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ADAPTIVE ESTIMATION OF DISTRIBUTION DENSITY IN
THE BASIS OF ALGEBRAIC POLYNOMIALS¹⁾

Статья посвящена задаче адаптивного статистического оценивания плотности распределения, определенной на конечном интервале. Рассматриваются оценки проективного типа, основанные на многочленах Якоби. Построена адаптивная статистическая оценка, являющаяся асимптотически минимаксной в смысле среднеквадратических потерь для всех множеств из семейства сжимающихся множеств функций различной гладкости. Условия гладкости формулируются в терминах L_2 -норм остатков при аппроксимации плотностей распределений линейными комбинациями конечного числа первых многочленов Якоби. Обсуждается также обобщение этого результата на другие ортонормированные базисы, обладающие некоторыми естественными свойствами регулярности.

Ключевые слова и фразы: адаптивное оценивание, локально минимаксное оценивание, многочлены Якоби, оценки проективного типа, среднеквадратические потери.

1. Introduction. In the statistical estimation of an unknown parameter (finite- or infinite-dimensional) one seeks to use optimal estimators. For this a mathematical definition of optimality is needed. In the Bayes setting this problem is solved, however, practical realization of this solution runs into another problem: to select a prior distribution of the parameter. In the recent decades the notion of minimax estimators has found increasing favour. From the viewpoint of applications, the main criticism is that their construction is guided by the worst (in the sense of identification accuracy) case of possible models. This results in a strong dependence of the estimating quality on *a priori* information. Significantly more flexible are adaptive estimators which have the same construction as minimax estimators, but they use *more exact* prior information specified on the basis of the sample. These estimators are far less sensitive to *a priori* information yet retaining the asymptotic minimax optimality.

Let us introduce necessary definitions. Let $\mathbf{X} = \{X_1, \dots, X_N\}$ be a sample of independent and identically distributed (i.i.d.) random variables X_i

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taking values in a compact convex set $K \subset \mathbf{R}^d$, and having a distribution function F . Let $f = dF/d\mu$ be a distribution density (d.d.) with respect to a finite measure μ . For simplicity, we assume that $\mu(K) = 1$.

Let W be some set of d.d.'s and $\varrho(\psi, f)$ a loss function (pseudodistance). For a given statistical estimator $\hat{f}(\cdot) = \hat{f}(\cdot, \mathbf{X})$, denote $\delta(\hat{f}, f) = \mathbf{E}\varrho(\hat{f}, f)$, $\delta(\hat{f}, W) = \sup_{f \in W} \mathbf{E}\delta(\hat{f}, f)$. Then the minimax risk is defined by the equality

$$\delta(W) = \inf_{\hat{f}} \delta(\hat{f}, W),$$

where infimum is taken over all the estimators $\hat{f}$.

Definition 1. An estimator $f^*(\cdot) = f^*(\cdot, \mathbf{X})$ is called an asymptotically minimax one (with respect to the pair (ϱ, W)) if and only if $\delta(f^*, W) \sim \delta(W)$.

Here and in the sequel the symbol $\sim$ designates the asymptotic equivalence of two sequences, i.e., $y_N \sim z_N$ means $\lim_{N \rightarrow \infty} y_N/z_N = 1$. Here and everywhere below, unless otherwise stated, the asymptotics as $N \rightarrow \infty$ is considered.

Let $\{W_\alpha = W_\alpha(N), \alpha \in A\}$ be a given, dependent on N , family of neighborhoods of the distribution density.

Definition 2. An estimator f^* is called an adaptive estimator with respect to the family of neighborhoods $\{W_\alpha, \alpha \in A\}$ if

$$\forall \alpha \in A \quad \delta(f^*, W_\alpha) \sim \delta(W_\alpha).$$

Usually the parameter α characterizes the «smoothness» of the set W_α .

There is a wide bibliography devoted to the minimax and, especially, adaptive estimation. Let us give some relevant references. In the pioneering paper [31] M. S. Pinsker constructed the minimax estimator of a signal in noise for the squared L_2 distance $\varrho(f, g) = \|f - g\|_2^2$ and W being an ellipsoid in L_2 . This result was extended to other nonparametric models: distribution and spectral densities [13], [14] and regression function [21], [12] (see also [37]). A. P. Korostelev [26] have found the minimax estimator for the Hölder classes with L_∞ risk (see also [5]). In the case where W is a certain class of analytical functions, the minimax estimators for the L_p risk were proposed in [20] ($p = \infty$) and [23] ($1 \leq p < \infty$). To the authors' knowledge, these three special cases exhaust the list of minimax estimating problems solved. So-called rate minimax estimators that have optimal rate of convergence of the minimax risk have been investigated far more extensively. See, for instance, [24], [4], [36], [2], [3], [9], [25], [10], [8], and references cited therein. The respective optimal rate adaptive estimators (we use the terminology of [38]) have been proposed in [28], [29], [30], [22], [6], [7] among others. Note that, for the pointwise risk on the scale of Hölder or L_2 -Sobolev

classes even optimal rate adaptive estimators do not exist: a logarithmic factor is a necessary payment for adaptation [38]. The nonasymptotic theory of adaptive estimation and a review of recent results on adaptivity are presented in [1]. The adaptive minimax estimators are obtained only in the case of estimating in L_2 [15], [21], [18] (see also [34], [11], [16]). As it is shown in [30], adaptive estimators can not achieve the asymptotic minimax risk for the sup-norm losses on the scale of Hölder classes (Korestev's case).

The presented paper is closely related to [33]. Although [33] deals with a more general weighted mean-square risk, for simplicity, we discuss the main result only in a special case of usual L_2 -losses $\varrho(\psi, f) = \|\psi - f\|_2^2$. Then it would be natural to define the neighborhoods W_α , $\alpha \in A$, as contracting balls in L_2 . However, since the estimating accuracy heavily depends upon the smoothness of the unknown d.d., the definition of neighborhoods under consideration should also include certain smoothness conditions.

Let $u = \{u_k\}_0^\infty$ be an orthonormal basis in the space $L_2 = L_2(K, \mu)$ and $u_0(\cdot) \equiv 1$. Denote

$$W_b = \left\{ f: \|f - (f)_{k-1}\|_2^2 \leq b_k, \quad k = 1, 2, \dots; \right. \\ \left. 0 \leq f(x) \leq C_1, \quad \int_K f d\mu = 1 \right\}, \quad (1)$$

$$W_{g,b} = W_{g,b}(N) = \{f: f \in W_b, \|f - g\|_2^2 \leq \varepsilon_N\}.$$

Here $b = \{b_k\}_1^\infty$ is a given nonnegative sequence, $(f)_k$ signifies the orthogonal projection of the d.d. f onto the linear subspace $\text{span}\{u_0, \dots, u_k\}$, and $\varepsilon_N \rightarrow 0$ as $N \rightarrow \infty$. In paper [33], under quite general conditions for the orthobasis u and the set $\mathcal{B}$ of sequences b , it is established that for each $b \in \mathcal{B}$ there exists an estimator $f^*(\cdot) = f^*(\cdot, b, \mathbf{X})$ possessing the property of local asymptotic minimaxity:

$$\forall g \in W_b^\circ \quad \delta(f^*, W_{g,b}) \sim \delta(W_{g,b}), \quad (2)$$

where W_b° represents the «interior» of W_b ,

$$W_b^\circ = \left\{ g: u_0 + \lambda(g - u_0) \in W_b \text{ for some } \lambda > 1, \|g - (g)_k\|_\infty \rightarrow 0 \text{ as } k \rightarrow \infty \right\},$$

$\|\cdot\|_\infty$ is the supremum norm. Thus, the estimator f^* achieves the asymptotic efficiency in both parametric (the sequence b has only a finite number of nonzero terms) and nonparametric (b is strictly positive) cases. (Let us stress, however, that f^* is not an adaptive estimator since it requires *a priori* knowledge of the sequence b which determines the smoothness of the d.d. f .) It seems likely to be the first estimator having this property. The estimator involves the statistical analysis of the Fisher information matrix and this feature distinguishes it from the traditionally used linear projective estimators (see [14], [11], [16], [18], [19], and others).

Proceeding from [33], conceptually it is not difficult to build up an adaptive estimator $f^*(\cdot) = f^*(\cdot, \mathbf{X})$ possessing the same asymptotic efficiency without prior knowledge of the sequence b . However, the proof of this fact is rather complicate and cumbersome. Therefore we present here a simpler adaptive estimating procedure that fulfils (2) for strictly positive sequences b but is not efficient in the parametric case. The procedure ignores subtleties of the structure of the Fisher information matrix since for the basis of Jacobi polynomials in the nonparametric case it can be replaced by a certain diagonal matrix without violating (2).

The extension of the results to more general orthonormal bases and to both the nonparametric and parametric cases will be presented in the next paper of the authors.

The rest of the paper is organized as follows. We conclude this section with some notation. The main results are formulated in the next section. The last section contains the proofs.

General notation. By C , with or without indices, we denote absolute positive constants not necessarily the same in different places; for instance, C_f stands for any constant dependent on f only. Further, $\mathbf{N} = \{1, 2, \dots\}$, $a \vee b = \max(a, b)$, $a \wedge b = \min(a, b)$, $x^+ = x \vee 0$, $\underline{x} = 1 - x$, $\text{Int}(x)$ is the integer part of x . The indicator (characteristic function) of an event A is denoted by $\mathbf{I}_A$. The expression $\alpha_N \asymp \beta_N$ indicates that the sequences α_N and β_N are of the same order, i.e., $\limsup_{N \rightarrow \infty} (|\alpha_N/\beta_N| + |\beta_N/\alpha_N|) < \infty$. For any finite set T , $|T|$ is the number of elements of the set T .

2. Main results. We will write $b \in \mathcal{B}_\nu$, $\nu \geq 0$, ($\mathcal{B} \stackrel{\text{def}}{=} \mathcal{B}_0$) if the following conditions are fulfilled:

- 1) $b_k > 0$ for all $k \in \mathbf{N}$ (W_b is actually nonparametric);
- 2) $b_k \downarrow 0$ as $k \uparrow \infty$ and

$$\lim_{n \rightarrow \infty} \sup_{k \in \mathbf{N}} \frac{b_{kn}}{b_k} = 0; \quad (3)$$

- 3) the parameter ν governs the decreasing rate:

$$b_k = o(k^{-2\nu} \ln^{-1} k), \quad k \rightarrow \infty. \quad (4)$$

It is not difficult to check that (3) implies (4) for some $\nu = \nu(b) > 0$. Indeed, in view of (3) $\sup_{k \in \mathbf{N}} b_{ks}/b_k \leq \frac{1}{2}$ for some $s \in \mathbf{N}$. This inequality and monotonicity of b_k imply $b_k \leq 2b_1 k^{-h}$ with $h = \ln_s 2 > 0$.

For $b \in \mathcal{B}$, set

$$M = M(b, N) = \max\{k: b_k \geq kN^{-1}\}. \quad (5)$$

Then (see, e.g., [32], [33])

$$\delta(W_b) \asymp MN^{-1}.$$

As already pointed out, in the paper our attention is focused on the case, where $u = \{u_k\}_0^\infty$ is the orthonormal basis of Jacobi polynomials. This means that $K = [-1, 1]$, $u_k(\cdot)$ is a k th order polynomial, $k = 0, 1, \dots$, and

$$d\mu(x) = 2^{-(\beta+\gamma+1)} B^{-1}(\beta+1, \gamma+1) (1-x)^\beta (1+x)^\gamma dx, \quad x \in [-1, 1],$$

where $B(\cdot, \cdot)$ is the Beta function, $\gamma > -1$, $\beta > -1$. Let us recall that special cases of Jacobi polynomials are Chebyshev ($\gamma = \beta = -\frac{1}{2}$) and Legendre ($\gamma = \beta = 0$) polynomials.

Let $c_k = c_k(f)$, $k \in \mathbf{N}$, denote the Fourier coefficients of a function $f \in W_b$ and let $\hat{c}_k = N^{-1} \sum_{j=1}^N u_k(X_j)$, $k \in \mathbf{N}$, be the empirical Fourier coefficients. Set

$$a_g = \frac{1}{\pi} \int_{-1}^1 (1-x^2)^{-1/2} g(x) dx. \quad (6)$$

From [32] it is easy to derive the following statement.

Theorem 1. Suppose that $q \stackrel{\text{def}}{=} \max(\gamma, \beta) + \frac{1}{2} \geq 0$ and $b \in \mathcal{B}_q$. Let $\varepsilon_N = \varepsilon_N(b) \rightarrow 0$ and let $n = n(N, b)$ be chosen so that $M/N = o(\varepsilon_N)$, $b_n = o(M/N)$. Then the estimator

$$f^* = u_0 + \sum_{j=1}^n \frac{s_j^* \hat{c}_j}{s_j^* + a_g/N} u_j \quad (7)$$

is asymptotically minimax for all $W_{g,b}$, $g \in W_b^\circ$, and

$$\delta(W_{g,b}) \sim \delta(f^*, W_{g,b}) \sim \Delta(g, b) \stackrel{\text{def}}{=} \frac{a_g}{N} \sum_{j=1}^\infty \frac{s_j^*}{s_j^* + a_g/N}.$$

Here $\{s_j^*\}$ is a sequence maximizing all the sums $\sum_{j \geq k} s_j$, $k \in \mathbf{N}$, over the set of all nonincreasing nonnegative sequences $\{s_j\}_1^\infty$ satisfying $\sum_{j \geq k} s_j \leq b_k$ for all k .

The optimal estimator f^* given in (7) is obtained in the following way. Let the set Ω consist of all nonnegative nonincreasing sequences $\omega = \{\omega_k\}_1^\infty$ with $\omega_1 \leq 1$. Consider projective-type estimators

$$\hat{f} = u_0 + \sum_{j=1}^n \omega_j \hat{c}_j u_j, \quad (8)$$

where $\omega \in \Omega$ are determinate (nonrandom) weight coefficients. Then the mean-square risk $\mathbf{E} \|\hat{f} - f\|^2$ of $\hat{f}$ for appropriately chosen (sufficiently large) n is well approximated by the quantity

$$\Psi_{a,n}(\omega, c) = \sum_{k=1}^n \left[\omega_k^2 \frac{a}{N} + (1 - \omega_k)^2 c_k^2 \right] \quad (9)$$

with $a = a_f$. Since $a_f \sim a_g$ for $f \in W_{g,b}$, the optimal estimator f^* among estimators (8) can be defined by the equality

$$f^* = u_0 + \sum_{j=1}^n \omega_j^* \hat{c}_j u_j, \quad (10)$$

where

$$\omega^* = \arg \min_{\omega \in \Omega} \max_{f \in W_{g,b}} \Psi_{a_g,n}(\omega, c). \quad (11)$$

It is not difficult to verify that $\omega_j^* = s_j^*/(s_j^* + a_g/N)$, $j \in \mathbf{N}$, and

$$\Delta(g, b) \sim \Psi_{a_g,n}(\omega^*, s^*) \asymp MN^{-1}.$$

The following theorem is the main result of the paper.

Theorem 2. *There exists an estimator f^* (independent of g and b) such that*

$$\delta(f^*, W_{g,b}) \leq \Delta(g, b)(1 + o(1))$$

for all $b \in \mathcal{B}$ and $g \in W_b^\circ$.

Corollary 1. *Let the sequence $\varepsilon_N = \varepsilon_N(b)$ satisfy the relation $M/N = o(\varepsilon_N)$. Then there exists an adaptive estimator f^* for the class of neighborhoods $\{W_{g,b}, b \in \mathcal{B}_q, g \in W_b^\circ\}$, where $q \geq 0$ is defined in Theorem 1.*

Remark 1. If $b \in \mathcal{B}_\nu$ where $\nu > q + \frac{1}{2}$, then $\|g - (g)_k\|_\infty \rightarrow 0$ as $k \rightarrow \infty$ for each $g \in W_b$ (see, e.g., [35]). Thus, this condition can be omitted in the definition of W_b° provided $\nu > q + \frac{1}{2}$.

Construction of the adaptive estimator. The construction of the adaptive estimator f^* is based on formulas (9)–(11) with the following substitutions.

Let Π_i , $i = 0, 1, 2, 3$, be subsets of $\Pi = \{1, \dots, N\}$ and $\Pi = \bigcup_{i=0}^3 \Pi_i$. They define the four subsamples $\mathbf{X}(\Pi_i) = \{X_t, t \in \Pi_i\} \subset \mathbf{X}$ of the size $N_i = |\Pi_i|$.

1) A statistical estimator $\hat{a}$ based on the subsample $\mathbf{X}(\Pi_0)$ is substituted for a_g .

2) Instead of the determinate number n a certain statistics n^* of the subsample $\mathbf{X}(\Pi_1)$ is used.

3) The optimal weights ω^* are replaced by adaptive ones (we retained for them the same notation) calculated on the basis of the subsample $\mathbf{X}(\Pi_2)$. The adaptive weights are obtained via minimization of the estimators of $\Psi_{a_g,n}(\omega, c)$ given by

$$\hat{\Psi}_{\hat{a},n^*,D}(\omega) = \sum_{k=1}^{n^*} \left[\omega_k^2 \frac{\hat{a}}{N} + (1 - \omega_k)^2 \hat{c}_k^2(D) \right], \quad (12)$$

where $D \subset \Pi_2$ and

$$\widehat{c}_k^2(D) = \frac{1}{|D|(|D| - 1)} \sum_{t, \tau \in D, t \neq \tau} u_k(X_t) u_k(X_\tau) \quad (13)$$

is the unbiased estimator of c_k^2 , $k \in \mathbf{N}$, based on the observations $\mathbf{X}(D) = \{X_i, i \in D\}$.

4) The empirical Fourier coefficients $\widehat{c}_k$ are replaced by statistical estimators c_k^* , asymptotically equivalent to the former (in the sense of mean-square error). Actually, $c^* = \{c_k^*\}_1^\infty$ are the empirical Fourier coefficients based on the subsample $\mathbf{X}(\Pi_3)$. The asymptotic equivalence is ensured by the condition $N_3 \sim N$.

It is natural to take $\Pi_i \equiv \Pi$. However, in this case, the proof of adaptivity of f^* is very complicated in view of statistical dependence of the statistics $\widehat{a}, n^*, \omega^*, c^*$. Therefore, we take the disjoint sets Π_i , so $N_0 + \dots + N_3 = N$.

Now let us give explicit description of the estimators involved. Fix $\alpha \in (\frac{1}{2}, 1)$ and set $N_0 = \text{Int}(N^\alpha)$.

In view of (6)

$$a_f = \int_{-1}^1 \psi(x) dF(x),$$

where

$$\psi(x) = \frac{1}{\pi \sqrt{1 - x^2}} \left(\frac{d\mu}{dx}(x) \right)^{-1}.$$

Define

$$\widehat{a} = \frac{1}{N_0} \sum_{t \in \Pi_0} \psi_N(X_t) \quad (14)$$

with $\psi_N(x) = \psi(x) \wedge \sqrt{N}$. For estimation of n^* , we take a sequence of index sets D_i such that $D_s \subset \dots \subset D_2 \subset D_1 \subset \Pi \setminus \Pi_0$, $|D_k| = \text{Int}(N/(4k))$, and $s = \text{Int}(\ln_2 N)$. For a fixed $\beta \in (\frac{1}{2}, 1)$, define

$$k^* = 1 \vee \max \left\{ k: k \in \{1, \dots, s\}, \sum_{i=2^{k-1}}^{2^k} \widehat{c}_i^2(D_k) > \frac{2^{k\beta}}{N} \right\} \quad (15)$$

and $n^* = 2^{k^*}$, $\Pi_1 = D_{k^*}$.

Set $N_2 = 2N_1$ and split the set Π_2 into two nonintersecting subsets of equal cardinalities, $\Pi_2 = D \cup \widetilde{D}$, and set $\omega_{(\cdot)}^* = \widehat{\omega}_{(\cdot)} \vee \widetilde{\omega}_{(\cdot)}$, where $\widehat{\omega} = \omega(\widehat{\Psi}_{\widehat{a}, n^*, D})$, $\widetilde{\omega} = \omega(\widehat{\Psi}_{\widehat{a}, n^*, \widetilde{D}})$, and

$$\omega(\Psi_{(\cdot)}) = \arg \min_{\omega \in \Omega} \Psi_{(\cdot)}(\omega). \quad (16)$$

Finally, we take

$$c_j^* = N_3^{-1} \sum_{t \in \Pi_3} u_j(X_t), \quad j \in \mathbf{N}.$$

Generalization. Now we will describe the extensions of the results to more general orthonormal bases in $L_2(K, \mu)$ possessing some additional regularity properties.

(U0) The functions u_k , $k = 0, 1, 2, \dots$, are differentiable with the bounded gradients ∇u_k .

(U1) There exist a function $\psi \in L_p$, $p > 1$, and $\tau > 0$ such that, for each $g \in W_b$, $b \in \mathcal{B}$,

$$a_g \stackrel{\text{def}}{=} \lim_{k \rightarrow \infty} \int_K u_k^2(x) g(x) \mu(dx) = \int_K \psi(x) g(x) \mu(dx) \geq \tau \quad (17)$$

and the convergence in (17) is uniform with respect to $g \in W_b$.

(U2) There exists a sequence $\{m_k\} \subset \mathbf{N}$, $m_k = o(\sqrt{k})$, such that for each $l \in \mathbf{N}$,

$$\sum_{i=0}^k \sum_{j>i+m_k} \left(\int_K u_l(x) u_i(x) u_j(x) \mu(dx) \right)^2 = o(k), \quad k \rightarrow \infty.$$

We write $b \in \mathcal{B}_+$ if $b \in \mathcal{B}$ and

$$\frac{1}{N} \left\| \sum_{k=1}^M u_k^2(\cdot) \right\|_{\infty} \ln \left(2 + \frac{1}{N} \left\| \sum_{j=1}^M |\nabla u_j|^2 \right\|_{\infty} \right) \rightarrow 0, \quad N \rightarrow \infty, \quad (18)$$

where $M = M(b, N)$ is introduced in (5).

Theorem 3. *The statements of Theorems 1, 2 and Corollary 1 are valid for any orthonormal basis u satisfying conditions (U0)–(U2) provided the set $\mathcal{B}_q$ of sequences b in Theorem 1 and Corollary 1 is replaced by $\mathcal{B}_+$.*

3. Proof.

3.1. Auxiliary results. Let $v_j(m)$ denote the conditional variance of c_j^* under the condition $N_3 = m$.

Lemma 1. *Let $f \in W_b$, $b \in \mathcal{B}$, and let a_f be defined by (6). Then $v_j(m) \leq \|f\|_{\infty}/N_3$ and*

$$\lim_{j \rightarrow \infty} \lim_{m \rightarrow \infty} m v_j(m) = a_f. \quad (19)$$

The convergence in (19) is uniform with respect to $f \in W_b$.

P r o o f. The proof follows directly from the equality

$$m v_j(m) = \int_K u_j^2(x) dF(x) - c_j^2$$

and the fact that, for Jacobi polynomials, the limit functional a_g in (17) takes the form (6) (see [32, Lemma 6, p. 32]).

Further we need additional notation. $\lambda(x)$ denotes any function of the type $\lambda(x) = C \ln^2(x^+ + 2)/(x^+ + 2)$. A centered random variable ξ is denoted by $\widehat{\xi}$, i.e., $\widehat{\xi} = \xi - \mathbf{E}\xi$.

For any $T \subset \mathbf{R}^1$ and any sequence $Y = \{Y_t\}_1^\infty$, let $S_T(Y) = \sum_{t \in T \cap \mathbf{N}} Y_t$ and, assuming that T is bounded, $\bar{Y}_T = S_T(Y)/|T \cap \mathbf{N}|$. Put

$$\Gamma_T = S_T(c^2), \quad \gamma_T = \Gamma_T |T|^{-1}. \quad (20)$$

Thus $\gamma_{[a,b]}$ stands for γ_T with $T = [a, b]$. For $n, k \in \mathbf{N}$, define

$$\gamma_k = \gamma_k(n) = \begin{cases} \gamma_{[p,q]}, & \text{if } k \leq n, \\ 0, & \text{if } k > n; \end{cases} \quad (21)$$

here the integers $p = p(k, n)$ and $q = q(k, n)$ are defined by the relations

$$1 \leq p \leq k \leq q \leq n: \gamma_{[p,q]} = \min_{1 \leq i \leq k} \gamma_{[i,q]} = \max_{n \geq j \geq k} \gamma_{[p,j]}. \quad (22)$$

Let $D \subset \Pi$. The quantities $\widehat{\Gamma}_T(D)$, $\widehat{\gamma}_T(D)$, $\widehat{\gamma}_k(n, D)$, $\widehat{p}(k, n)$ and $\widehat{q}(k, n)$ are defined by the same equations (20)–(22) but with $\{c_k^2\}$ replaced by $\{\widehat{c}_k^2(D)\}$. Also set $\widehat{\gamma}_k = \widehat{\gamma}_k(n, D) \stackrel{\text{def}}{=} \gamma_k - \widehat{\gamma}_k(n, D)$, $k \in \mathbf{N}$.

A bit different definition of $\Psi_{a,n}(\omega)$ is more convenient in the proof. Redefine (9) as follows

$$\Psi_{a,n}(\omega) = \sum_{k=1}^n \left[\omega_k^2 \frac{a}{N} + (1 - \omega_k)^2 c_k^2 \right] + \sum_{k>n} c_k^2. \quad (23)$$

It easy to verify, using this expression, (12) and (16), that, for any $a > 0$ and $n \in \mathbf{N}$,

$$\omega_k(\Psi_{a,n}) = \frac{\gamma_k(n)}{\gamma_k(n) + aN^{-1}} \quad \text{and} \quad \omega_k(\widehat{\Psi}_{\widehat{a}, \widehat{n}, D}) = \frac{\widehat{\gamma}_k^+(\widehat{n}, D)}{\widehat{\gamma}_k^+(\widehat{n}, D) + \widehat{a}N^{-1}}.$$

By the definition of $\{\widehat{c}_k^2\}$ in (13), we have $\widehat{c}_k^2(D) = c_k^2 + 2\xi_k(D) + \zeta_k(D)$ where

$$\xi_k(D) = \frac{c_k}{|D|} \sum_{t \in D} \widehat{u}_k(X_t), \quad (24)$$

$$\zeta_k(D) = \frac{1}{|D|(|D| - 1)} \sum_{t, \tau \in D, t \neq \tau} \widehat{u}_k(X_t) \widehat{u}_k(X_\tau), \quad (25)$$

and $\widehat{u}_k(X_t) = u_k(X_t) - c_k$.

Lemma 2. For any $T \subset \mathbf{N}$ and $D \subset \Pi$,

$$\mathbf{D}S_T(\xi(D)) \leq \frac{C\|f\|_\infty \Gamma_T}{|D|}, \quad \mathbf{D}S_T(\zeta(D)) \leq \frac{C\|f\|_\infty^2 |T|}{|D|^2}.$$

Proof. Put $X = X_1$. Then by definition (25)

$$\begin{aligned} \mathbf{D}S_T(\zeta) &= \frac{2}{|D|(|D| - 1)} \sum_{i,j \in T} [\mathbf{E} \widehat{u}_i(X) \widehat{u}_j(X)]^2 \\ &\leq \frac{2}{|D|(|D| - 1)} \sum_{i,j \in T} \left(\int u_i u_j dF \right)^2 \\ &\leq \frac{C|T|}{|D|^2} \max_{j \in T} \{\|u_j f\|_2^2\} \leq \frac{C|T| \|f\|_\infty^2}{|D|^2}. \end{aligned}$$

Further, by (24)

$$|D| \mathbf{D}S_T(\xi) = \sum_{i,j \in T} c_i c_j \mathbf{E} \widehat{u}_i(X) \widehat{u}_j(X) \leq \int \left(\sum_{i \in T} c_i u_i \right)^2 dF \leq C \|f\|_\infty \cdot \Gamma_T.$$

Proposition 1. Suppose that for a nonnegative sequence $\beta = \{\beta_k\}_1^\infty$ and a sequence $Y = \{Y_k\}_1^\infty$ of random variables the inequalities

$$\mathbf{E} S_T^2(Y) \leq S_T(\beta), \quad T \subset \mathbf{N}, \quad (26)$$

are fulfilled. Then

$$\forall l \in \mathbf{N} \quad \mathbf{E} \max_{j=1, \dots, l} S_{[1,j]}^2(Y) \leq C \ln^2(1+l) S_{[1,l]}(\beta).$$

P r o o f. The statement is obvious for $l = 2$. Now, suppose that it holds for some $l \geq 2$. Then

$$\mathbf{E} \max_{j=1, \dots, l} S_{(k, k+j]}^2(Y) \leq C_0 \ln^2 l \cdot S_{(k, k+l]}(\beta) \quad \forall k = 0, 1, 2, \dots \quad (27)$$

We show that (27) implies

$$\mathbf{E} \max_{j=1, \dots, r} S_{[1,j]}^2(Y) \leq C_0 \ln^2 r \cdot S_{[1,r]}(\beta)$$

for $r = 2l$ provided $C_0 \geq 1/\ln^2 2$. In fact, from the Cauchy-Schwartz inequality and the elementary inequality $2xy \leq x^2 + y^2$, it follows that

$$\begin{aligned} \mathbf{E} \max_{j \leq r} S_{[1,j]}^2(Y) &\leq \mathbf{E} \max_{j \leq l} S_{[1,j]}^2(Y) + \mathbf{E} \max_{l < j \leq r} (S_{[1,l]}(Y) + S_{(l,j]}(Y))^2 \\ &\leq (C_0 \ln^2 l + 1) S_{[1,l]}(\beta) + 2 \ln l \sqrt{C_0 S_{[1,l]}(\beta) S_{(l,r]}(\beta)} \\ &\quad + C_0 \ln^2 l \cdot S_{(l,r]}(\beta) \leq (C_0 \ln^2 l + \sqrt{C_0} \ln l + 1) S_{[1,r]}(\beta) \\ &\leq C_0 \ln^2 r \cdot S_{[1,r]}(\beta). \end{aligned}$$

Proposition 2. If (26) holds, then

$$\mathbf{E} \sup_{i \in J, i \geq k+l} \bar{Y}_{[k,i]}^2 \leq \lambda(l) \cdot \sup_{i \in J, i \geq k} \bar{\beta}_{[k,i]}$$

for any $J \subset \mathbf{N}$, $k \in \mathbf{N}$, and $l \geq 0$.

P r o o f. Let $l \geq 1$, $J_r = [k+2^{r-1}l, k+2^r l) \cap J$, and $H = \max_{i \in J, i \geq k} \bar{\beta}_{[k,i]}$. In view of Proposition 1 we obtain

$$\begin{aligned} \mathbf{E} \max_{j \in J, j \geq k+l} \bar{Y}_{[k,j]}^2 &\leq \mathbf{E} \sum_{r=1}^{\infty} \frac{4^{1-r}}{l^2} \max_{j \in J_r} S_{[k,j]}^2(Y) \\ &\leq C \sum_{r=1}^{\infty} \frac{4^{-r}}{l^2} 2^r l \ln^2(2^r l) H \leq \frac{CH \ln^2 l}{l}. \end{aligned}$$

Corollary 2. For any $k < r$, $k, r \in \mathbf{N}$, and $D \subset \Pi$,

$$\mathbf{E} \max_{k \leq j \leq r} S_{[k,j]}^2(\xi(D)) \vee S_{[j,r]}^2(\xi(D)) \leq C_f \ln^2(1+r-k) \Gamma_{[k,r]} |D|^{-1}$$

and

$$\mathbf{E} \max_{k \leq j \leq r} S_{[k,j]}^2(\zeta(D)) \vee S_{[j,r]}^2(\zeta(D)) \leq C_f \ln^2(1+r-k) \cdot (r-k) |D|^{-2}.$$

Corollary 3. Let $\widehat{\gamma}_{(\cdot)} = \gamma_{(\cdot)} - \widehat{\gamma}_{(\cdot)}(D)$, and $T \subset \mathbf{N}$, $r \in \mathbf{N}$, $l \geq 0$. Then

$$\mathbf{E} \max_{j \in T} \widehat{\gamma}_{[j,r]}^2 \leq C_f \lambda(l) \left(\frac{\max_{j \in T} \gamma_{[j,r]}}{|D|} + \frac{1}{|D|^2} \right), \quad \text{if } T \subset [1, r-l],$$

and

$$\mathbf{E} \sup_{j \in T} \widehat{\gamma}_{[r,j]}^2 \leq C_f \lambda(l) \left(\frac{\max_{j \in T} \gamma_{[r,j]}}{|D|} + \frac{1}{|D|^2} \right), \quad \text{if } T \subset [r+l, \infty).$$

Corollary 2 follows from Lemma 2 and Proposition 1. Corollary 3 is a consequence of Lemma 2 and Proposition 2. Given $a > 0$, $n \in \mathbf{N}$, and $D \subset \Pi$, set $N_1 = |D|$, $\widehat{\omega} = \omega(\widehat{\Psi}_{a,n,D})$.

Lemma 3. Let $\omega = \omega(\Psi_{a,n})$. For each $s, r \in \mathbf{N}$, $s \leq r \leq n$, such that $s = p(s, n)$, $r = q(r, n)$, and $\Gamma_{[s,r]} \leq C r \gamma_r$, the following inequality holds:

$$\mathbf{E} \sum_{k=s}^r (\omega_r - \widehat{\omega}_k)_+^2 c_k^2 \leq C_f \gamma_s \ln^4(1+r) \cdot \frac{\gamma_r/N_1 + N_1^{-2}}{(\gamma_r + a/N)^2}. \quad (28)$$

P r o o f. Let $\widehat{\gamma}_{(\cdot)} = \widehat{\gamma}_{(\cdot)}(D)$. For a fixed $k \in \mathbf{N}$, set $\eta = (\gamma_r - \min_{i \leq k} \widehat{\gamma}_{[i,r]})_+$. For some $p, q \in \mathbf{N}$ we have $p \leq k \leq q$ and $\widehat{\gamma}_k = \widehat{\gamma}_{[p,q]} = \max_{n \geq j \geq k} \widehat{\gamma}_{[p,j]} \geq \widehat{\gamma}_{[p,r]} \geq \min_{i \leq k} \widehat{\gamma}_{[i,r]}$, and hence $(\omega_r - \widehat{\omega}_k)_+ \leq (\gamma_r - \widehat{\gamma}_k)_+ / (\gamma_r + a/N) \leq \eta / (\gamma_r + a/N)$. Consequently,

$$\mathbf{E} (\omega_r - \widehat{\omega}_k)_+^2 \leq \frac{\mathbf{E} \eta^2}{(\gamma_r + aN^{-1})^2}. \quad (29)$$

Denote

$$I_j = \{i: 1 \leq i \leq k, 2^j \leq \gamma_{[i,r]} \gamma_r^{-1} < 2^{j+1}\}, \quad (30)$$

$$l_j = 1 + r - \max\{i: i \in I_j\}. \quad (31)$$

By the definitions of I_j , l_j , and $\Gamma_{[k,r]}$ (see (20)),

$$l_j \geq \frac{\Gamma_{[k,r]}}{\gamma_r 2^{j+1}}. \quad (32)$$

Since $\min_{i \leq r} \gamma_{[i,r]} \leq \gamma_r$, we have $\eta = \max_{j \geq 0} \eta_j$, where $\eta_j = (\gamma_r - \min_{i \in I_j} \widehat{\gamma}_{[i,r]})_+$. Thus,

$$\mathbf{E} \eta^2 \leq \sum_{j=0}^{\infty} \mathbf{E} \eta_j^2. \quad (33)$$

In view of Corollary 3 and the definitions of I_j and $\widehat{\gamma}$, we can write

$$\mathbf{E}\eta_0^2 \leq \mathbf{E} \max_{i \in I_0} \widehat{\gamma}_{[i,r]}^2 \leq C_f \lambda(l_0) (\gamma_r N_1^{-1} + N_1^{-2}) \quad (34)$$

and, applying in addition the Chebyshev inequality,

$$\begin{aligned} \mathbf{E}\eta_j^2 &\leq \gamma_r^2 \mathbf{P}\{\eta_j > 0\} \leq \gamma_r^2 \mathbf{P}\left\{-\min_{i \in I_j} \widehat{\gamma}_{[i,r]} > (2^j - 1) \gamma_r\right\} \\ &\leq C_f \lambda(l_j) 4^{-j} (2^j \gamma_r N_1^{-1} + N_1^{-2}). \end{aligned} \quad (35)$$

Since $\Gamma_{[k,r]} \leq Cr\gamma_r$, the last inequality and formulas (29)–(34) yield

$$\mathbf{E}(\omega_r - \widehat{\omega}_k)_+^2 \leq C_f \ln^3(1+r) \cdot \frac{\gamma_r^2 N_1^{-1} + \gamma_r N_1^{-2}}{\Gamma_{[k,r]} (\gamma_r + aN^{-1})^2}. \quad (36)$$

Further,

$$\sum_{k=s}^r \frac{c_k^2}{\Gamma_{[k,r]}} \leq C \ln(1+r) \frac{\gamma_s}{\gamma_r}. \quad (37)$$

It is not difficult to verify this inequality taking into account the condition $s = p(s, r)$, $r = q(r, n)$. Inequalities (36) and (37) imply (28).

Corollary 4. *For each $k \in \mathbf{N}$, we have $\mathbf{E} \widehat{\gamma}_k^2 \leq C_f (\gamma_k/|D| + |D|^{-2})$.*

P r o o f. By making use of the definitions of $\gamma(\cdot)$, $\widehat{\gamma}(\cdot)$, p , and q (see (21), and (22)), we conclude that

$$\Delta_1 \stackrel{\text{def}}{=} \min_{i \leq q} \widehat{\gamma}_{[i,q]} \leq \widehat{\gamma}_k \leq \max_{j \geq p} \widehat{\gamma}_{[p,j]} \stackrel{\text{def}}{=} \Delta_2.$$

The required inequality for $\mathbf{E}\Delta_1^2$ follows from (33)–(35) and for $\mathbf{E}\Delta_2^2$ from Corollary 3.

Let the notation introduced before Lemma 3 hold, $\widetilde{n} = n \vee M$, and let $\widetilde{D} \subset \Pi$ be an arbitrary set satisfying $|\widetilde{D}| = |D| = N_1$, $\widetilde{D} \cap D = \emptyset$. Set $\widehat{\widehat{\omega}}_{(\cdot)} = \widehat{\omega}_{(\cdot)} \vee \omega_{(\cdot)}(\Psi_{a,n,\widetilde{D}})$.

Lemma 4. *For any $\tau \in (0, \frac{1}{2}]$,*

$$\mathbf{E}\Psi_{a,n}(\widehat{\widehat{\omega}}) \leq \Psi\left[\left(1 + \sqrt{\frac{C_f \ln^5 \widetilde{n}}{\tau M} \cdot \frac{N}{N_1}}\right)^2 + C\tau + \frac{C_f N^2}{\widetilde{n} N_1^2}\right], \quad (38)$$

where

$$\Psi \stackrel{\text{def}}{=} \min_{\omega \in \Omega} \Psi_{a,n}(\omega).$$

P r o o f. Let $\omega = \omega(\Psi_{a,n})$. First we estimate $\mathbf{E}\Delta$, where

$$\Delta = \sum_{j=1}^n \left(\widehat{\underline{\omega}}_j^2 - \underline{\omega}_j^2 \right) c_j^2. \quad (39)$$

(Recall that $\underline{\omega}_j = 1 - \omega_j$.) To this end, define a decreasing sequence of natural numbers r_k , $k \in \mathbf{N}$, by the equalities $r_0 = n$, $r_k = \max\{j: \underline{\omega}_j < (1 - \tau) \underline{\omega}_{r_{k-1}}\}$. Next, put $h = \max\{k: \underline{\omega}_{r_k} \geq \tilde{n}^{-2} \vee \underline{\omega}_1\}$, $s_k = r_{k+1} + 1$.

By the definition, for any $k = 0, \dots, h$,

$$p(s_k, n) = s_k, \quad q(r_k, n) = r_k, \quad (40)$$

$$(1 - \tau)^k \geq \underline{\omega}_{r_k} \geq \underline{\omega}_{s_k} \geq (1 - \tau) \underline{\omega}_{r_k}. \quad (41)$$

For $s_k \leq j \leq r_k$ denoting $\nu_j = \nu_j(k) = (\widehat{\underline{\omega}}_j - \underline{\omega}_{r_k})_+$ and applying the inequality $\underline{\omega}_{r_k} \leq 2\underline{\omega}_j$, we obtain

$$\Delta \leq \sum_{j < s_h} \widehat{\underline{\omega}}_j^2 c_j^2 + \sum_{k=1}^h \sum_{j=s_k}^{r_k} \left[(\omega_{r_k}^2 - \underline{\omega}_j^2) + 4\nu_j \underline{\omega}_j + \nu_j^2 \right] c_j^2 \stackrel{\text{def}}{=} \Delta_1 + \dots + \Delta_4. \quad (42)$$

When estimating $\mathbf{E}\Delta_i$, $i = 1, \dots, 4$, we frequently use the following obvious relations:

$$\sum_{j=1}^n \omega_j \gamma_j = \sum_{j=1}^n \frac{\omega_j a}{N} \leq \Psi, \quad C\Psi \geq \frac{M}{N}. \quad (43)$$

We have

$$\mathbf{E}\Delta_1 \leq C \sum_{j < s_h} \left[\underline{\omega}_j^2 + \mathbf{P}^2 \left\{ \widehat{\gamma}_j < \frac{\gamma_j}{2} \right\} \right] c_j^2. \quad (44)$$

By virtue of Corollary 4 and the Chebyshev inequality, $\mathbf{P}\{\widehat{\gamma}_j < \gamma_j/2\} \leq C_f/(N_1 \gamma_j)$. Thus, from (43), (44), and the inequalities $\underline{\omega}_j \leq \tilde{n}^{-2}$, $\gamma_j \geq (\tilde{n}^2 - 1)a/N$, where $j < s_h$, we conclude that

$$\mathbf{E}\Delta_1 \leq C_f \left(\frac{N}{N_1} \right)^2 \frac{\Psi}{\tilde{n}}. \quad (45)$$

Further, (41) and (43) imply

$$\Delta_2 \leq C\tau \sum_{j=1}^n \omega_j^2 c_j^2 \leq C\tau \Psi. \quad (46)$$

Application of the Cauchy–Schwartz inequality yields

$$\mathbf{E}\Delta_3 \leq 4 \left(\sum_{j=s_h}^n \mathbf{E}\nu_j^2 c_j^2 \right)^{1/2} \left(\sum_{j=1}^n \mathbf{E}\underline{\omega}_j^2 c_j^2 \right)^{1/2} \leq C(\mathbf{E}\Delta_4)^{1/2} \Psi^{1/2}. \quad (47)$$

It remains to estimate the last term $\mathbf{E}\Delta_4$. Since $\gamma_j = \gamma_{[s_k, r_k]}$ for $s_k \leq j \leq r_k$, from Lemma 3 and (41) we derive that

$$\mathbf{E} \sum_{j=s_k}^{r_k} \nu_j^2 c_j^2 \leq C_f \ln^4 \tilde{n} \cdot \frac{N}{N_1^2} \leq \frac{C_f \ln^4 \tilde{n}}{M} \left(\frac{N}{N_1} \right)^2 \Psi. \quad (48)$$

Because $h \leq \ln \tilde{n}/\tau$, from (42) and (45)–(48) it follows that

$$\mathbf{E}\Delta \leq \left(\frac{C_f N^2}{\tilde{n} N_1^2} + C\tau + \max_{\alpha=1,1/2} \left\{ \left[\frac{c_f \ln^5 \tilde{n}}{\tau M} \left(\frac{N}{N_1} \right)^2 \right]^\alpha \right\} \right) \Psi. \quad (49)$$

Now let

$$\Delta = \frac{a}{N} \sum_{j=1}^n (\hat{\omega}_j^2 - \omega_j^2). \quad (50)$$

The expectation of (50) is estimated in a similar way as that of (39). The natural numbers s_j and r_j are defined sequentially by equalities $s_0 = 1$, $s_k = \min\{j: \omega_j < (1 - \tau)\omega_{s_{k-1}}\}$, $h = \min\{k: \omega_{s_k} \leq \tilde{n}^{-2} \vee \omega_n\}$, $r_h = n$, $r_k = s_{k+1} - 1$ for $k < h$. It is clear that (40) holds for $k = 0, \dots, h$, and

$$(1 - \tau)^k \geq \omega_{s_k} \geq \omega_{r_k} (1 - \tau)^{-1}, \quad \text{if } k < h. \quad (51)$$

Set $\nu_j = \nu_j(k) = (\hat{\omega}_j - \omega_{s_k})_+$ for $s_k \leq j \leq r_k$. Then

$$\Delta = \frac{a}{N} \sum_{k=1}^h \sum_{j=s_k}^{r_k} \left[(\omega_{s_k}^2 - \omega_j^2) + 2\nu_j \omega_{s_k} + \nu_j^2 \right] \stackrel{\text{def}}{=} \Delta_1 + \Delta_2 + \Delta_3. \quad (52)$$

In view of (43), (51), and the definition of h ,

$$\Delta_1 \leq \frac{a}{N} \sum_{j=1}^n \frac{1}{\tilde{n}^2} \vee (C\tau\omega_j^2) \leq C(\tau + \tilde{n}^{-1}) \Psi. \quad (53)$$

For any $j \in [s_k, r_k]$, we obtain $\nu_j \leq (\hat{\gamma}_j - \gamma_{s_k})_+ / (\gamma_{s_k} + a/N)$ and $\hat{\gamma}_j = \min_{i \leq j} \hat{\gamma}_{[i, \hat{q}(j, n)]} \leq \hat{\gamma}_{[s_k, \hat{q}(j, n)]} \leq \gamma_{s_k} + \max_{i \geq j} \hat{\gamma}_{[s_k, i]}$. From these inequalities and Corollary 3, it follows that

$$\frac{a}{N} \sum_{j=s_k}^{r_k} \mathbf{E}\nu_j^2 \leq \sum_{j=s_k}^{r_k} C_f \lambda(j - s_k) \frac{N}{N_1^2} \leq C_f \ln^3 \tilde{n} \cdot \frac{N}{N_1^2}. \quad (54)$$

Taking into account (43) and the inequality $h < C \ln \tilde{n}/\tau$, we find that

$$\Delta_3 \leq \frac{C \ln^4 \tilde{n}}{\tau M} \left(\frac{N}{N_1} \right)^2 \Psi. \quad (55)$$

Finally, (43), (51), and the Cauchy–Schwartz inequality lead to the estimate

$$\Delta_2 \leq 4 \left(\sum_j \omega_j^2 \frac{a}{N} \right)^{1/2} \Delta_3^{1/2}. \quad (56)$$

Inequalities (52)–(56) demonstrate that (49) is valid for (50) as well, since $\hat{\omega}_j \leq \hat{\omega}_j$, $j \in \mathbf{N}$. Thus, (38) is proved.

Lemma 5. *Let $\hat{a}$ be given in (14), where $N_0 \geq N^{-\alpha}/C$ for some fixed $\alpha \in (\frac{1}{2}, 1)$. There exists $\nu > 0$ such that*

$$\mathbf{P}\{|\hat{a} - a_f| > N^{-\nu}\} \leq C_f N^{-2}. \quad (57)$$

P r o o f. We have $\hat{a} = (\pi N_0)^{-1} \sum_{t \in \Pi_0} Z_t$, where $Z_t = \psi_N(X_t)$, $t \in \Pi_0$. Obviously,

$$0 \leq a_f - \mathbf{E}\hat{a} \leq C_f \int_0^{N^{-1/(2q)}} \frac{dz}{\sqrt{z}} \leq C_f N^{-1/(4q)}. \quad (58)$$

In view of the independence of Z_j , $j \in \Pi_0$, and the well-known relationships between moments and cumulants, one can write

$$\begin{aligned} \mathbf{E}(\hat{a} - \mathbf{E}\hat{a})^{2k} &\leq C_1(k) \max_{2 \leq j \leq 2k} |\text{Cum}_j(\hat{a})|^{2k/j} \\ &= C_1(k) \max_{2 \leq j \leq 2k} \left| \frac{1}{\pi^j N_0^{j-1}} \text{Cum}_j(\widehat{Z}_1) \right|^{2k/j} \\ &\leq C_2(k) \max_{2 \leq j \leq 2k} \left(\mathbf{E}|\widehat{Z}_1|^j \cdot N_0^{1-j} \right)^{2k/j}, \end{aligned} \quad (59)$$

where $C_1(k)$, $C_2(k)$ are constants dependent on k only. Since

$$\mathbf{E}|\widehat{Z}_1|^j \leq \|f\|_\infty N^{j/2-1} \int_{-1}^1 \psi_N(x) \mu(dx) \leq C_f N^{j/2-1}, \quad (60)$$

inequality (57) follows from (58)–(60) and the Chebyshev inequality.

Let $\|V\|_2$ denote the Euclidean norm of a matrix $V = (v_{ij})_{i,j=1}^k$ and let $|V| = \max_{i,j} |v_{ij}|$.

Lemma 6. *Let V be a symmetric matrix and its elements $v_{ij} = 0$ for all i, j provided $|i - j| > m$. Let $\Lambda = \text{diag}\{\Lambda_1, \dots, \Lambda_k\}$ be a diagonal matrix with the elements satisfying inequalities $0 \leq \Lambda_1 \leq \dots \leq \Lambda_k \leq 1$. Then for $k > m$,*

$$\|\Lambda V - \Lambda^{1/2} V \Lambda^{1/2}\|_2^2 \leq \|V\|^2 m(m+1). \quad (61)$$

P r o o f. According to the definition of Λ we have

$$\begin{aligned} \|\Lambda V - \Lambda^{1/2} V \Lambda^{1/2}\|_2^2 &= \sum_{i,j=1}^k v_{ij}^2 \Lambda_i \left(\Lambda_i^{1/2} - \Lambda_j^{1/2} \right)^2 \leq |V|^2 \sum_{i=1}^k \sum_{|i-j| \leq m} (\Lambda_i - \Lambda_j)^2 \\ &\leq |V|^2 \sum_{i=1}^k \sum_{|i-j| \leq m} |\Lambda_i - \Lambda_j|. \end{aligned} \quad (62)$$

A simple algebra shows that

$$\sum_{i=1}^k \sum_{|i-j| \leq m} |\Lambda_i - \Lambda_j| = 2 \sum_{l=1}^m \sum_{r=1}^l (\Lambda_{k-l+r} - \Lambda_r) \leq m(m+1).$$

To complete the proof substitute this inequality into (62).

3.2. Proof of the main results.

P r o o f o f T h e o r e m 1. The analogous result (with a statistical estimate substituted for the functional a_g in the definition of f^*) is established in [32, Theorem 4 and Example] under the additional restrictions on the sequence b . Lemma 6 shows that these restrictions are redundant. In fact, to make sure of this, it suffices to replace reference [32, (A.29)] by (61).

P r o o f o f T h e o r e m 2. Since we are considering the asymptotic behavior as $N \rightarrow \infty$, everywhere below we assume that N is a sufficiently large number. Further C denotes various constants dependent only on C_1 (see (1)). Let $f \in W_{g,b}$, $b \in B$, $a = a_f$, $M = M(b, N)$ (see (5)), and let $\Psi_{(\cdot)}$ be defined by (23). Set $\Psi_n = \min_{\omega \in \Omega} \Psi_{a,n}(\omega)$. The sequence Ψ_n is nondecreasing and its limit $\Psi_\infty = \Delta(f, c) \leq \Delta(g, b)(1 + o(1))$. Suppose that a natural number $m = m(N)$ is chosen so that $M = o(m)$. Then by virtue of (3) we obtain $b_m = o(b_{M+1}) = o(M/N)$. Since $\Psi_m - \Psi_\infty \leq b_m$,

$$\Psi_m \leq \Delta(g, b)(1 + o(1)), \quad \Delta(g, b) \asymp MN^{-1}, \quad (63)$$

by Theorem 1. Here and everywhere below the convergence is uniform with respect to $f \in W_{g,b}$. The construction of the estimator

$$f^* = u_0 + \sum_{j=1}^{n^*} \omega_j^* c_j^* u_j$$

ensures the equality

$$\delta(f, f^*) = \mathbf{E} \left[\Psi_{a,n^*}(\omega^*) \frac{N}{N_3} + \sum_{j=1}^{n^*} (\omega_j^*)^2 \left(\frac{a}{N_3} - v_j(N_3) \right) \right].$$

Hence, due to Lemma 1,

$$\delta(f, f^*) \leq \mathbf{E}[\Psi_{a,n^*}(\omega^*) N N_3^{-1}] \cdot (1 + o(1)) + o(MN^{-1}). \quad (64)$$

Note that

$$\frac{N}{N_3} \leq C \wedge \left(1 - N^{\alpha-1} - \frac{C}{k^*} \right) \stackrel{\text{def}}{=} \rho(k^*). \quad (65)$$

Let $\tilde{\omega}$ be defined as ω^* with $\hat{a}$ replaced by a . By virtue of Lemma 5, from (64) and (65) we get

$$\delta(f, f^*) \leq \mathbf{E}[\Psi_{a,n^*}(\tilde{\omega}) \rho(k^*)] \cdot (1 + o(1)) + o(MN^{-1}). \quad (66)$$

Take $\hat{\omega}$ as in Lemma 4 with $n = n(k) = 2^k$, $N_1 = N_1(k) = \text{Int}(N/(4k))$. Inequalities (63) and (66) show that to conclude the proof it suffices to verify that

$$\mathbf{E} \sum_{k=1}^s \Psi_{a,n}(\hat{\omega}) \rho(k) \mathbf{P}\{k^* = k\} \leq \Psi_m(1 + o(1)) + o\left(\frac{M}{N}\right), \quad (67)$$

where $s = \text{Int}(\ln_2 N)$. Since $M \rightarrow \infty$, $N/M \rightarrow \infty$, one can assume in addition that $m \leq (M \ln M) \wedge (N/2)$. Lemma 4 implies that

$$\mathbf{E}\Psi_{a,n}(\widehat{\omega}) \leq \Psi_n(1 + o(1)) \leq (\Psi_m + \Gamma_{(n,m)})(1 + o(1)), \quad \text{if } n \leq M^2, \quad (68)$$

and

$$\mathbf{E}\Psi_{a,n}(\widehat{\omega}) \leq C\Psi_m \ln^7 n, \quad \text{if } n > M^2. \quad (69)$$

For $n > M^2$, one has $\Gamma_{[n/2,n]} = o(\sqrt{n}/N)$. Therefore, applying notation $K_1 = (\ln_2 M^2, s] \cap \mathbf{N}$, $\widehat{\Gamma}_k = \Gamma_{[2^{k-1}, 2^k]}(D_k)$, the definition of k^* , Lemma 2, and Chebyshev inequality, we obtain

$$\begin{aligned} \sum_{k \in K_1} \ln^7 n(k) \mathbf{P}\{k^* = k\} &\leq C \sum_{k \in K_1} k^7 \mathbf{P}\left\{\widehat{\Gamma}_k > \frac{2^{k\beta-1}}{N}\right\} \\ &\leq C \sum_{k \in K_1} k^9 2^{k(1-2\beta)} = o(1). \end{aligned} \quad (70)$$

Set $r = \text{Int}(\ln_2 m + 1)$. Now we will estimate

$$\Phi = \sum_{k < r} \Gamma_{(n,m)} \mathbf{P}\{k^* = k\}. \quad (71)$$

Fix $k < r$. If $\Gamma_{(n,m)} > 2r2^{\beta r}/N$, then for some integer j , $k < j \leq r$,

$$\Gamma_j \stackrel{\text{def}}{=} \Gamma_{[2^{j-1}, 2^j]} \geq \frac{\Gamma_{(n,m)}}{r} \geq 2 \cdot \frac{2^{\beta r}}{N}. \quad (72)$$

In this case Lemma 2 and the definition of k^* imply

$$\begin{aligned} \mathbf{P}\{k^* = k\} &\leq \mathbf{P}\left\{\widehat{\Gamma}_j \leq \frac{2^{j\beta}}{N}\right\} \leq \mathbf{P}\left\{-\widehat{\Gamma}_j \geq \frac{\Gamma_j}{2}\right\} \\ &\leq \frac{C[\Gamma_j |D_j|^{-1} + |D_j|^{-2}]}{\Gamma_j^2} \leq \frac{Cj}{N\Gamma_j} \leq \frac{Cr^2}{N\Gamma_{(n,m)}}. \end{aligned} \quad (73)$$

From the above mentioned and the equality $\sum_k \mathbf{P}\{k^* = k\} = 1$, it follows that

$$\Phi \leq \frac{Cr^3}{N} + \frac{Cm^{\beta r}}{N} = o\left(\frac{M}{N}\right). \quad (74)$$

Since $\rho(k) = 1 + o(1)$ as $k \rightarrow \infty$, to finish the proof it suffices to show that

$$\Psi_m \sum_{k < l} \rho(k) \mathbf{P}\{k^* = k\} = o\left(\frac{M}{N}\right) \quad (75)$$

for some $l = l(N) \rightarrow \infty$. Indeed, (68)–(75) imply (67).

If $\Psi_m = o(M/N)$, then (75) follows from the inequality $\rho(k) \leq C$. Let $\Psi_m > \tau M/N$ for some fixed $\tau > 0$. For any $h < m$, one can write $\Psi_m \leq Ch/N + \Gamma_{[h,m]} + \Gamma_{[m,\infty)}$ and $\Gamma_{[m,\infty)} = o(M/N)$. Thus, there exists $h = 2^l$, $l \in \mathbf{N}$, such that $h > \tau M/C$ and $\Gamma_l \geq \tau M/(CN \ln M)$. Again, (15),

Lemma 2, and Chebyshev inequality yield that $\mathbf{P}\{k^* < l\} = o(1)$. Since $\tau > 0$ is arbitrary, we arrive at (75).

P r o o f o f T h e o r e m 3 repeats essentially the arguments in the proofs of Theorem 1 and Theorem 2. In [32, Theorem 4] the analog of Theorem 1 is proved for more general orthonormal bases satisfying conditions [32, (A)–(D)] which are very similar to (U0)–(U2) and (18) for $b \in \mathcal{B}_+$ except condition (B). Condition (B) in [32] is used in the derivation of the upper bound for the minimax risk of f^* and can be successfully replaced by condition (U1). This fact is apparent, in particular, from the proof of Theorem 2. The relaxation of (U2) as compared to [32, (C)] is stipulated by the application of Lemma 6 instead of [32, (A.29)].

The demonstration of the adaptivity of f^* does not use any special property of the basis u except the explicit form (17) of the functional a_g for Jacobi polynomials given in (6). By making use of (U1) it easy to build up an estimator of a_g with the desirable properties. The details are omitted.

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