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d-DIMENSIONAL PRESSURELESS GAS EQUATIONS¹⁾

Пусть $x \in \mathbf{R}^d \rightarrow u(x, 0)$ — непрерывная ограниченная функция и $\rho(dx, 0)$ вероятностная мера на $\mathbf{R}^d$. Мы показываем, что для любой случайной величины X_0 с распределением вероятностей $\rho(dx, 0)$ стохастическое дифференциальное уравнение

$$X_t = X_0 + \int_0^t \mathbf{E}[u(X_0, 0) | X_s] ds, \quad t \geq 0,$$

имеет решение, которое является $\sigma(X_0)$ -измеримым марковским процессом. Мы выводим, для $d \geq 1$, слабое решение уравнения для газа без давления с начальным распределением масс $\rho(dx, 0)$ и начальной скоростью $u(\cdot, 0)$. Для $d = 1$ мы доказываем существование единственного марковского процесса (X_t) , являющегося решением нашего стохастического дифференциального уравнения.

Ключевые слова и фразы: уравнения для газа без давления, вариационные принципы.

1. The main result. The model of one-dimensional pressureless gases, with sticky particles, can be described at a discrete level by a finite collection of particles that stick together after they collide, with conservation of mass and momentum. At a continuous level, the gas can be described by a density field $\rho(dx, t)$ and a velocity field $u(x, t)$ that satisfy the mass and momentum conservation laws

$$(\mathcal{S}(1)) \begin{cases} \partial_t(\rho(dx, t)) + \partial_x(u\rho(dx, t)) = 0, \\ \partial_t(u\rho(dx, t)) + \partial_x(u^2\rho(dx, t)) = 0, \\ \rho(dx, t) \rightarrow \rho(dx, 0), \quad u(x, t)\rho(dx, t) \rightarrow u(x, 0)\rho(dx, 0), \quad \text{weakly, as } t \rightarrow 0+. \end{cases}$$

This set of equations is known as the system of pressureless gases and can be viewed as the limit, when pressure goes to zero, of the usual Euler equations.

Global existence theorems in dimension one for solutions of the system $\mathcal{S}(1)$ have been obtained by W. E, Yu. G. Rykov, and Ya. G. Sinai [3]. Later A. Dermoune and B. Djehiche [1], [2] pointed out a connection between equation $\mathcal{S}(1)$ and the solution of the following nonlinear stochastic differential equation (SDE):

$$dX_t = \mathbf{E}[U(X_0) | X_t] dt, \quad \mathcal{L}(X_0) = \rho(dx, 0), \quad (1)$$

where $(U(x), x \in \mathbf{R})$ is any real continuous process such that $\mathbf{E}[U(X_0) | X_0] = u(X_0, 0)$.

The aim of this work is to construct a solution of (1) in d -dimensional case, namely, in the case, where $\rho(dx, 0)$ is a probability measure on $\mathbf{R}^d$. As a consequence we get a global weak solution of d -dimensional pressureless gas equations. We end this section by announcing our main results.

Theorem 1.1. *Let X_0 be a random variable with values on $\mathbf{R}^d$, and let $x \in \mathbf{R}^d \rightarrow u(x, 0)$ be a bounded continuous function. There exists at least one Markov process $(X_t, t \geq 0)$ which is a solution of the following SDE:*

$$X_t = X_0 + \int_0^t \mathbf{E}[u(X_0, 0) | X_s] ds, \quad t \geq 0. \quad (2)$$

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Corollary 1.1. *Let (X_t) be a solution of (2). For all $t \geq 0$, let $\rho(dx, t)$ be the probability distribution of X_t and $u(x, t) = (u_i(x, t): 1 \leq i \leq d) := \mathbf{E}[u(X_0, 0) | X_t = x]$. Then $(\rho(dx, t), u(x, t) \rho(dx, t))_{t \geq 0}$ is a weak solution of the d -dimensional pressureless gas equations*

$$(\mathcal{S}(d)) \left\{ \begin{array}{l} \partial_t(\rho(dx, t)) + \sum_{i=1}^d \partial_{x_i}(u_i \rho(dx, t)) = 0, \\ \partial_t(u \rho(dx, t)) + \sum_{i=1}^d \partial_{x_i}(u_i u \rho(dx, t)) = 0, \\ \rho(dx, t) \rightarrow \rho(dx, 0), \quad u(x, t) \rho(dx, t) \rightarrow u(x, 0) \rho(dx, 0), \quad \text{weakly, as } t \rightarrow 0+. \end{array} \right.$$

Theorem 1.2. *For $d = 1$, there exists only one Markov process which is $\sigma(X_0)$ -measurable solution of the SDE*

$$X_t = X_0 + \int_0^t \mathbf{E}[u(X_0, 0) | X_s] ds, \quad \forall t \geq 0.$$

The rest of this work is consecrated to the proof of Theorems 1.1 and 1.2. The proof of Corollary 1.1 is easy and works as follows. First we recall the definition of weak solution which was given in [3]. Let $(\rho(dx, t), I_t)_{t \geq 0}$ be a family of Borel measures, weakly continuous with respect to t , such that I_t is absolutely continuous with respect to $\rho(dx, t)$ for each fixed t . This family is a weak solution of $\mathcal{S}(d)$ if, for any $f \in C_0^1(\mathbf{R}^d)$, the space of C^1 -functions on $\mathbf{R}^d$ with compact support, and $0 < t_1 < t_2$,

$$(D1) \quad \int f(x) \rho(dx, t_2) - \int f(x) \rho(dx, t_1) = \int_{t_1}^{t_2} \int \sum_{i=1}^d \partial_{x_i} f(x) u_i(x, t) \rho(dx, t) dt,$$

$$(D2) \quad \begin{aligned} & \int f(x) u(x, t_2) \rho(dx, t_2) - \int f(x) u(x, t_1) \rho(dx, t_1) \\ &= \int_{t_1}^{t_2} \int \sum_{i=1}^d \partial_{x_i} f(x) u_i(x, t) u(x, t) \rho(dx, t) dt. \end{aligned}$$

Let $(X_t, t \geq 0)$ be a solution of (2), $0 < t_1 < t_2$, and f a smooth function. From the formula of change of the variables we have

$$f(X_{t_2}) - f(X_{t_1}) = \int_{t_1}^{t_2} \sum_{i=1}^d \partial_{x_i} f(X_s) dX_s^i,$$

where $X_s = (X_s^i: 1 \leq i \leq d)$. Using the fact that $dX_s^i = \mathbf{E}[u_i(X_0, 0) | X_s] ds$, we get

$$\mathbf{E}[f(X_{t_2}) - f(X_{t_1})] = \int_{t_1}^{t_2} \sum_{i=1}^d \mathbf{E}[\partial_{x_i} f(X_s) \mathbf{E}[u_i(X_0, 0) | X_s]] ds,$$

namely,

$$\int f(x) \rho(dx, t_2) - \int f(x) \rho(dx, t_1) = \int_{t_1}^{t_2} \sum_{i=1}^d \int \partial_{x_i} f(x) u_i(x, s) \rho(dx, s) ds.$$

Then we get (D1). Now, we use the equation

$$u(X_0, 0) f(X_{t_2}) - u(X_0, 0) f(X_{t_1}) = \int_{t_1}^{t_2} \sum_{i=1}^d u(X_0, 0) \partial_{x_i} f(X_s) dX_s^i$$

and the property of the conditional expectation in order to get (D2).

2. Preliminary results.

Proposition 2.1. *Let (X_t) be a $\sigma(X_0)$ -measurable Markov process which is a solution of (2). Then the following assertions are equivalent:*

- 1) $\forall s < t, \sigma(X_t) \subset \sigma(X_s)$;
- 2) *the process (X_t) has the Markov property.*

P r o o f. 1) $\implies$ 2). If $\sigma(X_t) \subset \sigma(X_s)$ for all $s < t$, then

$$\mathbf{E}[f(X_t) | \sigma(X_r, r \leq s)] = f(X_t) = \mathbf{E}[f(X_t) | X_s],$$

for all continuous bounded function. So X has the Markov property.

- 2) $\implies$ 1). X is Markov $\sigma(X_0)$ -measurable, then for all $s < t$,

$$\mathbf{E}[X_t | X_s] = \mathbf{E}[X_t | \sigma(X_r, r \leq s)] = X_t.$$

It follows that X_t is $\sigma(X_s)$ -measurable.

We remark that if X is a Markov process, $\sigma(X_0)$ -measurable, and solution of (2), then

$$X_t = \mathbf{E}[X_0 + tu(X_0, 0) | X_t] \quad \forall t \geq 0. \quad (3)$$

From Proposition 2.1 we have the following proposition.

Proposition 2.2. *Let X be a $\sigma(X_0)$ -measurable Markov process which is a solution of (2). There exist measurable maps $\varphi(s, t, \cdot)$, $s < t$, such that $X_t = \varphi(s, t, X_s)$, and $\varphi(s, t, \cdot)$ depends only on the law of (X_t, X_s) .*

Proposition 2.3. *Let X be a Markov process which is a solution of (2) and Y another Markov process, such that $\text{law}(X_t, Y_s) = \text{law}(Y_t, Y_s)$ for all s, t . Then Y also satisfies*

$$Y_t = Y_0 + \int_0^t \mathbf{E}[u(Y_0, 0) | Y_s] ds \quad \forall t \geq 0.$$

P r o o f. We have $\mathbf{E}[f(X_t) | X_s] = f(\varphi(s, t, X_s))$, as Y is a Markov process such that $\text{law}(X_t, X_s) = \text{law}(Y_t, Y_s)$, then

$$\mathbf{E}[f(Y_t) | Y_s] = f(\varphi(s, t, Y_s)) \quad \forall s < t.$$

In particular, $\mathbf{E}[X_t | X_s] = \varphi(s, t, X_s) = X_t$ for any $s < t$, which implies that

$$\mathbf{E}[Y_t | Y_s] = \varphi(s, t, Y_s) = Y_t,$$

then $\sigma(Y_t) \subset \sigma(Y_s)$ and Y is solution of $Y_t = Y_0 + \int_0^t \mathbf{E}[u(Y_0, 0) | Y_s] ds$.

3. Proof of Theorem 1.1. First suppose that $\rho(dx, 0) = \sum_{i=1}^N m_i \delta(x - x_i)$. Let us consider sticky particles with initial locations $(x_i, 1 \leq i \leq N)$, masses $(m_i, 1 \leq i \leq N)$ and velocities $(u(x_i, 0) = v_i, i \leq N)$. For each i we denote by $(\varphi(t, x_i), t \geq 0)$ the trajectory of the particle x_i . We show easily that the process $(X_t(x) = \varphi(t, x), t \geq 0)$ defined on the probability space $(\mathbf{R}^d, \mathcal{B}, \rho(dx, 0))$, is solution of the following SDE:

$$X_t = X_0 + \int_0^t \mathbf{E}[u(X_0, 0) | X_s] ds.$$

The process X , from its construction, has the Markov property (see [1, Section 3] for more details).

Now let $\rho(dx, 0)$ be any probability measure on $\mathbf{R}^d$. Consider a sequence $(\rho_n(dx, 0))$ of discrete probability measures such that $\rho_n(dx, 0) \rightarrow \rho(dx, 0)$ weakly. For each n we have a Markov process which is solution of

$$X_t^{(n)} = X_0^{(n)} + \int_0^t \mathbf{E}[u(X_0^{(n)}, 0) | X_s^{(n)}] ds,$$

where $X_0^{(n)}$ has the probability distribution $\rho_n(dx, 0)$.

Since $u(\cdot, 0)$ is bounded, the law of $(X_t^{(n)}, t \in [0, T])$ is tight in $C([0, T], \mathbf{R}^d)$ for all $T > 0$. Then there exists a subsequence noted also by n such that $X^{(n)}$ converges weakly in $C([0, T], \mathbf{R}^d)$ to some stochastic process X . From the Skorokhod representation theorem there exist a sequence $(Y^{(n)})$ and a process Y such that $\sup_{t \in [0, T]} |Y_t^{(n)} - Y_t|$

converges a.s. to 0 as $n \rightarrow +\infty$. The sequence $(Y^{(n)}, Y)$ has the same law as $(X^{(n)}, X)$ in $C([0, T], \mathbf{R}^{d^2})$. By Proposition 2.3 $Y^{(n)}$ is also a Markov process and a solution of

$$Y_t^{(n)} = Y_0^{(n)} + \int_0^t \mathbf{E}[u(Y_0^{(n)}, 0) | Y_s^{(n)}] ds$$

and

$$\sigma(Y_t^{(n)}) \subset \sigma(Y_s^{(n)}) \quad \forall s < t. \quad (4)$$

The process Y has the representation $Y_t = Y_0 + \int_0^t \varphi(s) ds$, where φ is some bounded process. We want to prove that

$$Y_t = Y_0 + \int_0^t \mathbf{E}[u(Y_0, 0) | Y_s] ds.$$

The proof is the same as in [1, Lemma 4.1]. It works as follows. We have for all smooth functions f, g ,

$$\begin{aligned} \mathbf{E} \left[\int g(t) f(Y_t^{(n)}) u(Y_0^{(n)}, 0) dt \right] &= \mathbf{E} \left[\int g(t) f(Y_t^{(n)}) \frac{dY_t^{(n)}}{dt} dt \right] \\ &\longrightarrow \mathbf{E} \left[\int g(t) f(Y_t) u(Y_0, 0) dt \right] = \mathbf{E} \left[\int g(t) f(Y_t) \varphi(t) dt \right]. \end{aligned}$$

We derive that

$$\mathbf{E}[\varphi(t) | Y_t] = \mathbf{E}[u(Y_0, 0) | Y_t] \quad \mathbf{P} \otimes dt\text{-a.e.} \quad (5)$$

Now, from (4) and the convergence of $Y_t^{(n)}$ to Y_t for all $t \geq 0$, we have $\sigma(Y_t) = \sigma(Y_s)$ for all $s < t$. From that and the equality $\varphi(t) = \lim_{\varepsilon \rightarrow 0+} [(Y(t+\varepsilon) - Y(t))/\varepsilon] dt\text{-a.e.}$ we conclude that $\varphi(t)$ is $\sigma(Y_t)$ -measurable $dt\text{-a.e.}$ Finally from (5) we have $\varphi(t) = \mathbf{E}[u(Y_0, 0) | Y_t] dt\text{-a.e.}$

4. Proof of Theorem 1.3. Now we consider the case $d = 1$, and we will show the uniqueness of Markov process

$$X_t = \varphi(t, X_0)$$

which is solution of

$$X_t = X_0 + \int_0^t \mathbf{E}[u(X_0, 0) | X_s] ds.$$

Lemma 4.1. *For all $t \geq 0$, the map $x \rightarrow \varphi(t, x)$ is nondecreasing. If $\varphi(s, x) = \varphi(s, y)$, then $\varphi(t, x) = \varphi(t, y)$ for all $t > s$.*

P r o o f. Let $x < y$ be two points of $\text{supp}(\rho_0)$. Suppose that $\varphi(t, x) > \varphi(t, y)$. As $s \rightarrow \varphi(s, x) - \varphi(s, y)$ is continuous, then there exists $\tau < t$ such that $\varphi(\tau, x) = \varphi(\tau, y)$. On the other hand, $\mathbf{E}[\varphi(t, X_0) | \varphi(\tau, X_0)] = \varphi(t, X_0)$, and then

$$\mathbf{E}[\varphi(t, X_0) | \varphi(\tau, X_0) = \varphi(\tau, x)] = \varphi(t, x) = \varphi(t, y),$$

which is absurd.

From (3) we have

$$\varphi(t, x) = \frac{\int_{D(t,x)} (\eta + tu(\eta, 0)) \rho(d\eta, 0)}{\rho(D(t, x), 0)}, \quad (6)$$

where $D(t, x) = \varphi^{-1}(t, \varphi(t, x))$. We are going to show that φ is unique.

Lemma 4.2. *For any subset G of $\mathbf{R}$ and $t \geq 0$, we denote*

$$C(G, t) = \mathbf{E}[X_0 + tu(X_0, 0) | X_0 \in G].$$

Let $A(x, t), B(x, t)$ be two intervals, a partition of $D(t, x) := \varphi^{-1}(t, \varphi(t, x))$, such that $\sup A(x, t) \leq \inf B(x, t)$. Then

- 1) $C(A(x, t), t) \geq C(B(x, t), t)$, and
- 2) $\beta + tu(\beta, 0) \leq \varphi(t, x) \leq \alpha + tu(\alpha, 0)$, where $\alpha = \inf A(x, t)$, $\beta = \sup B(x, t)$.

P r o o f. The proof is the same as in [3, Lemma 3]. It works as follows. Let us consider the evolution of the set of particles $A(x, t)$ on the interval $[0, t]$. There exists $s \leq t$ such that the point $a := \sup A(x, t)$ is hit from the right by $b := \inf B(x, t)$, namely $\varphi(s, a) = \varphi(s, b)$. It is easy to show that for all $r \geq s$

$$C(D(s, b), r) \leq C(D(s, a) \cup D(s, b), r) \leq C(D(s, a), r).$$

In particular at $s = t$, we have $D(t, b) = B(t, x)$, $D(t, a) = A(t, x)$ and

$$C(B(x, t), t) \leq \varphi(t, x) \leq C(A(x, t), t).$$

From the continuity of $u(\cdot, 0)$, we derive the assertion 2) by taking successively $A(x, t) = (\alpha, \alpha + \varepsilon)$ and $B(x, t) = (\beta - \varepsilon, \beta)$.

Lemma 4.3. *If $x \in S_+ := \{y \in \text{supp}(\rho(dx, 0)) : \rho([y, y + \varepsilon], 0) > 0 \text{ for any } \varepsilon > 0\}$, then $\varphi(t, x + 0) = \varphi(t, x)$ and if $x \in S_- := \{y \in \text{supp}(\rho(dx, 0)) : \rho((y - \varepsilon, y], 0) > 0 \text{ for any } \varepsilon > 0\}$, then $\varphi(t, x - 0) = \varphi(t, x)$.*

P r o o f. First we remark that $\varphi(t, x)$ is only well defined for $x \in \text{supp}(\rho(dx, 0))$. If $x, y \in \text{supp}(\rho(dx, 0))$ are such that $(x, y) \cap \text{supp}(\rho(dx, 0)) = \emptyset$, then we define, for all $z \in (x, y)$, $\varphi(t, z)$ as linear interpolation of $\varphi(t, x)$ and $\varphi(t, y)$.

Let $\alpha = \inf\{y \in \text{supp}(\rho(dx, 0)) : \varphi(t, x) = \varphi(t, y)\}$ and $\beta = \sup\{y \in \text{supp}(\rho(dx, 0)) : \varphi(t, x) = \varphi(t, y)\}$. We have only to study the case where $x = \alpha$ or $x = \beta$.

Suppose that there exists $\varepsilon > 0$ such that $[\alpha, \alpha + \varepsilon) \cap S_+ = \{\alpha\}$, then by construction of $\varphi(t, \cdot)$, we have $\varphi(t, \alpha + 0) = \varphi(t, \alpha)$.

By the same argument, we have $\varphi(t, \beta - 0) = \varphi(t, \beta)$, if $(\beta - \varepsilon, \beta] \cap S_- = \{\beta\}$.

Suppose that $\rho_0(\alpha, \alpha + \varepsilon) > 0$ and $\varphi(t, \alpha + 0) > \varphi(t, \alpha)$. Then α is a right endpoint of the set $C(t, \alpha)$. From Lemma 4.2 we have $\alpha + tu(\alpha, 0) \leq \varphi(t, \alpha) < \varphi(t, \alpha + 0) \leq \alpha + tu(\alpha, 0)$ which is absurd. The same argument works for β .

Corollary 4.1. *For all $x \in \text{supp}(\rho(dx, 0))$ there exist α, β , $\alpha \leq \beta$, such that $\varphi^{-1}(t, \varphi(t, x)) = [\alpha, \beta]$ and*

$$\varphi(t, x) = \frac{\int_{\alpha}^{\beta} (\eta + tu(\eta, 0)) \rho(d\eta, 0)}{\rho([\alpha, \beta], 0)}.$$

Moreover, α, β are endpoints of the partition ξ_t given by the generalized variational principle (GVP) introduced in [3]. It follows that φ is the forward flow defined by E, Rykov, and Sinai in [3].

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