Then for all $m \leq r$*

$$\mathbf{ED}_m^\pi(\nu_B) \sim \left(\frac{\ln B}{q_d} \right)^m \quad \text{as } B \rightarrow \infty. \quad (3.30)$$

To prove this theorem we will need two auxiliary results that we formulate in the form of Lemmas 2 and 3 below. Lemma 2 is similar to, and to some extent is a particular case of, Lemma 1. Lemma 3 is related to convergence of moments of one-sided stopping times that bound the stopping time ν_B from above. The first lemma will be used to obtain lower bounds for the moments of the detection delay, while the second one will be used for deriving the corresponding upper bounds.

The following notation is used throughout this subsection:

$$\begin{aligned} L_B &= q_d^{-1} \ln B, \\ \gamma_\varepsilon^{(k)}(B) &= \mathbf{P}_k\{k \leq \nu_B < k + (1 - \varepsilon)L_B\}, \\ \gamma_\varepsilon(B) &= \mathbf{P}^\pi\{\lambda \leq \nu_B < \lambda + (1 - \varepsilon)L_B\}. \end{aligned}$$

Note that $\gamma_\varepsilon(B) = \sum_{k=1}^\infty \pi_k \gamma_\varepsilon^{(k)}(B)$.

Lemma 2. *Suppose condition (3.2) holds for some $q > 0$. Then*

$$\lim_{B \rightarrow \infty} \gamma_\varepsilon^{(k)}(B) = 0 \quad \text{for all } 0 < \varepsilon < 1 \text{ and } k \geq 1; \quad (3.31)$$

$$\lim_{B \rightarrow \infty} \gamma_\varepsilon(B) = 0 \quad \text{for every } 0 < \varepsilon < 1. \quad (3.32)$$

P r o o f. The proof runs along the lines of the proof of Lemma 1. It suffices to note that $\mathbf{P}^\pi\{\nu_B < \lambda\} \leq 1/(1 + B) \leq 1/B$. Therefore, replacing α by $1/B$ in Lemma 1 completes the proof.

We now formulate the second important result that will be used to obtain upper bounds for the moments of the stopping time ν_B . Write $S_{k+n-1}^k = Z_{k+n-1}^k + nw_{n,k}$, $w_{n,k} = n^{-1} \ln(\pi_k(\Pi_{k+n-1})^{-1})$ and, for $b > 0$, introduce the sequence of one-sided stopping times

$$\eta_b(k) = \inf\{n \geq 1: S_{k+n-1}^k \geq b\}, \quad k = 1, 2, \dots \quad (3.33)$$

Lemma 3. *Let m be a positive, not necessarily integer number and suppose that for some $r > 0$ the condition (3.25) is satisfied. Then, for all $m \leq r$,*

$$\mathbf{E}_k(\eta_b(k) b^{-1})^m \rightarrow (q_d)^{-m} \quad \text{as } b \rightarrow \infty. \quad (3.34)$$

P r o o f. For the geometric prior distribution with $\pi_0 = 0$, this lemma can be directly derived from Theorem 4.2 of [8], since in this case $(\ln \Pi_n)/n = d = |\ln(1 - \rho)|$ and $w_{n,k} = n \ln(1 - \rho)$ for all $n \geq 1$. In the general case, the proof requires a modification which is given below.

By condition (3.25), $Z_{k+n}^k/n \rightarrow q$ as $n \rightarrow \infty$ $\mathbf{P}_k$ -a.s. Since by assumptions (2.6)–(2.7), $w_{n,k} \rightarrow d$ as $n \rightarrow \infty$ ($d \geq 0$), it follows that $S_{k+n}^k/n \rightarrow q_d$ as $n \rightarrow \infty$ $\mathbf{P}_k$ -a.s. for all $k \geq 1$. Therefore,

$$\mathbf{P}_k \left\{ \frac{1}{M} \max_{0 \leq n < M} S_{k+n}^k \geq (1 + \varepsilon) q_d \right\} \xrightarrow{M \rightarrow \infty} 0 \quad \text{for all } \varepsilon > 0 \text{ and } k \geq 1$$

and the argument identical to that used in the proof of Lemma 1 (with α replaced by e^{-b}) yields

$$\mathbf{P}_k \{ \eta_b(k) \geq (1 - \varepsilon) b (q_d)^{-1} \} \rightarrow 1 \quad \text{as } b \rightarrow \infty.$$

Hence Chebyshev's inequality (for any $m > 0$) $\mathbf{E}_k[\eta_b(k)]^m \geq [(1 - \varepsilon) b (q_d)^{-1}]^m \mathbf{P}_k \{ \eta_b(k) \geq (1 - \varepsilon) b (q_d)^{-1} \}$ applies to show that

$$\mathbf{E}_k[\eta_b(k)]^m \geq (b(q_d)^{-1})^m (1 + o(1)) \quad \text{as } b \rightarrow \infty. \quad (3.35)$$

To obtain the upper bound, define

$$\tilde{T}_\varepsilon^{(k)} = \sup \{ n \geq 1: |n^{-1} S_{k+n-1}^k - q_d| > \varepsilon \}.$$

It is easy to see that $S_{k+\eta_b(k)-2}^k < b$ and $S_{k+\eta_b(k)-2}^k \geq (\eta_b(k) - 1)(q_d - \varepsilon)$ on $\{ \eta_b(k) - 1 > \tilde{T}_\varepsilon^{(k)} \}$. It follows that for every $0 < \varepsilon < q_d$

$$\begin{aligned} \eta_b(k) &\leq 1 + \frac{b}{q_d - \varepsilon} \mathbb{1}_{\{ \eta_b(k) > \tilde{T}_\varepsilon^{(k)} + 1 \}} + (\tilde{T}_\varepsilon^{(k)} + 1) \mathbb{1}_{\{ \eta_b(k) \leq \tilde{T}_\varepsilon^{(k)} + 1 \}} \\ &\leq \tilde{T}_\varepsilon^{(k)} + 2 + \frac{b}{q_d - \varepsilon}. \end{aligned} \quad (3.36)$$

Since $w_{n,k} \rightarrow d$ as $n \rightarrow \infty$, the condition (3.25) implies $\mathbf{E}_k[\tilde{T}_\varepsilon^{(k)}]^r < \infty$. Since ε can be arbitrarily small, letting $\varepsilon \rightarrow 0$ we obtain that for all $m \leq r$

$$\mathbf{E}_k[\eta_b(k)]^m \leq (b(q_d)^{-1})^m (1 + o(1)) \quad \text{as } b \rightarrow \infty,$$

which, along with the reverse inequality (3.35), completes the proof.

We are now ready to prove Theorem 2.

P r o o f o f T h e o r e m 2. (i) For $b = \ln B$, define the sequence of stopping times $\eta_b(k)$ as in (3.33). It is easily verified that the statistic $\ln \Lambda_n$ can be written in the form

$$\ln \Lambda_n = Z_n^k + (n - k + 1)w_{k+n-1,k} + \ell_{n,k}, \quad (3.37)$$

where the random variable $\ell_{n,k}$ is nonnegative. Thus, for $b = \ln B$ and for every $k \geq 1$

$$\nu_B - k \leq \eta_b(k) \quad \text{on } \{\nu_B \geq k\} \quad (3.38)$$

and

$$\mathbf{E}_k[(\nu_B - k)^+]^m \leq \mathbf{E}_k\{[\eta_b(k)]^m; \nu_B \geq k\} \leq \mathbf{E}_k[\eta_b(k)]^m. \quad (3.39)$$

By (2.4) and (3.8), $\mathbf{P}_k\{\nu_B \geq k\} \geq 1 - [(1 + B)\Pi_k]^{-1}$, and hence, for $B > (1 - \Pi_k)/\Pi_k$,

$$\mathbf{ED}_m^{(k)}(\nu_B) = \frac{\mathbf{E}_k[(\nu_B - k)^+]^m}{\mathbf{P}_k\{\nu_B \geq k\}} \leq \frac{\mathbf{E}_k[\eta_b(k)]^m}{1 - [(1 + B)\Pi_k]^{-1}}.$$

Applying Lemma 3, we obtain the following estimate for the upper bound:

$$\mathbf{ED}_m^{(k)}(\nu_B) \leq \left(\frac{\ln B}{q_d}\right)^m (1 + o(1)) \quad \text{as } B \rightarrow \infty \text{ for all } k \geq 1. \quad (3.40)$$

In order to prove (3.29) it remains to show that the right-hand side of the latter inequality is a lower bound for $\mathbf{ED}_m^{(k)}(\nu_B)$ as $B \rightarrow \infty$. To that end, we use Lemma 2 and the Chebyshev inequality.

Indeed, applying the Chebyshev inequality and the inequality $\mathbf{P}_k\{\nu_B \geq k\} \geq 1 - [(1 + B)\Pi_k]^{-1}$ yields, for $B > (1 - \Pi_k)/\Pi_k$,

$$\begin{aligned} \mathbf{ED}_m^{(k)}(\nu_B) &\geq [(1 - \varepsilon)L_B]^m [\mathbf{P}_k\{\nu_B \geq k\} - \gamma_\varepsilon^{(k)}(B)] \\ &\geq [(1 - \varepsilon)L_B]^m [1 - (\Pi_k B)^{-1} - \gamma_\varepsilon^{(k)}(B)]. \end{aligned}$$

Since ε is arbitrary and, by Lemma 2, $\gamma_\varepsilon^{(k)}(B) \rightarrow 0$ as $B \rightarrow \infty$ for every $k \geq 1$, we obtain that the right-hand side in (3.40) is the asymptotic lower bound for $\mathbf{ED}_m^{(k)}(\nu_B)$. The asymptotic formula (3.29) follows.

(ii) Similar to (3.36), for every $0 < \varepsilon < q_d$ and $k \geq 1$,

$$\nu_B - k \leq \eta_b(k) \leq \tilde{T}_\varepsilon^{(k)} + 2 + \frac{b}{q_d - \varepsilon} \quad \text{on } \{\nu_B \geq k\}. \quad (3.41)$$

Applying (3.41) along with the fact that $\mathbf{P}^\pi\{\nu_B \geq \lambda\} \geq B/(1 + B)$ yields

$$\mathbf{ED}_m^\pi(\nu_B) \leq \frac{B + 1}{B} \sum_{k=1}^{\infty} \pi_k \mathbf{E}_k \left(\frac{\ln B}{q_d - \varepsilon} + \tilde{T}_\varepsilon^{(k)} + 2 \right)^m.$$

Since ε can be arbitrarily small and, by the assumption of the theorem, $\sum_{k=1}^{\infty} \pi_k \mathbf{E}_k [T_{\varepsilon}^{(k)}]^m < \infty$, which as before implies $\sum_{k=1}^{\infty} \pi_k \mathbf{E}_k [\tilde{T}_{\varepsilon}^{(k)}]^m < \infty$, it follows that

$$\mathbf{ED}_m^{\pi}(\nu_B) \leq \left(\frac{\ln B}{q_d} \right)^m (1 + o(1)) \quad \text{as } B \rightarrow \infty. \quad (3.42)$$

To obtain a lower bound for $\mathbf{ED}_m^{\pi}(\nu_B)$, we again use the Chebyshev inequality, which yields

$$\begin{aligned} \mathbf{ED}_m^{\pi}(\nu_B) &\geq \mathbf{E}^{\pi}[(\nu_B - \lambda)^+]^m \geq [(1 - \varepsilon) L_B]^m [\mathbf{P}^{\pi}\{\nu_B \geq \lambda\} - \gamma_{\varepsilon}(B)] \\ &\geq [(1 - \varepsilon) L_B]^m [B(1 + B)^{-1} - \gamma_{\varepsilon}(B)], \end{aligned}$$

where $\gamma_{\varepsilon}(B) \rightarrow 0$ as $B \rightarrow \infty$ by Lemma 2. Recall that $L_B = (\ln B)/q_d$. Since ε is arbitrary, it follows that

$$\mathbf{ED}_m^{\pi}(\nu_B) \geq \left(\frac{\ln B}{q_d} \right)^m (1 + o(1)) \quad \text{as } B \rightarrow \infty, \quad (3.43)$$

which along with the upper bound (3.42) proves (3.30). The proof is complete.

Remark 2. The proof suggests that (3.29) holds for all $k < K_{\alpha}$ where K_{α} is such integer number for which $\mathbf{P}\{\lambda > K_{\alpha}\} < \alpha$. From a theoretical viewpoint this does not cause the problem, since $K_{\alpha} \rightarrow \infty$ as $\alpha \rightarrow 0$. However, from the point of view of practical applications it is important to investigate the behavior of $\mathbf{ED}_m^{(k)}(\nu_B)$ for large values of k .

3.3. Asymptotic optimality. We are now in a position to prove the asymptotic optimality of the detection procedure ν_B with the threshold $B = (1 - \alpha)/\alpha$ in the class $\Delta(\alpha)$ for small values of the PFA α . Since everything we need has been prepared, the proof is immediate.

Theorem 3. *If $B = B_{\alpha} = (1 - \alpha)/\alpha$, then the detection procedure $\nu_{B_{\alpha}}$ belongs to the class $\Delta(\alpha)$, and the following two assertions hold.*

(i) *Let condition (3.27) hold for some positive q and $r \geq 1$. Then for all $m \leq r$ and $k \geq 1$,*

$$\inf_{\tau \in \Delta(\alpha)} \mathbf{ED}_m^{(k)}(\tau) \sim \mathbf{ED}_m^{(k)}(\nu_{B_{\alpha}}) \sim \left(\frac{|\ln \alpha|}{q_d} \right)^m \quad \text{as } \alpha \rightarrow 0. \quad (3.44)$$

(ii) *Let condition (3.28) hold for some positive q and $r \geq 1$. Then for all $m \leq r$,*

$$\inf_{\tau \in \Delta(\alpha)} \mathbf{ED}_m^{\pi}(\tau) \sim \mathbf{ED}_m^{\pi}(\nu_{B_{\alpha}}) \sim \left(\frac{|\ln \alpha|}{q_d} \right)^m \quad \text{as } \alpha \rightarrow 0. \quad (3.45)$$

Proof. Applying Theorems 1 and 2 yields (3.44) and (3.45).

Corollary 1. *Let $B = B_{\alpha} = (1 - \alpha)/\alpha$. If condition (3.23) is satisfied for some positive q , then*

$$\inf_{\tau \in \Delta(\alpha)} \text{CADD}_k(\tau) \sim \text{CADD}_k(\nu_{B_{\alpha}}) \sim \frac{|\ln \alpha|}{q_d} \quad \text{as } \alpha \rightarrow 0 \text{ for all } k \geq 1. \quad (3.46)$$

If condition (3.24) is satisfied for some positive q , then

$$\inf_{\tau \in \Delta(\alpha)} \text{ADD}(\tau) \sim \text{ADD}(\nu_{B_\alpha}) \sim \frac{|\ln \alpha|}{q_d} \quad \text{as } \alpha \rightarrow 0. \quad (3.47)$$

We stress that the detection procedure ν_B with the threshold $B = B_\alpha = (1 - \alpha)/\alpha$ is asymptotically optimal not only relative to the ADD, but also uniformly asymptotically optimal with respect to the conditional ADD for all values of $\lambda = k$, $k = 1, 2, \dots$. We obtain this strong optimality result primarily because the constraint on false alarms in the Bayesian formulation that we consider is averaged over all possible realizations of the change point. Such a strong optimality result is not available for the minimax formulation of the problem with the constraint on the mean time to false alarm $\mathbf{E}_\infty \tau$ [16], [17], [21], [34].

It is also interesting to observe from Theorem 3 that for prior distributions with exponential tail ($d > 0$) if $q \gg d$, then the observations contain more information about the change than the prior distribution, and the performance is determined by q . On the other hand, if $q \ll d$, then the decision about the change-point can be made based solely on the prior distribution to yield an ADD of $|\ln \alpha|/d$. For heavy-tailed distributions $\pi \in \mathcal{H}$, prior information does not affect asymptotic performance, as could be expected.

R e m a r k 3. The results of Theorem 3 remain true in a more general case, where the r -quick convergence condition (3.27) is satisfied with an increasing function $\phi(n)$ in place of n , i.e.,

$$\frac{1}{\phi(n)} Z_{k+n}^k \xrightarrow[n \rightarrow \infty]{\mathbf{P}_k\text{-}r\text{-quickly}} q \quad \text{for all } k \geq 1. \quad (3.48)$$

In particular, if (3.48) holds with $\phi(n) = n^p$, $p > 0$, then, similar to (3.44), the following more general result can be proved:

$$\inf_{\tau \in \Delta(\alpha)} \mathbf{ED}_m^{(k)}(\tau) \sim \mathbf{ED}_m^{(k)}(\nu_{B_\alpha}) \sim \left(\frac{|\ln \alpha|}{q_d} \right)^{m/p} \quad \text{as } \alpha \rightarrow 0.$$

It is worth noting that the r -quick convergence conditions (3.27) and (3.28) are only sufficient and by no means are necessary. Indeed, the proof of Lemma 3 suggests that the upper bound holds whenever left-sided versions of r -quick conditions are satisfied, i.e., $\mathbf{E}_k[t_\varepsilon^{(k)}]^r < \infty$ and $\sum_{k=1}^\infty \pi_k \mathbf{E}_k[t_\varepsilon^{(k)}]^r < \infty$, where $t_\varepsilon^{(k)} = \sup\{n \geq 1: n^{-1}Z_{k+n-1}^k - q < -\varepsilon\}$. Moreover, even these latter conditions can be substantially relaxed into the following condition:

$$\lim_{n \rightarrow \infty} n^{r-1} \mathbf{P}_k \{n^{-1}Z_{k+n-1}^k - q \leq -\varepsilon\} = 0 \quad (3.49)$$

for all $\varepsilon > 0$, $k \geq 1$, and some $r \geq 1$.

A proof can be built upon a generalization of a «trick» exploited recently by Lai [16] for studying the asymptotic optimality of the CUSUM and window-limited CUSUM tests.

However, we find it natural and convenient from the methodological standpoint to formulate conditions in terms of rates of convergence in the strong law of large numbers for the LLR process. Besides, r -quick convergence implies the right-tail condition (3.2) which is the key for obtaining the lower bounds.

4. Asymptotic operating characteristics of the detection procedure ν_B in the i.i.d. case

In this section, we will deal with the i.i.d. case where $f_{0,n}(X_n | \mathbf{X}^{n-1}) = f_0(X_n)$ and $f_{1,n}(X_n | \mathbf{X}^{n-1}) = f_1(X_n)$. Then $\mathbf{P}_\infty$ is the probability measure under which the PDF of X_n is $f_0(x)$ for every $n \geq 1$ and for $\lambda = k$, $k \geq 1$, $\mathbf{P}_k$ is the probability measure under which the PDF of X_n is $f_0(x)$ if $n \leq k-1$ and is $f_1(x)$ if $n \geq k$ (with respect to a σ -finite measure $\mu(x)$). The LLR defined in (2.5) gets modified to

$$Z_n^k := \ln \frac{d\mathbf{P}_k}{d\mathbf{P}_\infty}(X^n) = \sum_{t=k}^n \ln \frac{f_1(X_t)}{f_0(X_t)}, \quad n \geq k, \quad (4.1)$$

and the decision statistic Λ_n obeys (2.9) with Z_n^k defined in (4.1).

Note also that in the i.i.d. case the statistic Λ_n satisfies the recursion

$$\Lambda_n = \left(\frac{\Pi_{n-1}}{\Pi_n} \Lambda_{n-1} + \frac{\pi_n}{\Pi_n} \right) \frac{f_1(X_n)}{f_0(X_n)}, \quad n \geq 1, \quad \Lambda_0 = \frac{\pi_0}{1 - \pi_0}, \quad (4.2)$$

which may be deployed for practical implementation and simulations.

As it was mentioned in Section 2, in the i.i.d. case and for the geometric prior distribution, the Shiryaev procedure (2.10) is optimal when the threshold B can be chosen in such a way that $\text{PFA}(\nu_B) = \alpha$. Since it is difficult to meet this exact requirement, we will study the properties of the detection procedure ν_B with $B = (1 - \alpha)/\alpha$, which guarantees the inequality $\text{PFA}(\nu_B) \leq \alpha$. Since this choice neglects the overshoot, it is expected that the actual false alarm probability may be substantially smaller than α (see Sections 4.2 and 6).

Let

$$D(f_1, f_0) = \mathbf{E}_1 Z_1^1 = \int \ln \left(\frac{f_1(x)}{f_0(x)} \right) f_1(x) \mu(dx)$$

be the Kullback-Leibler (K-L) information number between the densities $f_1(x)$ and $f_0(x)$ (also called the K-L «distance»). In the i.i.d. case, the K-L number $D(f_1, f_0)$ plays the role of the number q that appeared in Theorems 1–3 of the previous section.

By analogy with (3.20) we define the last entry times

$$T_\varepsilon^{(k)} = \sup \left\{ n \geq 1: \left| \frac{1}{n} \sum_{t=k}^{k+n-1} Y_t - D(f_1, f_0) \right| > \varepsilon \right\}. \quad (4.3)$$

Since $Y_t, t = 1, 2, \dots$, are i.i.d. random variables, $T_\epsilon^{(k)}$ have the same statistical properties for all $k \geq 1$. Therefore, $\mathbf{E}_k[T_\epsilon^{(k)}]^r = \mathbf{E}_1[T_\epsilon^{(1)}]^r$ and the condition $\mathbf{E}_1[T_\epsilon^{(1)}]^r < \infty$ for all $\epsilon > 0$ (i.e., the r -quick convergence condition of Z_n^1/n to $D(f_1, f_0)$ under $\mathbf{P}_1$) implies

$$\sum_{k=1}^\infty \pi_k \mathbf{E}_k[T_\epsilon^{(k)}]^r = (1 - \pi_0) \mathbf{E}_1[T_\epsilon^{(1)}]^r < \infty \quad \text{for all } \epsilon > 0. \tag{4.4}$$

In the i.i.d. case, the condition $\mathbf{E}_1|Z_1^1|^{r+1} < \infty$ is both necessary and sufficient for the r -quick convergence of Z_n^1/n to $D(f_1, f_0)$. Indeed, by the Baum–Katz rate of convergence in the strong law [2], the following statements are equivalent for any $r > 0$:

$$\begin{aligned} \mathbf{E}_1|Z_1^1|^{r+1} < \infty &\iff \sum_{n=1}^\infty n^{r-1} \mathbf{P}_1\left\{\left|\sum_{t=1}^n \tilde{Y}_t\right| \geq \epsilon n\right\} < \infty \quad \text{for some } \epsilon > 0, \\ &\iff \sum_{n=1}^\infty n^{r-1} \mathbf{P}_1\left\{\left|\sup_{t \geq n} t^{-1} \tilde{Y}_t\right| \geq \epsilon\right\} < \infty \quad \text{for all } \epsilon > 0, \end{aligned} \tag{4.5}$$

where $\tilde{Y}_t = Y_t - D(f_1, f_0)$. Since $\mathbf{P}_1\{T_\epsilon^{(1)} > n\} \leq \mathbf{P}_1\{\sup_{t \geq n} t^{-1} \tilde{Y}_t \geq \epsilon\}$, the implication $\mathbf{E}_1|Z_1^1|^{r+1} < \infty \Leftrightarrow \mathbf{E}_1[T_\epsilon^{(1)}]^r < \infty$ for all $\epsilon > 0$ follows.

Applying Theorem 3, one can conclude that if the K-L number $D(f_1, f_0)$ is strictly positive and the $(r + 1)$ -st absolute moment of the LLR Z_1^1 is finite, then the Shiryaev detection procedure ν_{B_α} asymptotically minimizes moments of the detection delay up to the order r . However, below we show that the Shiryaev procedure minimizes all positive moments of the detection delay under weaker conditions. As we demonstrate, all that is required is positiveness and finiteness of the K-L numbers.

Details are given in the next subsection.

4.1. Asymptotic optimality. We will impose the following mild condition on the K-L information numbers:

$$0 < D(f_1, f_0) < \infty \quad \text{and} \quad 0 < D(f_0, f_1) < \infty. \tag{4.6}$$

Positiveness of the K-L information numbers is not at all restrictive, since it holds whenever the PDFs $f_0(x)$ and $f_1(x)$ do not coincide almost everywhere, i.e., $\mu\{x: f_0(x) \neq f_1(x)\} > 0$. If it does not hold, the LLR Z_1^1 is equal to zero almost surely, in which case the detection problem is degenerate. The second condition (finiteness) is quite natural and it holds in most cases. However, there are reasonable models for which it does not hold. The problem of detecting a change in mean value of a rectangular distribution may serve as a good example. In the latter case, the moments of the LLR are infinite and the detection problem becomes degenerate at least in the asymptotic setting. On the other hand, if K-L numbers are finite, then all the moments of the negative part of the LLR are finite, $\mathbf{E}_1\{-\min(0, Z_1^1)\}^m < \infty$, since the

PDFs $f_0(x)$ and $f_1(x)$ are mutually absolutely continuous (i.e., if $f_0(x) = 0$ so is $f_1(x)$), which implies that

$$\mathbf{E}_{f_1} \frac{f_0(X_1)}{f_1(X_1)} = \mathbf{E}_1 \exp\{-Z_1^1\} = 1. \quad (4.7)$$

In the rest of this section, for the sake of simplicity, we will restrict our attention to the geometric prior distribution given in (2.8), in which case $d = |\ln(1 - \rho)|$ and the statistic Λ_n obeys the recursion

$$\Lambda_n = \frac{1}{1 - \rho} (\Lambda_{n-1} + \rho) \frac{f_1(X_n)}{f_0(X_n)}, \quad n \geq 1, \quad \Lambda_0 = \frac{\pi_0}{1 - \pi_0}. \quad (4.8)$$

The latter recursion follows from (4.2) observing that $\pi_n/\Pi_n = \rho/(1 - \rho)$ and $\Pi_{n-1}/\Pi_n = 1/(1 - \rho)$.

The following theorem establishes asymptotic optimality properties of ν_{B_α} in the class $\Delta(\alpha)$ with respect to all positive moments of the detection delay.

Theorem 4. *Let, conditioned on $\lambda = k$, the observations $X_1, \dots, X_{k-1}$ be i.i.d. with the PDF $f_0(x)$ and let X_k, X_{k+1} be i.i.d. with the PDF $f_1(x)$. Further, let prior distribution of the change point λ be geometric. Suppose that conditions (4.6) are satisfied. If $B_\alpha = (1 - \alpha)/\alpha$, then $\text{PFA}(\nu_{B_\alpha}) \leq \alpha$ and, as $\alpha \rightarrow 0$, for all $m \geq 1$*

$$\inf_{\tau \in \Delta(\alpha)} \mathbf{ED}_m^\pi(\tau) \sim \mathbf{ED}_m^\pi(\nu_{B_\alpha}) \sim \left(\frac{|\ln \alpha|}{D(f_1, f_0) + |\ln(1 - \rho)|} \right)^m, \quad (4.9)$$

$$\begin{aligned} \inf_{\tau \in \Delta(\alpha)} \mathbf{ED}_m^{(k)}(\tau) &\sim \mathbf{ED}_m^{(k)}(\nu_{B_\alpha}) \\ &\sim \left(\frac{|\ln \alpha|}{D(f_1, f_0) + |\ln(1 - \rho)|} \right)^m \quad \forall k \geq 1. \end{aligned} \quad (4.10)$$

Proof. For $b > 0$, define $\eta_b(k)$ as in (3.33) and note that in the i.i.d. case and for the geometric prior distribution the statistic $S_{k+n-1}^k(\rho) = Z_{k+n-1}^k + n|\ln(1 - \rho)|$, $n \geq 1$, is a random walk with mean $\mathbf{E}_k S_k^k(\rho) = D(f_1, f_0) + |\ln(1 - \rho)|$.

By conditions (4.6), $D(f_1, f_0)$ is positive and finite and hence $\mathbf{E}_k \{-\min(0, Z_k^k)\}^m < \infty$ for all $m > 0$. Indeed,

$$\mathbf{E}_k \exp\{-\min(0, Z_k^k)\} = \mathbf{E}_k e^{-Z_k^k} \mathbf{1}_{\{Z_k^k < 0\}} + \mathbf{E}_k \mathbf{1}_{\{Z_k^k \geq 0\}} \leq \mathbf{E}_k e^{-Z_k^k} + 1 = 2,$$

where the last equality follows from (4.7). Therefore, Theorem III.8.1 of Gut [10] applies to show that for all $m \geq 1$

$$\mathbf{E}_k [\eta_b(k)]^m = \left(\frac{b}{D(f_1, f_0) + |\ln(1 - \rho)|} \right)^m (1 + o(1)) \quad \text{as } b \rightarrow \infty.$$

Since $\nu_B - k \leq \eta_{\ln B}(k)$ on $\{\nu_B \geq k\}$ and $\mathbf{P}_k\{\nu_B \geq k\} \rightarrow 1$ as $B \rightarrow \infty$, it follows that

$$\mathbf{ED}_m^{(k)}(\nu_{B_\alpha}) \leq \left(\frac{|\ln \alpha|}{D(f_1, f_0) + |\ln(1 - \rho)|} \right)^m (1 + o(1)) \quad \text{as } \alpha \rightarrow 0.$$

Note that in the i.i.d. case condition (3.2) holds trivially with $q = D(f_1, f_0)$, since Z_{k+n}^k/n converges to $D(f_1, f_0)$ ($\mathbf{P}_k$ -a.s.) by the strong law of large numbers and since $\mathbf{P}_k\{M^{-1} \max_{0 \leq n < M} Z_{k+n}^k \geq (1 + \varepsilon) D(f_1, f_0)\}$ does not depend on k . Thus, the lower bound follows from (3.14):

$$\inf_{\tau \in \Delta(\alpha)} \mathbf{ED}_m^{(k)}(\tau) \geq \left(\frac{|\ln \alpha|}{D(f_1, f_0) + |\ln(1 - \rho)|} \right)^m (1 + o(1)) \quad \text{as } \alpha \rightarrow 0,$$

which completes the proof of (4.10).

The proof of (4.9) is quite similar and is therefore omitted.

4.2. Higher-order asymptotic approximations for ADD and PFA. In this subsection, we use the nonlinear renewal theory developed by Woodroffe [42] (see also Siegmund [29]) to improve the first-order approximations for the ADD and PFA.

We will also suppose (with minor loss of generality) that $\pi_0 = 0$, i.e., in the rest of this subsection the «pure» geometric prior distribution, $\pi_k = \rho(1 - \rho)^{k-1}$, $k \geq 1$, will be considered. We first observe that, in this case, $\text{CADD}_1(\nu_B) \geq \text{CADD}_k(\nu_B)$ for all $k \geq 1$. To understand why, it is sufficient to consider the recursion (4.8) and note that, for $\lambda = k = 1$, the initial condition $\Lambda_0 = 0$ while, for $\lambda = k \geq 2$, $0 \leq \Lambda_{k-1} < B$ on $\nu_B \geq k$. Moreover, for large B , the difference between $\mathbf{E}_1(\nu_B - 1)$ and $\mathbf{E}_k(\nu_B - k | \nu_B \geq k)$ is a constant that is approximately equal to the mean of the initial condition, $\mathbf{E}_\infty \ln \Lambda_{k-1}$ (for $k = 1$ this value is equal to 0). This constant varies for different models and its calculation is usually problematic. For this reason, we will concentrate on the evaluation of the worst-case delay $\text{CADD}_1(\nu_B)$.

In order to apply relevant results from nonlinear renewal theory, we rewrite the stopping time ν_B in the form of a random walk crossing a constant threshold plus a nonlinear term that is «slowly changing» in the sense defined by Woodroffe [42] and Siegmund [29]. Indeed, the stopping time ν_B can be written in the following form:

$$\nu_B = \inf\{n \geq 1: S_n(\rho) + \ell_n \geq b\}, \quad b = \ln(B\rho^{-1}), \quad (4.11)$$

where $S_n(\rho) = Z_n + n|\ln(1 - \rho)|$ is a random walk with mean $\mathbf{E}_1 S_1(\rho) = D(f_1, f_0) + |\ln(1 - \rho)|$ and

$$\ell_n = \ln \left\{ 1 + \sum_{i=1}^{n-1} (1 - \rho)^i \prod_{s=1}^i \frac{f_0(X_s)}{f_1(X_s)} \right\} = \ln \left\{ 1 + \sum_{i=1}^{n-1} (1 - \rho)^i e^{-Z_i} \right\}. \quad (4.12)$$

Here and in the rest of this subsection, we write Z_n in place of Z_n^1 .

For $b > 0$, define $\eta_b(1) = \eta_b$ as in (3.33), i.e.,

$$\eta_b = \inf\{n \geq 1: S_n(\rho) \geq b\}, \quad (4.13)$$

and let $\varkappa_b = S_{\eta_b}(\rho) - b$ (on $\{\eta_b < \infty\}$) denote the excess (overshoot) of the statistic $S_n(\rho)$ over the threshold b at time $n = \eta_b$. Let

$$G(y, \rho, D) = \lim_{b \rightarrow \infty} \mathbf{P}_1\{\varkappa_b \leq y\} \quad (4.14)$$

be the limiting distribution of the overshoot and let

$$\bar{\pi}(\rho, D) = \lim_{b \rightarrow \infty} \mathbf{E}_1 \chi_b = \int_0^\infty y dG(y, \rho, D)$$

denote the related limiting average overshoot. Let us also define

$$\begin{aligned} \zeta(\rho, D) &= \lim_{b \rightarrow \infty} \mathbf{E}_1 e^{-\chi_b} = \int_0^\infty e^{-y} dG(y, \rho, D) \quad \text{and} \\ C(\rho, D) &= \mathbf{E}_1 \ln \left\{ 1 + \sum_{i=1}^{\infty} (1 - \rho)^i e^{-Z_i} \right\}. \end{aligned} \quad (4.15)$$

Note that by (4.11), $S_{\nu_B}(\rho) = b - \ell_{\nu_B} + \chi_b$ on $\{\nu_B < \infty\}$, where $\chi_b = S_{\nu_B}(\rho) + \ell_{\nu_B} - b$ is the excess of the process $S_n(\rho) + \ell_n$ over the level b at time ν_B . Taking the expectations on both sides and applying Wald's identity, we obtain

$$(D + |\ln(1 - \rho)|) \mathbf{E}_1 \nu_B = b - \mathbf{E}_1 \ell_{\nu_B} + \mathbf{E}_1 \chi_b. \quad (4.16)$$

The crucial observations are that the sequence $\{\ell_n, n \geq 1\}$ is slowly changing, and that ℓ_n converges $\mathbf{P}_1$ -a.s. as $n \rightarrow \infty$ to the random variable $\ell = \ln\{1 + \sum_{i=1}^{\infty} (1 - \rho)^i e^{-Z_i}\}$ with finite expectation $\mathbf{E}_1 \ell = C(\rho, D)$. In fact, applying Jensen's inequality yields

$$C(\rho, D) = \mathbf{E}_1 \ell \leq \ln \left(1 + \sum_{k=1}^{\infty} (1 - \rho)^k \right) = \ln \left(\frac{1}{\rho} \right). \quad (4.17)$$

Moreover, $\lim_{n \rightarrow \infty} \mathbf{E}_1 \ell_n = C(\rho, D)$ due to the fact that $\ell_n \leq \ell$.

An important consequence of the slowly changing property is that, under mild conditions, the limiting distribution of the excess of a random walk over a fixed threshold does not change by the addition of a slowly changing nonlinear term (see [42, Theorem 4.1]). Furthermore, since $\ell_n \rightarrow \ell$ and $\mathbf{E}_1 \ell_n \rightarrow C(\rho, D)$, using (4.16) we expect that for large b ,

$$\mathbf{E}_1 \nu_B \approx \frac{1}{D(f_1, f_0) + |\ln(1 - \rho)|} (b - C(\rho, D) + \bar{\pi}(\rho, D)).$$

The mathematical details are given in Theorem 5 below.

More importantly, nonlinear renewal theory allows us to obtain an asymptotically accurate approximation for $\text{PFA}(\nu_B)$ that takes the overshoot into account. This approximation is important for practical applications where the value of $D(f_1, f_0)$ is moderate. (For small values of ρ and $D(f_1, f_0)$ the overshoot can be neglected, and formula (2.11) will be reasonably accurate.)

Theorem 5. *Let the prior distribution of the change point λ be geometric, $\pi_k = \rho(1 - \rho)^{k-1}$, $k \geq 1$, and assume that Z_n , $n \geq 1$, are nonarithmetic with respect to $\mathbf{P}_\infty$ and $\mathbf{P}_1$.*

(i) *If conditions (4.6) hold, then*

$$\text{PFA}(\nu_B) = \frac{\zeta(\rho, D)}{B} (1 + o(1)) \quad \text{as } B \rightarrow \infty. \quad (4.18)$$

(ii) If, in addition, the second moment $\mathbf{E}_1|Z_1|^2$ is finite, then as $B \rightarrow \infty$

$$\mathbf{E}_1 \nu_B = \frac{1}{D(f_1, f_0) + |\ln(1 - \rho)|} \left[\ln \frac{B}{\rho} - C(\rho, D) + \bar{\kappa}(\rho, D) \right] + o(1). \quad (4.19)$$

P r o o f. (i) Obviously,

$$\begin{aligned} \text{PFA}(\nu_B) &= \mathbf{E}^\pi(1 - p_{\nu_B}) = \mathbf{E}^\pi(1 + \Lambda_{\nu_B})^{-1} = \mathbf{E}^\pi \left[1 + B \left(\frac{\Lambda_{\nu_B}}{B} \right) \right]^{-1} \\ &= \frac{1}{B} \mathbf{E}^\pi e^{-\chi_b} (1 + o(1)) \quad \text{as } B \rightarrow \infty, \end{aligned}$$

where $\chi_b = \ln \Lambda_{\nu_B} - b$. Since $\chi_b \geq 0$ and $\text{PFA}(\nu_B) \leq 1/(1 + B) < 1/B$, it follows that

$$\begin{aligned} \mathbf{E}^\pi e^{-\chi_b} &= \mathbf{E}^\pi \{e^{-\chi_b} \mid \nu_B < \lambda\} \text{PFA}(\nu_B) + \mathbf{E}^\pi \{e^{-\chi_b} \mid \nu_B \geq \lambda\} (1 - \text{PFA}(\nu_B)) \\ &= \mathbf{E}^\pi \{e^{-\chi_b} \mid \nu_B \geq \lambda\} + O(B^{-1}) \quad \text{as } B \rightarrow \infty. \end{aligned}$$

Therefore, it suffices to evaluate the value of

$$\mathbf{E}^\pi \{e^{-\chi_b} \mid \nu_B \geq \lambda\} = \sum_{k=1}^{\infty} \mathbf{P}\{\lambda = k \mid \nu_B \geq k\} \mathbf{E}_k \{e^{-\chi_b} \mid \nu_B \geq k\}.$$

To this end, we recall that, by (3.37), for any $1 \leq k < \infty$,

$$\nu_B = \{n \geq 1: S_n^k(\rho) + \ell_{n,k} \geq b\},$$

where $S_n^k(\rho) = Z_n^k + (n - k + 1)|\ln(1 - \rho)|$, $n \geq k$, is a random walk with the expectation $\mathbf{E}_k S_k^k(\rho) = D(f_1, f_0) + |\ln(1 - \rho)|$ and $\ell_{n,k}$, $n \geq k$, are slowly changing under $\mathbf{P}_k$. Since, by conditions (4.6), $0 < D(f_1, f_0) < \infty$, we can apply Theorem 4.1 of Woodroffe [42] to obtain

$$\lim_{B \rightarrow \infty} \mathbf{E}_k \{e^{-\chi_b} \mid \nu_B \geq k\} = \int_0^\infty e^{-y} dG(y, \rho, D) = \zeta(\rho, D).$$

Also,

$$\lim_{B \rightarrow \infty} \mathbf{P}\{\lambda = k \mid \nu_B \geq k\} = \lim_{B \rightarrow \infty} \frac{\pi_k \mathbf{P}_\infty \{\nu_B \geq k\}}{\mathbf{P}^\pi \{\nu_B \geq \lambda\}} = \pi_k.$$

Consequently,

$$\lim_{B \rightarrow \infty} \mathbf{E}^\pi \{e^{-\chi_b} \mid \nu_B \geq \lambda\} = \lim_{B \rightarrow \infty} \mathbf{E}^\pi e^{-\chi_b} = \zeta(\rho, D),$$

which completes the proof of (4.18).

(ii) The proof of (4.19) rests on Woodroffe's Nonlinear Renewal Theorem (see [42, Theorem 4.5]). Indeed, by (4.11), the stopping time ν_B is based on the thresholding of the sum of the random walk $S_n(\rho)$ and the nonlinear term ℓ_n . Since

$$\ell_n \xrightarrow[n \rightarrow \infty]{\mathbf{P}_{1-\text{a.s.}}} \ell \quad \text{and} \quad \mathbf{E}_1 \ell_n \xrightarrow[n \rightarrow \infty]{} \mathbf{E}_1 \ell = C(\rho, D),$$

ℓ_n , $n \geq 1$, are slowly changing under $\mathbf{P}_1$. In order to apply this theorem we have to check the validity of the following three conditions:

$$\sum_{n=1}^{\infty} \mathbf{P}_1\{\ell_n \leq -\varepsilon n\} < \infty \quad \text{for some } 0 < \varepsilon < D; \quad (4.20)$$

$$\max_{0 \leq k \leq n} |\ell_{n+k}|, \quad n \geq 1, \quad \text{are } \mathbf{P}_1\text{-uniformly integrable;} \quad (4.21)$$

$$\lim_{B \rightarrow \infty} b \mathbf{P}_1\{\nu_B \leq \varepsilon b(D + \mu)^{-1}\} = 0 \quad \text{for some } 0 < \varepsilon < 1. \quad (4.22)$$

Condition (4.20) holds trivially, since $\ell_n \geq 0$. The value of $\max_{0 \leq k \leq n} |\ell_{n+k}|$ is equal to ℓ_{2n} because ℓ_n , $n = 1, 2, \dots$, are nondecreasing, and to prove (4.21) it suffices to show that ℓ_n , $n \geq 1$, are $\mathbf{P}_1$ -uniformly integrable. Since $\ell_n \leq \ell$ and, by (4.17), $\mathbf{E}_1 \ell < \infty$, the desired uniform integrability follows. Therefore, condition (4.21) is satisfied.

We now turn to checking condition (4.22). Noting that in the notation of Subsection 3.2,

$$\mathbf{P}_1\{\nu_B < (1 - \varepsilon)b(D_\rho)^{-1}\} = \gamma_\varepsilon^{(1)}(B), \quad \text{where } D_\rho = D + |\ln(1 - \rho)|,$$

and using inequalities (3.6) and (3.9) with $\alpha = e^{-b}$, we obtain

$$\mathbf{P}_1\{\nu_B < (1 - \varepsilon)b(D_\rho)^{-1}\} \leq e^{-y_\varepsilon b} + \beta_1(\varepsilon, B),$$

where $y_\varepsilon > 0$ for all $\varepsilon > 0$ and $\beta_1(\varepsilon, B) = \mathbf{P}_1\{\max_{1 \leq n < K_{\varepsilon, B}} Z_n \geq (1 + \varepsilon)DK_{\varepsilon, B}\}$, $K_{\varepsilon, B} = (1 - \varepsilon)b(D_\rho)^{-1}$. The first term in the above inequality is $o(1/b)$ as $B \rightarrow \infty$. All it remains to do is to show that the second term is $o(1/b)$.

To this end, we apply Theorem 1 of Chow and Lai [6], according to which for all $\varepsilon > 0$ and $r \geq 0$

$$\sum_{n=1}^{\infty} n^{r-1} \mathbf{P}_1\left\{\max_{1 \leq k \leq n} (Z_k - Dk) \geq \varepsilon n\right\} \leq C_r \{\mathbf{E}_1[(Z_1 - D)^+]^{r+1} + [\mathbf{E}_1(Z_1 - D)^2]^r\},$$

where C_r is a universal constant. Recall that by the conditions of the theorem, $\mathbf{E}_1|Z_1|^2 < \infty$. Therefore, the sum on the left-hand side of the previous inequality is finite for $r = 1$ and all $\varepsilon > 0$, which implies that the summand should be $o(1/n)$. Since

$$\beta_1(\varepsilon, B) \leq \mathbf{P}_1\left\{\max_{n < K_{\varepsilon, B}} (Z_n - Dn) \geq \varepsilon D(f_1, f_0) K_{\varepsilon, B}\right\},$$

it follows that $\beta_1(\varepsilon, B) = o(b^{-1})$. Hence condition (4.22) holds for all $0 < \varepsilon < 1$.

Thus, all the conditions of Theorem 4.5 in [42] are satisfied. The use of this theorem yields (4.19) for large B .

R e m a r k 4. The constants $\bar{\kappa}(\rho, D)$ and $\zeta(\rho, D)$ are the subject of the renewal theory. The constant $C(\rho, D)$ is not easy to compute in general. For ρ close to 1, the upper bound (4.17) may be useful. Obviously, this bound is asymptotically accurate when $D \rightarrow 0$. Monte Carlo experiments may be used to estimate C with a reasonable accuracy (see Table 1 in Section 6).

The usefulness of Theorem 5 is two-fold. First, it provides accurate approximations for both the ADD and the PFA. Second, it allows us to study the important limiting case of $\rho \rightarrow 0$.

To analyze the latter case it is convenient to consider the statistic $R_{\rho,n} = \Lambda_n/\rho$. As $\rho \rightarrow 0$ this statistic converges to the so-called Shiryaev–Roberts statistic $R_n = \lim_{\rho \rightarrow 0} R_{\rho,n}$. Also, as $\rho \rightarrow 0$, PFA tends to 1 and $\lim_{\rho \rightarrow 0} \{[1 - \text{PFA}(\nu_B)]/\rho\} = \mathbf{E}_\infty \nu_B$. Thus, it is natural to consider asymptotics $\alpha \rightarrow 1$ and $\rho \rightarrow 0$ so that the ratio $(1 - \alpha)/\rho$ remains constant at T , $0 < T < \infty$, where T may be interpreted as a constraint on the mean time to false alarm $\mathbf{E}_\infty \nu_B$. The requirement $(1 - \alpha)/\rho \rightarrow T$ as $\alpha \rightarrow 1, \rho \rightarrow 0$ is similar to that used by Shiryaev [25] for detecting changes in stationary regimes (for the Wiener process).

The following corollary follows directly from Theorem 5 and the above argument.

Corollary 2. *Let $\nu_B = \inf\{n: R_{\rho,n} \geq B\}$ and $B = (1 - \alpha)/(\alpha\rho)$. Assume that $\alpha \rightarrow 1$ and $\rho \rightarrow 0$ in such a way that $(1 - \alpha)/\rho \rightarrow T$, where $0 < T < \infty$. Then, under the conditions of Theorem 5*

$$\mathbf{E}_1 \nu_T = \frac{1}{D(f_1, f_0)} [\ln T - C(D) + \overline{\kappa}(D)] + o(1) \quad \text{as } T \rightarrow \infty, \quad (4.23)$$

where ν_T represents ν_B under the above limit, and where $C(D) = C(0, D)$ and $\overline{\kappa}(D) = \overline{\kappa}(0, D)$.

It can also be shown that in the conditions of Theorem 5, $\mathbf{E}_\infty \nu_B \sim T$ as $B = T\zeta(D)$ and $T \rightarrow \infty$, where $\zeta(D) = \zeta(0, D)$. These results generalize similar results obtained previously by Pollak [22] for exponential families (see [22, Theorem 3]).

5. Asymptotic performance of other detection procedures

It is known that in the case where the observations are i.i.d. and the change point is modeled as deterministic but unknown, the cumulative sum (CUSUM) detection procedure of Page [19] and the randomized Shiryaev–Roberts detection procedure proposed by Pollak [21] are optimal with respect to the minimax expected detection lag, subject to a constraint on the mean time to false alarm.

More specifically, consider the following two detection procedures:

$$\hat{\tau}_B = \inf\{n \geq 0: R_n \geq B\} \quad \text{and} \quad \tau_B^* = \inf\{n \geq 1: U_n \geq B\},$$

where the statistics R_n and U_n are defined as follows:

$$R_n = \sum_{k=1}^n e^{Z_n^k} \quad \text{and} \quad U_n = \max_{1 \leq k \leq n} e^{Z_n^k}.$$

If $R_0 = 0$, the detection procedure $\hat{\tau}_B$ is the Shiryaev–Roberts procedure. Its randomized version, when R_0 is random with a certain distribution, has

been suggested by Pollak [21]. We will refer to this procedure as to the Shiryaev–Roberts–Pollak (SRP) test. The second test τ_B^* is nothing but the CUSUM algorithm.

In 1971, Lorden [17] proposed to measure the loss due to the detection delay by the «worst-worst» minimax risk $ES(\tau) = \sup_k \text{ess sup } \mathbf{E}_k\{(\tau - k + 1)^+ | X^{k-1}\}$ and showed that the CUSUM procedure τ_B^* with $B = T$ is asymptotically optimal for i.i.d. models, as $T \rightarrow \infty$, in the class $\Delta_T = \{\tau: \mathbf{E}_\infty \tau \geq T\}$ of procedures for which the mean time to false alarm $\mathbf{E}_\infty \tau$ exceeds a predefined number T . In 1986, Moustakides [18] proved that the CUSUM test is strictly optimal for any $T > 0$ with respect to the risk $ES(\tau)$ whenever the threshold B is chosen so that $\mathbf{E}_\infty \tau_B^* = T$. In 1996, Shiryaev [28] extended this result for continuous-time processes (detecting a change in the mean of the Brownian motion). In 1998, Lai [16] generalized previous results for a general non-i.i.d. case showing that the CUSUM procedure is asymptotically optimal in the class Δ_T as $T \rightarrow \infty$ with respect to the «worst-worst» risk $ES(\tau)$ as well as with respect to the «average-worst» risk (minimax delay) $MD(\tau) = \sup_k \mathbf{E}_k(\tau - k | \tau \geq k)$. The latter measure of the detection speed has been introduced earlier by Pollak [21].

In the mid of 1980s, Pollak [21] proposed the randomized version of the Shiryaev–Roberts procedure $\hat{\tau}_B$ where the statistic R_n is randomized at the zero point $n = 0$, i.e., R_0 is a random variable ($R_0 = 0$ for the standard Shiryaev–Roberts procedure). Pollak proved that this randomized procedure is nearly optimal with respect to the risk $MD(\tau)$ for i.i.d. data models. Pollak [22] also presented a comprehensive asymptotic analysis of the procedure $\hat{\tau}_B$ for exponential families. See [1], [30]–[39] for further extensions and details.

Therefore, both the CUSUM and the SRP detection procedures minimize $\sup_k \mathbf{E}_k(\tau - k | \tau \geq k)$ (the expected detection delay in the worst-case scenario) in the class of procedures for which the mean time to false alarm $\mathbf{E}_\infty \tau$ exceeds a predefined number T (at least as $T \rightarrow \infty$).

In this section, we study asymptotic properties of these two classical change-point detection procedures in the class $\Delta(\alpha)$. The results of previous sections are used to establish that they lose the asymptotic optimality property under the Bayesian criterion for prior distributions with exponential tail $\mathcal{E}(d)$, but remain optimal for heavy-tailed prior distributions $\mathcal{H}$.

In order to obtain an upper bound for PFA of the SRP procedure, we note that the statistic $R_n - n$ is a zero-mean $\mathbf{P}_\infty$ -martingale (with respect to $\mathcal{F}_n^X$). Therefore, R_n is a submartingale with mean $\mathbf{E}_\infty R_n = n$. Using Doob submartingale inequality, we get

$$\mathbf{P}_\infty\{\hat{\tau}_B < n\} = \mathbf{P}_\infty\left\{\max_{1 \leq k < n} R_k \geq B\right\} \leq nB^{-1},$$

which yields

$$\widehat{\text{PFA}}(B) = \mathbf{P}^\pi \{\hat{\tau}_B < \lambda\} = \sum_{n=1}^{\infty} \pi_n \mathbf{P}_\infty \{\hat{\tau}_B < n\} \leq \bar{\lambda} B^{-1},$$

where $\bar{\lambda} = \sum_{k=1}^{\infty} k\pi_k$ is the mean of the prior distribution. Thus, choosing $B = B_\alpha = \bar{\lambda}/\alpha$ guarantees $\hat{\tau}_{B_\alpha} \in \Delta(\alpha)$.

Since $\tau_B^* \geq \hat{\tau}_B$, for the CUSUM procedure we obtain

$$\mathbf{P}_\infty \{\tau_B^* < n\} \leq \mathbf{P}_\infty \{\hat{\tau}_B < n\} \leq nB^{-1}, \quad n \geq 1, \quad (5.1)$$

and hence,

$$\begin{aligned} \text{PFA}^*(B) &= \mathbf{P}^\pi \{\tau_B^* < \lambda\} = \sum_{n=1}^{\infty} \pi_n \mathbf{P}_\infty \{\tau_B^* < n\} \\ &\leq B^{-1} \sum_{n=1}^{\infty} n\pi_n = \bar{\lambda} B^{-1}. \end{aligned} \quad (5.2)$$

It follows that $B = B_\alpha = \bar{\lambda}/\alpha$ implies $\tau_{B_\alpha} \in \Delta(\alpha)$.

The following theorem, which is a prototype of Theorem 3, establishes the asymptotic operating characteristics of the CUSUM and SRP procedures in terms of moments of the detection delay.

Theorem 6. *Let $\bar{\lambda} = \sum_{k=1}^{\infty} k\pi_k < \infty$ and $B = B_\alpha = \bar{\lambda}/\alpha$. Then both detection procedures $\tau_{B_\alpha}^*$ and $\hat{\tau}_{B_\alpha}$ belong to the class $\Delta(\alpha)$ and the following two assertions hold.*

(i) *If the condition (3.25) is satisfied for some $r > 0$, then for all $m \leq r$ and $k \geq 1$ as $\alpha \rightarrow 0$*

$$\mathbf{ED}_m^{(k)}(\tau_{B_\alpha}^*) \sim \mathbf{ED}_m^{(k)}(\hat{\tau}_{B_\alpha}) \sim \left(\frac{|\ln \alpha|}{q} \right)^m. \quad (5.3)$$

(ii) *If the condition (3.26) is satisfied for some $r > 0$, then for all $m \leq r$ as $\alpha \rightarrow 0$*

$$\mathbf{ED}_m^\pi(\tau_{B_\alpha}^*) \sim \mathbf{ED}_m^\pi(\hat{\tau}_{B_\alpha}) \sim \left(\frac{|\ln \alpha|}{q} \right)^m. \quad (5.4)$$

P r o o f. We provide the proof only for the SRP procedure, since the proof for the CUSUM procedure is essentially the same.

For any $B > 0$, define the one-sided sequential probability ratio test by $\eta_B(k) = \inf\{n \geq 1: Z_{k+n-1}^k \geq \ln B\}$, $k \geq 1$.

Our first observation is that $\hat{\tau}_B - k \leq \eta_B(k)$ on $\{\hat{\tau}_B \geq k\}$. Consequently, for $B > k$

$$\mathbf{ED}_m^{(k)}(\hat{\tau}_{B_\alpha}) = \frac{\mathbf{E}_k[(\hat{\tau}_B - k)^+]^m}{\mathbf{P}_k\{\hat{\tau}_B \geq k\}} \leq \frac{\mathbf{E}_k\{\eta_B(k)^m; \hat{\tau}_B \geq k\}}{\mathbf{P}_k\{\hat{\tau}_B \geq k\}} \leq \frac{\mathbf{E}_k[\eta_B(k)]^m}{1 - kB^{-1}},$$

where the latter inequality follows from (5.1). By Lemma 3,

$$\mathbf{E}_k[\eta_B(k)]^m \sim \left(\frac{\ln B}{q} \right)^m \quad \text{as } B \rightarrow \infty, \quad (5.5)$$

and hence,

$$\mathbf{ED}_m^{(k)}(\hat{\tau}_{B_\alpha}) \leq \left(\frac{\ln B}{q}\right)^m (1 + o(1)) \quad \text{as } B \rightarrow \infty. \quad (5.6)$$

Next, similar to (3.41)

$$\hat{\tau}_B - k \leq \eta_B(k) \leq T_\varepsilon^{(k)} + 2 + \frac{\ln B}{q - \varepsilon} \quad \text{on } \{\hat{\tau}_B \geq k\}.$$

Applying the latter inequality along with $\mathbf{P}^\pi\{\hat{\tau}_B \geq \lambda\} \geq 1 - \bar{\lambda}/B$ yields (for $B > \bar{\lambda}$)

$$\mathbf{ED}_m^\pi(\hat{\tau}_{B_\alpha}) \leq \frac{B - \bar{\lambda}}{B} \sum_{k=1}^{\infty} \pi_k \mathbf{E}_k \left(\frac{\ln B}{q - \varepsilon} + T_\varepsilon^{(k)} + 2 \right)^m.$$

Since by the assumption of the theorem, $\sum_{k=1}^{\infty} \pi_k \mathbf{E}_k [T_\varepsilon^{(k)}]^r < \infty$ and ε can be arbitrarily small, it follows that for $m \leq r$

$$\mathbf{ED}_m^\pi(\hat{\tau}_B) \leq \left(\frac{\ln B}{q}\right)^m (1 + o(1)) \quad \text{as } B \rightarrow \infty. \quad (5.7)$$

We now proceed with deriving the lower bounds. Write

$$\begin{aligned} \hat{\gamma}_\varepsilon^{(k)}(B) &= \mathbf{P}_k\{k \leq \hat{\tau}_B < k + (1 - \varepsilon)q^{-1} \ln B\}, \\ \hat{\gamma}_\varepsilon(B) &= \mathbf{P}^\pi\{\lambda \leq \hat{\tau}_B < \lambda + (1 - \varepsilon)q^{-1} \ln B\}. \end{aligned}$$

An argument similar to that used in the proof of Lemma 2 shows that

$$\lim_{B \rightarrow \infty} \hat{\gamma}_\varepsilon^{(k)}(B) = 0 \quad \text{and} \quad \lim_{B \rightarrow \infty} \hat{\gamma}_\varepsilon(B) = 0 \quad \text{for every } 0 < \varepsilon < 1. \quad (5.8)$$

Therefore, applying Chebyshev's inequality, we obtain

$$\mathbf{E}^\pi[(\hat{\tau}_B - \lambda)^+]^m \geq [(1 - \varepsilon)q^{-1} \ln B]^m [\mathbf{P}^\pi\{\hat{\tau}_B \geq \lambda\} - \hat{\gamma}_\varepsilon(B)].$$

Since $\mathbf{ED}_m^\pi(\hat{\tau}_B) = \mathbf{E}^\pi[(\hat{\tau}_B - \lambda)^+]^m / \mathbf{P}^\pi\{\hat{\tau}_B \geq \lambda\}$ and $\mathbf{P}^\pi\{\hat{\tau}_B \geq \lambda\} \geq 1 - \bar{\lambda}/B$, it follows that, for $B > \bar{\lambda}$,

$$\mathbf{ED}_m^\pi(\hat{\tau}_B) \geq [(1 - \varepsilon)q^{-1} \ln B]^m \left[1 - \frac{\hat{\gamma}_\varepsilon(B)}{1 - \bar{\lambda}B^{-1}} \right],$$

where $\hat{\gamma}_\varepsilon(B) \rightarrow 0$ as $B \rightarrow \infty$ by (5.8). Since ε is arbitrary,

$$\mathbf{ED}_m^\pi(\hat{\tau}_B) \geq \left(\frac{\ln B}{q}\right)^m (1 + o(1)) \quad \text{as } B \rightarrow \infty,$$

which along with the upper bound (5.7) yields the asymptotic equality

$$\mathbf{ED}_m^\pi(\hat{\tau}_B) = \left(\frac{\ln B}{q}\right)^m (1 + o(1)) \quad \text{as } B \rightarrow \infty. \quad (5.9)$$

Setting $B = B_\alpha = \ln(\bar{\lambda}/\alpha)$ in (5.9) proves (5.4).

Analogously, for $B > k$,

$$\begin{aligned} \mathbf{ED}_m^{(k)}(\widehat{\tau}_{B_\alpha}) &\geq [(1 - \varepsilon) q^{-1} \ln B]^m \left[1 - \frac{\widehat{\gamma}_\varepsilon^{(k)}(B)}{\mathbf{P}_k\{\widehat{\tau}_B \geq k\}} \right] \\ &\geq [(1 - \varepsilon) q^{-1} \ln B]^m \left[1 - \frac{\widehat{\gamma}_\varepsilon^{(k)}(B)}{1 - kB^{-1}} \right]. \end{aligned}$$

Since ε can be arbitrarily small and, by (5.8), $\widehat{\gamma}_\varepsilon^{(k)}(B) \rightarrow 0$ as $B \rightarrow \infty$ for every $k \geq 1$, we can conclude that the right-hand side in (5.6) is the asymptotic lower bound for $\mathbf{ED}_m^{(k)}(\widehat{\tau}_{B_\alpha})$. Thus, the asymptotic approximation (5.9) holds for $\mathbf{ED}_m^{(k)}(\widehat{\tau}_{B_\alpha})$. Finally, setting $B = B_\alpha = \ln(\bar{\lambda}/\alpha)$ proves (5.3).

Comparing Theorem 3 and Theorem 6 shows that the CUSUM and SRP procedures are not asymptotically optimal in a Bayesian context for $\pi \in \mathcal{E}(d)$ but remain asymptotically optimal for $\pi \in \mathcal{H}$. In particular, for the geometric prior distribution and i.i.d. data models

$$\lim_{\alpha \rightarrow 0} \frac{\text{ADD}(\widehat{\tau}_{B_\alpha})}{\inf_{\tau \in \Delta(\alpha)} \text{ADD}(\tau)} = \lim_{\alpha \rightarrow 0} \frac{\text{ADD}(\tau_{B_\alpha}^*)}{\inf_{\tau \in \Delta(\alpha)} \text{ADD}(\tau)} = 1 + \frac{|\ln(1 - \rho)|}{D(f_1, f_0)}. \tag{5.10}$$

R e m a r k 5. While the standard CUSUM algorithm is not optimal, using the above techniques one can show that the following «Bayesian» modification of the CUSUM procedure is asymptotically optimal as $\alpha \rightarrow 0$. Define the weighted CUSUM detection procedure by

$$\tau_B = \inf \left\{ n \geq 1: \max_{0 \leq k \leq n} \left[(\Pi_n)^{-1} \pi_k \prod_{i=k}^n \frac{f_{1,i}(X_i \mid \mathbf{X}^{i-1})}{f_{0,i}(X_i \mid \mathbf{X}^{i-1})} \right] \geq B \right\},$$

$B > \pi_0/(1 - \pi_0)$. If $B = B_\alpha = (1 - \alpha)/\alpha$, then $\tau_{B_\alpha} \in \Delta(\alpha)$ and Theorems 3 and 4 hold true for τ_{B_α} .

6. Example 1: change detection in the mean of the autoregressive process

In this section, we consider an example that illustrates the results of the previous sections. This example is focused on single-sensor or centralized detection. In the next section, we will consider two more examples that are related to decentralized (distributed) detection in multi-sensor systems. In the rest of the paper, without special emphasis the prior distribution will be assumed geometric (2.8) with $\pi_0 = 0$.

Consider the «signal plus noise» model, in which we assume that $X_n = \mathbb{1}_{\{n \geq \lambda\}} \theta_n + \xi_n$, $n \geq 1$, where θ_n is a deterministic signal that appears at an unknown point in time λ , and $\{\xi_n, n \geq 1\}$ is a Markov Gaussian sequence (noise), which obeys the recursion

$$\xi_n = \delta \xi_{n-1} + w_n, \qquad n \geq 1, \quad \xi_0 = 0.$$

Here $w_1, w_2, \dots$ are i.i.d. Gaussian random variables with mean zero and variance σ^2 . The parameters $0 \leq \delta < 1$ and $\sigma > 0$ are assumed to be known

(δ is the correlation coefficient of noise). Let $\varphi(x) = (2\pi)^{-1/2} \exp\{-x^2/2\}$ denote the PDF of the standard normal distribution.

For this model, the conditional PDFs $f_{0,n}(X_n | \mathbf{X}^{n-1})$ and $f_{1,n}(X_n | \mathbf{X}^{n-1})$ introduced in Section 2 are of the form

$$\begin{aligned} f_{0,n}(X_n | \mathbf{X}^{n-1}) &= f_0(X_n | X_{n-1}) = \frac{1}{\sigma} \varphi\left(\frac{X_n - \delta X_{n-1}}{\sigma}\right) \quad \text{for all } n \geq 1, \\ f_{1,n}(X_n | \mathbf{X}^{n-1}) &= f_{1,n}(X_n | X_{n-1}) \\ &= \frac{1}{\sigma} \varphi\left(\frac{X_n - \delta X_{n-1} - \theta_n}{\sigma}\right) \quad \text{for } n = \lambda, \\ f_{1,n}(X_n | \mathbf{X}^{n-1}) &= f_{1,n}(X_n | X_{n-1}) \\ &= \frac{1}{\sigma} \varphi\left(\frac{X_n - \delta X_{n-1} - (\theta_n - \delta \theta_{n-1})}{\sigma}\right) \quad \text{for } n \geq \lambda + 1, \end{aligned}$$

where $X_0 = \theta_0 = 0$.

Write $\tilde{X}_i = X_i - \delta X_{i-1}$ and $\tilde{\theta}_i = \theta_i - \delta \theta_{i-1}$. It is easy to see that

$$Z_n^k = \frac{1}{\sigma^2} \left(\theta_k \tilde{X}_k + \sum_{i=k+1}^n \tilde{\theta}_i \tilde{X}_i \right) - \frac{1}{2\sigma^2} \left(\theta_k^2 + \sum_{i=k+1}^n \tilde{\theta}_i^2 \right), \quad 1 \leq k \leq n.$$

Next, obviously, conditioned on the change point $\lambda = k$, the values of $\tilde{X}_n$, $n = 1, 2, \dots$, are independent normal random variables with variance σ^2 , and $\mathbf{E}_k \tilde{X}_n = 0$ for $n < k$, $\mathbf{E}_k \tilde{X}_n = \theta_n$ for $n = k$, and $\mathbf{E}_k \tilde{X}_n = \tilde{\theta}_n$ for $n > k$. Therefore, conditioned on $\lambda = k$, $\{Z_n^k, n \geq k\}$ is a Gaussian process with independent increments and parameters

$$\begin{aligned} \mathbf{E}_k Z_n^k &= -\mathbf{E}_\infty Z_n^k = \frac{1}{2\sigma^2} \left(\theta_k^2 + \sum_{i=k+1}^n \tilde{\theta}_i^2 \right), \\ \mathbf{D}_k Z_n^k &= \mathbf{D}_\infty Z_n^k = \frac{1}{\sigma^2} \left(\theta_k^2 + \sum_{i=k+1}^n \tilde{\theta}_i^2 \right), \end{aligned}$$

where $\mathbf{D}$ stands for variance.

Write

$$Q_{k,n} = \frac{1}{\sigma^2 n} \sum_{i=k}^{k+n-1} \tilde{\theta}_i^2$$

and assume that

$$\lim_{n \rightarrow \infty} Q_{k,n} = Q \quad \text{for all } k \geq 1, \quad (6.1)$$

where Q characterizes the average «signal-to-noise ratio», $0 < Q < \infty$. It is easily verified that

$$\mathbf{P}_k \left\{ \left| Z_{k+n-1}^k - \frac{Qn}{2} \right| > \varepsilon n \right\} = 2\Phi \left\{ -\frac{(\varepsilon - \Delta_{k,n})\sqrt{n}}{\sqrt{Q_{k,n}}} \right\},$$

where $\Phi(x)$ is the standard normal distribution function and $\Delta_{k,n} = (Q_{k,n} - Q)/2$. By condition (6.1),

$$\sum_{n=1}^{\infty} n^{r-1} \mathbf{P}_k \left\{ \left| Z_{k+n-1}^k - \frac{Qn}{2} \right| > \varepsilon n \right\} < \infty \quad \text{for all } r > 0,$$

and hence, for all $r > 0$

$$n^{-1} Z_{k+n}^k \xrightarrow[n \rightarrow \infty]{\mathbf{P}_k\text{-}r\text{-quickly}} \frac{Q}{2} \quad \text{and} \quad n^{-1} Z_{k+n}^k \xrightarrow[n \rightarrow \infty]{\mathbf{P}_{\infty}\text{-}r\text{-quickly}} -\frac{Q}{2}.$$

Thus, condition (3.25) holds for all positive r .

Also, evidently,

$$\sum_{k=1}^{\infty} \pi_k \left(\sum_{n=1}^{\infty} n^{r-1} \mathbf{P}_k \left\{ \left| Z_{k+n-1}^k - \frac{Qn}{2} \right| > \varepsilon n \right\} \right) < \infty,$$

which implies condition (3.26).

Thus, under condition (6.1) with $0 < Q < \infty$, according to Theorem 3, the Shiryaev detection algorithm ν_B with $B = (1 - \alpha)/\alpha$ asymptotically minimizes all positive moments of the detection delay in the class $\Delta(\alpha)$, and the asymptotic formulas (3.44) and (3.45) hold with $q = Q/2$.

This result can easily be generalized for the problem of detecting a change in the mean of the p -th order Gaussian autoregressive process

$$\xi_n = \sum_{j=1}^p \delta_j \xi_{n-j} + w_n, \quad n \geq 1, \quad \xi_k = 0 \text{ for } k \leq 0,$$

where $w_n, n \geq 1$, are i.i.d. $\mathcal{N}(0, \sigma^2)$. Specifically, define $\tilde{\theta}_{p,n} = \theta_n - \sum_{j=1}^p \theta_{n-j}$ and assume that

$$\lim_{n \rightarrow \infty} \frac{1}{\sigma^2(n-p)} \sum_{k=p+1}^n \tilde{\theta}_{p,k}^2 = Q, \quad \text{where } 0 < Q < \infty.$$

Then Theorem 3 and Corollary 1 show that ν_B is asymptotically optimal, and asymptotic formulas (3.44)–(3.47) hold true with $q = Q/2$.

We now return to the Markov case and assume that the mean value is constant, $\theta_n = \theta \neq 0$. Then condition (6.1) is fulfilled with $Q = \theta^2(1 - \delta)^2/\sigma^2$.

In the latter case, the results of Subsection 4.2 can be applied. To show this, we first note that the LLR Z_n^1 can be written in the form

$$Z_n^1 = \sum_{k=2}^n \Delta W_k + \frac{\theta}{\sigma^2} X_1 - \frac{\theta^2}{2\sigma^2},$$

where

$$\Delta W_k = \frac{\theta(1 - \delta)}{\sigma^2} \tilde{X}_k - \frac{\theta^2(1 - \delta)^2}{2\sigma^2}, \quad k \geq 2,$$

are i.i.d. Gaussian random variables with parameters

$$\mathbf{E}_1 \Delta W_k = -\mathbf{E}_{\infty} \Delta W_k = \frac{Q}{2}, \quad \mathbf{D}_1 \Delta W_k = \mathbf{D}_{\infty} \Delta W_k = Q. \quad (6.2)$$

Therefore, by adding and subtracting the random variable ΔW_1 , which has the same distribution as $\Delta W_2, \Delta W_3, \dots$, one can represent Z_n^1 in the form

$$Z_n^1 = W_n + S$$

with $W_n = \Delta W_1 + \dots + \Delta W_n$ being a Gaussian random walk with the parameters given by (6.2), and $S = (\theta/\sigma^2) X_1 - \theta^2/(2\sigma^2) - \Delta W_1$ being a Gaussian random variable with $\mathbf{E}_1 S = -\mathbf{E}_\infty S = QA_\delta/2$, where $A_\delta = [1 - (1 - \delta)^2]/(1 - \delta)^2$.

In further calculations, including Monte Carlo experiments, the stopping time ν_B will be defined as $\nu_B = \inf\{n \geq 1: R_{\rho,n} \geq B\}$ with $R_{\rho,n} = \Lambda_n/\rho$.

A slight modification of the proof of Theorem 5 shows that the PFA obeys the asymptotic formula (4.18) and, as $B \rightarrow \infty$,

$$\mathbf{E}_1 \nu_B = \frac{1}{Q/2 + |\ln(1 - \rho)|} \left[\ln B - C(\rho, Q) + \bar{\kappa}(\rho, Q) - \frac{QA_\delta}{2} \right] + o(1). \quad (6.3)$$

As compared to the asymptotic expansion (4.19) for the i.i.d. case, here an additional term $-QA_\delta/2$ appears due to the random variable S in the decomposition of the LLR.

To guarantee the given PFA α in simulations, we used the following threshold value obtained by reverting to (4.18) in Theorem 5,

$$B = \frac{\zeta(\rho, Q)}{\alpha\rho}. \quad (6.4)$$

According to [42, Corollary 2.2.7], the constant $\zeta(\rho, Q)$ is computed from the formula

$$\zeta(\rho, Q) = \frac{2}{Q + 2|\ln(1 - \rho)|} \exp \left\{ - \sum_{k=1}^{\infty} \frac{1}{k} F_k(\rho, Q) \right\}, \quad (6.5)$$

where

$$\begin{aligned} F_k(\rho, Q) = & \Phi \left(- \frac{Q + 2|\ln(1 - \rho)|}{2\sqrt{Q}} \sqrt{k} \right) \\ & + (1 - \rho)^k \Phi \left(- \frac{Q - 2|\ln(1 - \rho)|}{2\sqrt{Q}} \sqrt{k} \right) \end{aligned} \quad (6.6)$$

and $\Phi(x) = \int_{-\infty}^x \varphi(t) dt$ is a standard normal distribution function.

To compute the CADD, we used the following higher order (HO) approximation:

$$\text{CADD}_1(\nu_B) \approx \max \left\{ 0, \frac{2}{Q_\rho} \left(\ln B - C(\rho, Q) + \bar{\kappa}(\rho, Q) - \frac{QA_\delta}{2} \right) - 1 \right\}, \quad (6.7)$$

where, according to [42, Corollary 2.2.7],

$$\bar{\kappa}(\rho, Q) = \frac{Q_\rho^2/4 + Q}{Q_\rho} - \sqrt{Q} \sum_{k=1}^{\infty} \left[k^{-1/2} \varphi \left(\frac{Q_\rho \sqrt{k}}{2\sqrt{Q}} \right) - \frac{Q_\rho}{2\sqrt{Q}} \Phi \left(- \frac{Q_\rho \sqrt{k}}{2\sqrt{Q}} \right) \right].$$

Here we used the notation $Q_\rho = Q + 2|\ln(1 - \rho)|$.

Formula (6.7) follows from the HO asymptotic (6.3). Note that this formula requires the computation of the constant $C(\rho, Q)$ using (4.15). As we observed in Remark 4, we usually have to resort to Monte Carlo methods to estimate $C(\rho, Q)$. Values of C for various choices of Q, ρ and δ are given

$\delta = 0$ (i.i.d.)			$\delta = 0.5$		
Q	ρ	C	Q	ρ	C
1.0	0.3	0.8366	1.0	0.3	0.5062
1.0	0.1	1.2396	1.0	0.1	0.7538
1.0	0.03	1.4036	1.0	0.03	0.8681
1.0	0.01	1.4647	1.0	0.01	0.9081
0.5	0.3	0.9949	0.5	0.3	0.7590
0.5	0.1	1.5859	0.5	0.1	1.2211
0.5	0.03	1.9001	0.5	0.03	1.5002
0.5	0.01	2.0371	0.5	0.01	1.5936
0.25	0.3	1.0913	0.25	0.3	0.9479
0.25	0.1	1.8694	0.25	0.1	1.6444
0.25	0.03	2.3827	0.25	0.03	2.1245
0.25	0.01	2.5992	0.25	0.01	2.3196
0.1	0.3	1.1630	0.1	0.3	1.0970
0.1	0.1	2.1290	0.1	0.1	2.0009
0.1	0.03	2.9164	0.1	0.03	2.7597
0.1	0.01	3.3528	0.1	0.01	3.1722

Table 1. Values of the constant C for different Q, ρ, δ

in Table 1. The number of trials were such that the estimate of the standard deviation of C was within 0.5% of the mean.

$\rho = 0.1, Q = 0.25$					
α	MCPFA	MCADD	MCCADD ₁	FOADD	HOCADD ₁
0.1000	0.0768	9.2315	12.3424	18.5338	11.9508
0.0600	0.0464	11.1187	14.4889	20.9400	14.4484
0.0300	0.0215	14.0684	17.6605	24.0854	17.5509
0.0100	0.0070	18.7026	22.4509	28.9431	22.3195
0.0060	0.0043	20.8690	24.6155	31.1781	24.6034
0.0030	0.0024	23.7940	27.6285	34.2001	27.5586
0.0010	0.0007	28.5247	32.3746	38.9779	32.3523

$\rho = 0.1, Q = 0.1$					
α	MCPFA	MCADD	MCCADD ₁	FOADD	HOCADD ₁
0.1000	0.0858	11.9405	16.4280	27.9637	15.8823
0.0600	0.0460	14.7162	19.7314	31.5316	19.4116
0.0300	0.0240	18.9581	24.2873	36.1953	24.0955
0.0100	0.0083	25.6559	31.3594	43.3981	31.2017
0.0060	0.0048	28.8515	34.6259	46.7120	34.5589
0.0030	0.0025	33.2216	39.0867	51.1929	39.0017
0.0010	0.0008	40.2447	46.1591	58.2772	46.1774

Table 2. Results for i.i.d. case with $B = (1 - \alpha)/(\rho\alpha)$

For the purpose of comparison, we also used the first order (FO) approximations for CADD (see (3.29))

$$\text{CADD}_1(\nu_B) \approx \max \left\{ 0, \frac{2 \ln B}{Q + 2|\ln(1 - \rho)|} - 1 \right\}. \quad (6.8)$$

$\rho = 0.1, Q = 1$					
α	MCPFA	MCADD	MCCADD ₁	FOADD	HOCADD ₁
0.1000	0.0914	3.9388	4.9192	5.6139	4.8214
0.0600	0.0554	4.6407	5.7084	6.4577	5.6622
0.0300	0.0293	5.7191	6.8523	7.6027	6.8195
0.0100	0.0100	7.4474	8.6344	9.4175	8.6221
0.0060	0.0059	8.2627	9.4719	10.2614	9.4494
0.0030	0.0030	9.3973	10.6116	11.4064	10.6225
0.0010	0.0010	11.1895	12.4177	13.2212	12.4328

$\rho = 0.01, Q = 1$					
α	MCPFA	MCADD	MCCADD ₁	FOADD	HOCADD ₁
0.1000	0.0907	8.5173	9.9681	11.4037	9.9424
0.0600	0.0547	9.5119	11.0090	12.4052	10.9459
0.0300	0.0290	10.7933	12.2914	13.7642	12.3139
0.0100	0.0100	12.9459	14.4763	15.9181	14.4788
0.0060	0.0058	13.9602	15.4800	16.9196	15.4538
0.0030	0.0030	15.2986	16.8320	18.2786	16.8507
0.0010	0.0010	17.4523	18.9875	20.4325	18.9818

$\rho = 0.1, Q = 0.25$					
α	MCPFA	MCADD	MCCADD ₁	FOADD	HOCADD ₁
0.1000	0.0915	8.4385	11.4574	17.6206	11.0010
0.0600	0.0558	10.2942	13.5159	19.8381	13.2243
0.0300	0.0284	12.9266	16.3797	22.8471	16.2042
0.0100	0.0096	17.4060	21.0897	27.6162	21.0265
0.0060	0.0060	19.4797	23.2396	29.8337	23.2110
0.0030	0.0029	22.4640	26.2587	32.8426	26.2497
0.0010	0.0010	27.1694	31.0175	37.6117	31.0254

$\rho = 0.1, Q = 0.1$					
α	MCPFA	MCADD	MCCADD ₁	FOADD	HOCADD ₁
0.1000	0.0914	11.3236	15.7955	27.2882	15.1322
0.0600	0.0549	13.8997	18.8115	30.5762	18.3927
0.0300	0.0298	17.8510	23.1455	35.0377	22.8742
0.0100	0.0097	24.3888	30.0665	42.1091	29.9915
0.0060	0.0060	27.5188	33.2905	45.3971	33.1729
0.0030	0.0030	31.9391	37.7784	49.8586	37.7489
0.0010	0.0010	38.9244	44.8407	56.9300	44.8114

Table 3. Results for i.i.d. case with $B = \zeta(\rho, Q)/(\rho\alpha)$

Extensive Monte Carlo simulations have been performed for different values of Q , ρ , δ , and α . The number of trials used for these results is given by $1000/\alpha$. Sample results are shown in Tables 2, 3 and 4. In these tables, we

present the Monte Carlo (MC) estimates of ADD along with the theoretical values computed according to (6.7) and (6.8). The abbreviations MCADD, MCCADD₁, FOADD, and HOCADD₁ are used for the ADD obtained by the MC experiment, CADD₁ obtained by the MC experiment, the FO approximation (6.8), and the HO approximation (6.7) for CADD₁, respectively. We also list MC estimates for the false alarm probability PFA.

Table 2 contains results of analysis in the i.i.d. case when the threshold $B = (1 - \alpha)/(\rho\alpha)$. This threshold value is based on the general upper bound that ignores the overshoot. It can be seen that the MC estimates for PFA in this case are substantially smaller than the design values α . This leads to an increase of the true values of the average detection delay, which is undesirable. It can also be seen that FO approximations are inaccurate even for relatively small α , while HO approximations are very accurate.

The results in Table 3 correspond to the i.i.d. case where the threshold B is set using (6.4). It is seen that the MC estimates for PFA match α very closely, especially for values smaller than 0.01. Thus, (6.4) provides an accurate method to design the threshold B to meet the PFA constraint α . It is also seen that, as expected, MCCADD₁ exceeds MCADD in all cases. The FOADD values are not good approximations even when PFA is small. On the other hand, the HO approximation for CADD₁ (given by HOCADD₁) is seen to be very accurate even for moderate values of PFA.

$\rho = 0.1, Q = 1$					
α	MCPFA	MCADD	MCCADD ₁	FOADD	HOCADD ₁
0.1000	0.0839	2.9519	3.3721	5.6139	3.1324
0.0600	0.0532	3.5588	4.0728	6.4577	3.9739
0.0300	0.0289	4.5526	5.1437	7.6027	5.1293
0.0100	0.0100	6.2505	6.9137	9.4175	6.9328
0.0060	0.0062	7.0655	7.7425	10.2614	7.7790
0.0030	0.0029	8.1902	8.8830	11.4064	8.9244
0.0010	0.0010	9.9847	10.6885	13.2212	10.7444

$\rho = 0.1, Q = 0.25$					
α	MCPFA	MCADD	MCCADD ₁	FOADD	HOCADD ₁
0.1000	0.0895	7.8914	10.6258	17.6206	10.3019
0.0600	0.0568	9.7496	12.7327	19.8381	12.6246
0.0300	0.0289	12.3524	15.5596	22.8471	15.6679
0.0100	0.0098	16.7599	20.2234	27.6162	20.3995
0.0060	0.0061	18.8899	22.4150	29.8337	22.5276
0.0030	0.0030	21.8352	25.3999	32.8426	25.6374
0.0010	0.0010	26.5485	30.1661	37.6117	30.4034

Table 4. Results for correlated case with $\delta = 0.5$

Results for the correlated case with $\delta = 0.5$ are presented in Table 4, with the threshold B being set using (6.4). Here again we see the accuracy of HO order approximations for PFA and ADD. Also, it is interesting to

see that for the same value of effective signal-to-noise ratio, Q , the ADD in the correlated case is slightly smaller than in the i.i.d. case. On the other hand, if we fix the value of «actual» signal-to-noise ratio θ^2/σ^2 , e.g., $Q = 1$ in the i.i.d. case and $Q = 0.25$ in the $\delta = 0.5$ case, then we can see that the correlation slows down the change detection.

7. More examples: decentralized quickest change detection

The results of the previous sections are particularly useful in analysis of the decentralized version of the change detection problem described in [41]. We first outline this interesting problem and related asymptotic optimality results for i.i.d. data models. Then we give two examples.

7.1. A decentralized detection problem. Assume that the information about the change is available through a set of L separate sensors. At time n an observation $X_{\ell,n}$ is made at sensor S_ℓ . Conditioned on the change point λ , the observation sequences $\{X_{1,n}\}, \dots, \{X_{L,n}\}$ are assumed to be mutually independent. Furthermore, throughout this section, we restrict our attention to the «i.i.d. case» where the observations in a particular sequence, say $\{X_{\ell,n}\}_{n \geq 1}$, are independent conditioned on λ , have a common PDF $f_\ell^{(0)}$ before the change, and a common PDF $f_\ell^{(1)}$ from the time of change. Note that we are assuming that all the sensors change distribution at the change time λ . As in Section 4, we will suppose that the prior distribution is geometric with the parameter ρ , $\rho > 0$.

Based on the information available at S_ℓ at time n , a message $U_{\ell,n}$, belonging to a finite alphabet of size V_ℓ , is formed and sent to the fusion center. We will use the vector notation: $\mathbf{X}_n = (X_{1,n}, \dots, X_{L,n})$ and $\mathbf{U}_n = (U_{1,n}, \dots, U_{L,n})$. Based on the sequence of sensor messages, a decision about the change is made at the fusion center. The fusion center picks a time τ , which is a *stopping time* on $\{\mathbf{U}_n\}_{n \geq 1}$, at which it is declared that a change has occurred.

Various information structures are possible for the decentralized configuration depending on how feedback and local information is used at the sensors [41]. Consider the simplest information structure, where the message $U_{\ell,n}$ formed by sensor S_ℓ at time n is a function of only its current observation $X_{\ell,n}$, i.e., $U_{\ell,n} = \psi_{\ell,n}(X_{\ell,n})$. Moreover, since for a particular ℓ , the sequence $\{X_{\ell,n}\}_{n \geq 1}$ is assumed to be i.i.d., it is natural to confine ourselves to *stationary* quantizers³⁾ for which the quantizing functions $\psi_{\ell,n}$ do not depend on n , i.e., $\psi_{\ell,n} = \psi_\ell$ for all $n \geq 1$.

The set of quantizing functions $\{\psi_\ell, \ell = 1, \dots, L\} = \Psi$, together with the fusion center stopping time τ , form a policy $\phi = (\tau, \Psi)$. The goal is

³⁾ We can prove the optimality of stationary quantizers under some mild conditions on the observations and the quantizers.

to choose the policy ϕ that minimizes the $\text{ADD}(\phi) = \mathbf{E}^\pi\{\tau - \lambda \mid \tau \geq \lambda\}$, or more generally the moments of the detection delay $\mathbf{ED}_m^\pi(\phi) = \mathbf{E}^\pi\{(\tau - \lambda)^m \mid \tau \geq \lambda\}$ for all $m \geq 1$, while maintaining the false alarm probability $\text{PFA}(\phi) = \mathbf{P}^\pi\{\tau < \lambda\}$ at a level not greater than α .

Let H_k be the hypothesis that the change occurs at time $\lambda = k \in \{1, 2, \dots\}$, and let H_∞ be the hypothesis that the change does not occur at all. Since the observations at each sensor S_ℓ , $\{X_{\ell,n}, n = 1, 2, \dots\}$, are i.i.d., for stationary sensor quantizers, the sensor outputs, $\{U_{\ell,n}, n = 1, 2, \dots\}$ will also be i.i.d. Let $g_\ell^{(j)}$ denote the PMF (probability mass function) induced on $U_{\ell,n}$ when the observation $X_{\ell,n}$ is distributed as $f_\ell^{(j)}$, $j = 0, 1$.

Then, for *fixed* stationary sensor quantizers, the LLRs between the hypotheses H_k and H_∞ at the sensor S_ℓ and at the fusion center are given by

$$Z_n^k(\ell) = \sum_{i=k}^n \ln \frac{g_\ell^{(1)}(U_{\ell,i})}{g_\ell^{(0)}(U_{\ell,i})} \quad \text{and} \quad Z_n^k = \sum_{\ell=1}^L Z_n^k(\ell).$$

For fixed sensor quantizers, the fusion center faces a standard change-point detection problem based on the vector observation sequence $\{\mathbf{U}_n\}$. Hence we can define the average likelihood ratio statistic Λ_n^{dc} and the corresponding statistic $R_{\rho,n}^{\text{dc}} = \Lambda_n^{\text{dc}}/\rho$ with $f_1(X_n)/f_0(X_n)$ now replaced by $\prod_{\ell=1}^L [g_\ell^{(1)}(U_{\ell,n})/g_\ell^{(0)}(U_{\ell,n})]$. The index «dc» will be used to denote parameters associated with the decentralized detection problem.

The decentralized Shiryaev detection procedure at the fusion center ν_B^{dc} is given by

$$\nu_B^{\text{dc}} = \inf\{n \geq 1: R_{\rho,n}^{\text{dc}} \geq B\}, \quad (7.1)$$

where B is a positive threshold which is selected so that $\text{PFA}(\nu_B) \leq \alpha$.

If $D(g_\ell^{(1)}, g_\ell^{(0)})$, the K-L distances between the $g_\ell^{(1)}$ and $g_\ell^{(0)}$ are positive and finite, then for fixed stationary sensor quantizers, an application of Theorem 4 gives us that the detection procedure ν_B^{dc} given in (7.1), with $B = B_\alpha = (1 - \alpha)/(\alpha\rho)$, is asymptotically optimal as $\alpha \rightarrow 0$ among all procedures with PFA no greater than α . To be specific, let $\Psi = \{\psi_1, \dots, \psi_L\}$ be a set of stationary quantizers. Then, as $\alpha \rightarrow 0$, for all $m \geq 1$

$$\inf_{\tau \in \Delta(\alpha)} \mathbf{ED}_m^\pi(\Psi, \tau) \sim \mathbf{ED}_m^\pi(\Psi, \nu_{B_\alpha}^{\text{dc}}) \sim \left(\frac{|\ln \alpha|}{\sum_{\ell=1}^L D(g_\ell^{(1)}, g_\ell^{(0)}) + |\ln(1 - \rho)|} \right)^m,$$

where $\mathbf{ED}_m^\pi(\Psi, \tau) = \mathbf{E}^\pi\{(\tau - \lambda)^m \mid \tau \geq \lambda\}$ is the m -th moment of the detection delay for the policy (Ψ, τ) .

This result immediately reveals how to optimize the sensor quantizers.

Corollary 3. *It is asymptotically optimum (as $\alpha \rightarrow 0$) for sensor S_ℓ to use the stationary quantizer that maximizes the K-L information distance at its output, i.e., $\psi_{\ell,\text{opt}} = \arg \max D(g_\ell^{(1)}, g_\ell^{(0)})$, $\ell = 1, \dots, L$.*

Based on the results of Tsitsiklis [40], it is easy to show that the optimum stationary quantizer $\psi_{\ell, \text{opt}}$ is a monotone likelihood ratio quantizer, i.e., there exist thresholds $h_{\ell, 1}, \dots, h_{\ell, V_\ell - 1}$ satisfying $0 = h_{\ell, 0} \leq h_{\ell, 1} \leq \dots \leq h_{\ell, V_\ell - 1} \leq \infty = h_{\ell, V_\ell}$ such that

$$\psi_{\ell, \text{opt}}(X) = i \quad \text{only if} \quad h_{\ell, i-1} < \frac{f_\ell^{(1)}(X)}{f_\ell^{(0)}(X)} \leq h_{\ell, i}, \quad i = 1, \dots, V_\ell.$$

Thus, the asymptotically optimal policy ϕ_{opt} for the decentralized change detection problem in the class of stationary (in time) quantizers consists of a set of monotone likelihood ratio quantizers at the sensors followed by Shiryaev's procedure based on $\{\mathbf{U}_n\}_{n \geq 1}$ at the fusion center (as described in (7.1)).

For each ℓ , let the PMFs induced on $U_{\ell, n}$ by the optimum monotone likelihood ratio quantizer $\psi_{\ell, \text{opt}}$ be given by $g_{\ell, \text{opt}}^{(1)}$ and $g_{\ell, \text{opt}}^{(0)}$. Then the effective K-L information distance between the «change» and «no change» hypotheses at the fusion center is given by

$$D_{\text{tot}} = \sum_{\ell=1}^L D(g_{\ell, \text{opt}}^{(1)}, g_{\ell, \text{opt}}^{(0)}).$$

Finally, denote by $\nu_{\text{opt}}^{\text{dc}}$ Shiryaev's stopping rule at the fusion center for the case where the sensor quantizers are chosen to be $\psi_{\ell, \text{opt}}$, and by $\Phi_{\text{st}}(\alpha)$ the class of policies ϕ with all stationary quantizers and stopping rules at the fusion center such that $\tau \in \Delta(\alpha)$.

The asymptotic performance of the asymptotically optimal solution to the decentralized change detection problem described above is given in the following theorem, which follows directly from Theorem 4 and the argument given above.

Theorem 7. *Suppose that*

$$0 < D(g_{\ell, \text{opt}}^{(1)}, g_{\ell, \text{opt}}^{(0)}) < \infty \quad \text{for } \ell = 1, \dots, L.$$

Then $B_\alpha = (1 - \alpha)/(\alpha\rho)$ implies that $\text{PFA}(\nu_{\text{opt}}^{\text{dc}}) \leq \alpha$ and for all $m \geq 1$

$$\inf_{\phi \in \Phi_{\text{st}}(\alpha)} \mathbf{ED}_m^\pi(\phi) \sim \mathbf{ED}_m^\pi(\phi_{\text{opt}}) \sim \left(\frac{|\ln \alpha|}{D_{\text{tot}}} + |\ln(1 - \rho)| \right)^m \quad \text{as } \alpha \rightarrow 0,$$

where $\phi_{\text{opt}} = (\nu_{\text{opt}}^{\text{dc}}, \{\psi_{\ell, \text{opt}}\})$.

Theorem 5 can also be applied to the problem in question. Specifically, if in addition to conditions of Theorem 7, we assume that the LLR at the fusion center $Z_1^1 = \sum_{\ell=1}^L \ln[g_{\ell, \text{opt}}^{(1)}(U_{\ell, 1})/g_{\ell, \text{opt}}^{(0)}(U_{\ell, 1})]$ is nonarithmetic, then the false alarm probability satisfies asymptotic formula (4.18) with B replaced by $B\rho$ and

$$\mathbf{E}_1 \nu_B = \frac{1}{D_{\text{tot}} + |\ln(1 - \rho)|} [\ln B - C(\rho, D_{\text{tot}}) + \overline{\kappa}(\rho, D_{\text{tot}})] + o(1) \quad (7.2)$$

with the corresponding modification of the definitions of $\overline{\pi}(\rho, D_{\text{tot}})$ and $C(\rho, D_{\text{tot}})$.

7.2. Example 2: Decentralized detection of a change in the mean of a normal population. Surveillance systems, such as those used in defense, deal with the detection and tracking of moving targets that appear and disappear at unknown points in time. As a result, the target detection problem can be naturally formulated as a multi-sensor abrupt change detection problem considered in Subsection 7.1. We now consider an example that is of interest in target detection theory. In the centralized setting, this example is a particular case of Example 1. Here we consider the decentralized problem discussed above.

Consider the problem of detecting a nonfluctuating target using L geographically separated sensors. The observations are corrupted by additive white Gaussian noise that is independent from sensor to sensor. The sensors preprocess the observations using a matched filter, matched to the signal corresponding to the target (see [24]). The output of the matched filter at sensor S_ℓ at time n (when the time of appearance of the target is λ) is given by

$$X_{\ell,n} = \begin{cases} \xi_{\ell,n} & \text{if } n < \lambda, \\ \mu_\ell + \xi_{\ell,n} & \text{if } n \geq \lambda, \end{cases}$$

where $\{\xi_{\ell,n}, n = 1, 2, \dots\}$ is a sequence of i.i.d. zero-mean Gaussian random variables with variance σ_ℓ^2 . Therefore, the likelihood ratio at sensor S_ℓ is given by

$$Y_\ell(x) = \frac{f_\ell^{(1)}(x)}{f_\ell^{(0)}(x)} = \exp \left\{ \frac{\mu_\ell(x - \mu_\ell/2)}{\sigma_\ell^2} \right\}. \quad (7.3)$$

Since Y_ℓ is monotonically increasing, we can characterize the optimum stationary sensor quantizers in terms of thresholds on the observations, rather than on their likelihood ratios. To further simplify the example, we assume that the sensor messages are binary, i.e., $V_\ell = 2$ for all ℓ . Then the quantizers reduce to binary tests that are characterized by a single threshold, i.e.,

$$U_{\ell,n} = \begin{cases} 1 & \text{if } X_{\ell,n} \geq h_\ell, \\ 0 & \text{otherwise.} \end{cases}$$

The distributions induced on $U_{\ell,n}$ by this quantizer are given by

$$g_\ell^{(j)}(0) = 1 - g_\ell^{(j)}(1) = \Phi \left(\frac{h_\ell - j\mu_\ell}{\sigma_\ell} \right) = q_\ell^{(j)}, \quad j = 0, 1, \quad (7.4)$$

where $\Phi(\cdot)$ is the distribution function of a standard Gaussian random variable. The optimum value of h_ℓ , i.e., the one that maximizes $D(g_\ell^{(1)}, g_\ell^{(0)})$, is easily found based on (7.4). Then we can compute the decision statistic $R_{\rho,n}$ at the fusion center, which obeys the recursion (assuming that $\pi_0 = 0$)

$$R_{\rho,n} = \frac{1}{1 - \rho} (1 + R_{\rho,n-1}) e^{Z_n^n}, \quad n \geq 1, \quad R_{\rho,0} = 0, \quad (7.5)$$

with

$$Z_n^n = \sum_{\ell=1}^L \left(U_{\ell,n} \ln \left[\frac{1 - q_{\ell}^{(1)}}{1 - q_{\ell}^{(0)}} \frac{q_{\ell}^{(0)}}{q_{\ell}^{(1)}} \right] - \ln \left[\frac{q_{\ell}^{(0)}}{q_{\ell}^{(1)}} \right] \right). \quad (7.6)$$

Based on (7.6), we may also compute higher order approximations for the PFA and ADD as given in Theorem 5 using the technique given in [42, Section 2.4].

The operating characteristics in an example with five sensors having identically distributed observations are illustrated in Figure 1. The parameter values are $\rho = 0.1$, $\mu_{\ell} = 0.4$ and $\sigma_{\ell}^2 = 1$. The K-L distance for the sensor observations is 0.08. The threshold that maximizes the K-L distance at the

Fig. 1. Operating characteristics for an example with five sensors with identically distributed Gaussian observations

output of the sensor is $h = 0.32$, and the corresponding maximum K-L distance is 0.0509. The fusion center threshold is set using $B = (1 - \alpha)/(\rho\alpha)$. Estimates for the PFA and ADD were obtained using MC methods with the number of trials being $1000/\alpha$. We plot ADD versus $-\ln(\text{PFA})$ for the optimum decentralized detection policy and compare the performance with that of a centralized policy that has direct access to the observations at the sensors. As we expect, for the centralized policy, the plot of ADD versus $-\ln(\text{PFA})$ is a straight line with a slope that is approximately equal to

$$\frac{1}{5D(f^{(1)}, f^{(0)}) + \ln(1 - \rho)} \approx 1.98.$$

For the optimum decentralized policy, the tradeoff curve has a slope that is

roughly equal to

$$\frac{1}{D_{\text{tot}} + \ln(1 - \rho)} \approx 2.78$$

as expected from Theorem 7. The decentralized policy of course suffers a performance degradation relative to the centralized policy. However, the bandwidth requirements for communication with the fusion center are considerably smaller in decentralized setting, especially with binary quantizers. Figure 1 also shows the tradeoff curve for a centralized detection policy with a *single* sensor. As expected, the slope of ADD versus $-\ln \text{PFA}$ is five times larger. Furthermore, it can be seen that even if the sensor observations are quantized to one bit, the decentralized policy with five sensors far outperforms the single sensor centralized policy.

7.3. Example 3: Decentralized detection of a change in a Poisson sequence. In distributed computer networks, large scale attacks in their final stages can readily be identified by observing very abrupt changes in the network traffic. However, in the early stage of an attack, these changes are hard to detect and difficult to distinguish from usual traffic patterns. In this subsection, we argue that the Shiryaev detection algorithm can be effectively deployed for an early detection of intrusions from the class of Denial-of-Service attacks. An efficient nonparametric approach to this problem has been recently proposed by Blažek, Kim, Rozovskii, and Tartakovsky [4]. Here we consider a parametric approach with a Poisson model for the observables.

Assume that sensor observations are Poisson random variables with different means before and after the disruption. For instance, in the network security applications, $X_{\ell,n}$ may correspond to the number of packets of a particular type (say, TCP-packets) at sensor S_ℓ in the n -th time interval of a certain length. Let the observations at sensor S_ℓ have mean $\mu_{0,\ell}$ before the disruption, and mean $\mu_{1,\ell}$ after the disruption. Without loss of generality assume that $\mu_{1,\ell} > \mu_{0,\ell}$. Then the likelihood ratio at S_ℓ is given by

$$Y_\ell(X_{\ell,n}) = \left(\frac{\mu_{1,\ell}}{\mu_{0,\ell}} \right)^{X_{\ell,n}} \exp \{ -(\mu_{1,\ell} - \mu_{0,\ell}) \}.$$

Note that the likelihood ratio is again monotonically increasing, and hence, we can characterize the optimum stationary sensor quantizers in terms of thresholds on the observations. For binary quantizers,

$$U_{\ell,n} = \begin{cases} 1 & \text{if } X_{\ell,n} \geq h_\ell, \\ 0 & \text{otherwise.} \end{cases}$$

The distributions induced on $U_{\ell,n}$ by this quantizer are given by

$$g_\ell^{(j)}(0) = 1 - g_\ell^{(j)}(1) = \sum_{k=0}^{\lfloor h_\ell \rfloor} \frac{\mu_{j,\ell}^k e^{-\mu_{j,\ell}}}{k!} = q_\ell^{(j)}, \quad j = 0, 1. \quad (7.7)$$

Here again, the optimum value of h_ℓ , i.e., the one that maximizes $D(g_\ell^{(1)}, g_\ell^{(0)})$, is easily found based on (7.7). The decision statistic at the fusion center is then given by (7.5) and (7.6).

Fig. 2. Operating characteristics for an example with three sensors with identically distributed Poisson observations

The operating characteristics in an example with three sensors having identically distributed observations are illustrated in Figure 2. The parameter values are $\rho = 0.1$, $\mu_{0,\ell} = 10$, and $\mu_{1,\ell} = 12$. The K-L distance for the sensor observations is 0.1879. The threshold that maximizes the K-L distance at the output of the sensor is $h = 11$, and the corresponding maximum K-L distance is 0.119. The fusion center threshold is set using $B = (1 - \alpha)/(\rho\alpha)$, and estimates for the PFA and ADD were obtained using MC methods. As in the previous example, we see that the plots of ADD versus $-\ln \text{PFA}$ in the three cases considered have the behavior predicted by the theory.

8. Concluding remarks

We end by giving the following concluding remarks.

1. Most of the asymptotic optimality results remain true for stochastic processes observed in continuous time. However, continuous-time problems have certain special features that should be handled carefully. A general asymptotic detection theory for continuous-time models will be presented elsewhere.

2. The general asymptotic theory that has been developed in this paper covers only simple hypotheses and can be considered as the first step. For most practical applications it is important to consider composite hypotheses,

especially in the post-change mode. Mixture-type and adaptive versions of the Shiryaev Bayesian rule are excellent candidates for composite-hypothesis problems. Adaptive Bayesian modifications seem to be especially attractive for on-line implementations.

3. For the decentralized detection problem discussed in Section 7, it is of interest to extend the asymptotic analysis to non-i.i.d. observations at the sensors, and to possible correlation across sensors (conditioned on the change point). The extension to non-i.i.d. observations is straightforward, whereas the extension to include correlation across sensors appears to be nontrivial.

4. The results of Section 7 show that fusion of data in decentralized multi-sensor systems with quantizers always leads to a certain loss of information which results in the performance degradation of the optimal decentralized policy. Specifically, for the geometric prior distribution the asymptotic relative efficiency of the optimal centralized detection procedure with respect to decentralized is equal to

$$\lim_{\alpha \rightarrow 0} \frac{\inf_{\tau \in \Delta(\alpha)} \text{ADD}_c(\tau)}{\inf_{\tau \in \Delta(\alpha)} \text{ADD}_{dc}(\tau)} = \frac{|\ln(1 - \rho)| + \sum_{\ell=1}^L D(g_\ell^{(1)}, g_\ell^{(0)})}{|\ln(1 - \rho)| + \sum_{\ell=1}^L D(f_\ell^{(1)}, f_\ell^{(0)})} < 1.$$

Interestingly, it is possible to construct decentralized detection procedures with no quantization that are asymptotically equivalent to the optimal centralized procedure (i.e., globally asymptotically optimal) and at the same time have bandwidth requirements for communications between sensors and the fusion center similar to decentralized policies with binary quantization. However, these procedures require significant processing capabilities at the sensors so that they can run individual change detection tests. Such procedures will be discussed in a separate paper.

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