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СПИСОК ЛИТЕРАТУРЫ

1. Белавкин В. П. Об обобщенных соотношениях неопределенности Гейзенберга и эффективных измерениях в квантовых системах. — Теорет. матем. физ., 1976, т. 26, № 3, с. 316–329.
2. Barndorff-Nielsen O. E., Gill R. D. Fisher information in quantum statistics. — J. Phys. A, 2000, v. 33, № 24, pp. 4481–4490.
3. Barndorff-Nielsen O. E., Gill R. D., Jupp P. E. On quantum statistical inference. — J. Roy. Statist. Soc. Ser. B, 2003, v. 65, № 4, p. 775–816.
4. Connes A. Noncommutative Geometry. San Diego: Academic Press, 1994, 661 p.
5. Крамер Г. Математические методы статистики. М.: Мир, 1975, 648 с.
6. Дирак П. Принципы квантовой механики. М.: Физматгиз, 1960, 434 с.
7. Fisher R. A. Theory of statistical estimation. — Proc. Cambridge Philos. Soc., 1925, v. 22, p. 700–725.
8. Helstrom C. W. Quantum Detection and Estimation Theory. New York: Academic Press, 1976.
9. Холеев А. С. Вероятностные и статистические аспекты квантовой теории. М.-Ижевск: Ин-т компьют. исслед., 2003, 404 с.
10. Холеев А. С. Асимптотическое оценивание параметра сдвига квантового состояния. — Теория вероятн. и ее примен., 2004, т. 49, в. 2, с. 335–350.
11. Luo S. Wigner–Yanase skew information vs. quantum Fisher information. — Proc. Amer. Math. Soc., 2004, v. 132, № 3, p. 885–890.
12. Petz D. Monotone metrics on matrix spaces. — Linear Algebra Appl., 1996, v. 244, p. 81–96.
13. Sakurai J. J. Modern Quantum Mechanics. Reading, MA: Addison-Wesley, 1994.
14. Schrödinger E. About Heisenberg uncertainty relation. — Proc. Prussian Acad. Sci., Phys.-Math. Sect., 1930, v. 19, p. 296–303; рус. пер.: К принципу неопределенности Гейзенберга. — Э. Шредингер. Избранные труды по квантовой механике. М.: Наука, 1976, с. 210–217.
15. фон Нейман Дж. Математические основы квантовой механики. М.: Наука, 1964, 367 с.

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LOWER BOUNDS FOR TAILS OF SUMS OF INDEPENDENT SYMMETRIC RANDOM VARIABLES¹⁾

Обобщая подход Клейтмана [9] и Кантера [8] к неравенствам для многомерных функций концентрации, мы получаем для вероятностей уклонений сумм независимых симметричных случайных величин оценку снизу, зависящую только от вероятностей уклонений слагаемых. Эта оценка оптимальна с точностью до эффектов дискретизации, улучшает результат Нагаева [11] и дополняет теоремы сравнения Бирнбаума [3] и Прусса [12]. Терема Бирнбаума для унимодальных случайных величин распространяется на решетчатый случай.

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1. Introduction. For deviation probabilities $\mathbf{P}\{|S| > t\}$ of sums $S = \sum_{i=1}^n X_i$ of independent, real-valued, and symmetrically distributed random variables X_i , Nagaev [11, Theorem 1] (in different notation) obtained the lower bound

$$\mathbf{P}\{|S| > t\} \geq \sum_{k > t/h} 2^{-k} B_p(\{k\}) \quad (t \in [0, nh[), \quad (1)$$

where $h \in]0, \infty[$ is a free parameter and

$$B_p := \bigstar_{i=1}^n B_{p_i} \quad (2)$$

is the convolution of the Bernoulli distributions $B_{p_i} = (1 - p_i) \delta_0 + p_i \delta_1$ with success probabilities $p_i := \mathbf{P}\{|X_i| \geq h\}$. Nagaev also provided analytically more tractable lower bounds for the right-hand side of (1) and showed that the resulting inequalities for $\mathbf{P}\{|S| > t\}$ effectively complement other bounds depending on second and third absolute moments of the random variables X_i .

The main purpose of the present note is to provide (see Theorem 2.2 below) a generalization of Kanter's [8] concentration function inequality for sums of independent and symmetric random vectors, which yields as Corollary 2.2 below in particular the following improvement of (1), under the same assumptions as above:

$$\mathbf{P}\{|S| > t\} \geq \sum_{k > t/h} \left(1 - 2^{-k} F_k\left(\left\lfloor \frac{t}{h} + 1 \right\rfloor\right)\right) B_p(\{k\}) \quad (t \in [0, nh[). \quad (3)$$

Here and below, we use the standard notation $[x] := \max\{k \in \mathbf{Z} : k \leq x\}$ and $\lceil x \rceil := -\lceil -x \rceil$, and write

$$F_n(m) := \max_{r \in \mathbf{Z}} \sum_{i=r}^{r+m-1} \binom{n}{i} \quad (n, m \in \mathbf{N}_0) \quad (4)$$

for the sum of the m largest binomial coefficients of order n . For $t \in [(n-1)h, nh[$, the inequalities in (1) and (3) are identical, while for $t \in [0, (n-1)h[$ and $B_p(\{n-1\}) > 0$, inequality (3) is strictly sharper than (1). Moreover, as follows from the proof of Corollary 2.2, inequality (3) is optimal up to discretization effects, in the sense that, subject to the stated assumptions, the right-hand side of (3) is the greatest lower bound for $\mathbf{P}\{|S| > t\} + \mathbf{P}\{|S| = t\}/2$ for every $t = mh$ with $m \in \{1, \dots, m\}$.

The rest of this note is structured as follows. Section 2 develops the Kleitman–Kanter approach to multivariate concentration function inequalities. A specialization to the one-dimensional case, namely Corollary 2.1, immediately yields the above-mentioned Corollary 2.2 improving Nagaev's result. Section 3 reformulates Corollary 2.1 as a comparison theorem, stated together with related results of Pruss [12] and Birnbaum [3]. The latter is generalized to the lattice case. Historical remarks are collected in Section 4.

2. A generalized Kanter inequality. Let $\|\cdot\|$ be a seminorm on an $\mathbf{R}$ -vector-space E , and let $|\cdot|$ denote the usual absolute value on $\mathbf{R}$. We write $\mathbf{N} := \{1, 2, 3, \dots\}$ and $\mathbf{N}_0 := \{0\} \cup \mathbf{N}$.

Lemma 2.1. *Let $a \in E$, $m \in \mathbf{N}$, and $C_1, \dots, C_m \subset E$ with*

$$x, y \in C_j \implies \|x - y\| < \|a\| \quad (5)$$

for each $j \in \{1, \dots, m\}$. Then for some $r \in \{1, \dots, m\}$ the translate $C_r - a$ is disjoint from $\bigcup_{j=1}^m C_j$.

P r o o f. We may assume that $D := \bigcup_{j=1}^m C_j \neq \emptyset$ and $\|a\| > 0$. In the special case $E = \mathbf{R}$, $\|\cdot\| = |\cdot|$ and $a > 0$, we choose r such that $\min D = \min C_r$ if $\min D$ exists, and $\inf D = \inf C_r$ otherwise. In the general case we apply the Hahn–Banach theorem (compare, e.g., [13, Theorem 3.3 and its corollary]) to yield a linear functional ℓ on E with $\ell(a) = \|a\|$ and $|\ell(x)| \leq \|x\|$ for every $x \in E$, so that the special case applied to $\ell(a) > 0$ and $\ell(C_1), \dots, \ell(C_m) \subset \mathbf{R}$ yields the claim.

Lemma 2.2. For $n, m \in \mathbf{N}_0$, we have $F_n(m) = \sum_{i=r}^s \binom{n}{i}$ with $r = r_{n,m} := \lfloor (n - m + 1)/2 \rfloor$ and $s = r + m - 1$, and also with $\lceil (n - m + 1)/2 \rceil$ in place of $r_{n,m}$. Further,

$$F_n(m) = F_{n-1}(m-1) + F_{n-1}(m+1) \quad (n, m \in \mathbf{N}) \quad (6)$$

and $n \mapsto 2^{-n}F_n(m)$ is for every $m \in \mathbf{N}_0$ a decreasing function.

P r o o f. The claim up to (6) follows easily from the symmetry, monotonicity and recursion properties of the binomial coefficients. The last claim follows, since the right-hand side of (6) is $\leq 2F_{n-1}(m)$.

We write $\sharp A$ for the cardinality of a set A .

Theorem 2.1 (essentially Theorem I in [9]). Let $n, m \in \mathbf{N}$, $a_1, \dots, a_n \in E$, and $C_1, \dots, C_m \subset E$ with

$$x, y \in C_j \implies \|x - y\| < \min_{i=1, \dots, n} \|a_i\| \quad (7)$$

for each $j \in \{1, \dots, m\}$. Then

$$\sharp \left\{ I \subset \{1, \dots, n\} : \sum_{i \in I} a_i \in \bigcup_{j=1}^m C_j \right\} \leq F_n(m) \quad (8)$$

with equality for $E = \mathbf{R}$, $\|\cdot\| = |\cdot|$, $a_1 = \dots = a_n = 1$, and $C_j = \{\lfloor (n - m + 1)/2 \rfloor + j - 1\}$.

P r o o f. We consider more generally $n, m \in \mathbf{N}_0$ and let $G_n(m)$ denote the supremum of the left-hand side of (8) subject to the stated assumptions on $a_1, \dots, a_n$ and $C_1, \dots, C_m$. Then

$$G_n(0) = 0 = F_n(0) \quad (n \in \mathbf{N}_0), \quad (9)$$

$$G_0(m) = 1 = F_0(m) \quad (m \in \mathbf{N}). \quad (10)$$

Let $n, m \in \mathbf{N}$. Given $a_1, \dots, a_n \in E$ and $C_1, \dots, C_m \subset E$ with (7), let $a := a_n$ and choose r according to Lemma 2.1. Then the left-hand side of (8) is $\sharp A$ with

$$A := \left\{ \varepsilon \in \{0, 1\}^n : \sum_{i=1}^n \varepsilon_i a_i \in \bigcup_{j=1}^m C_j \right\} = (A_1 \times \{0\}) \cup (A_2 \times \{1\}) \cup (A_3 \times \{1\}),$$

where

$$\begin{aligned} A_1 &:= \left\{ \varepsilon \in \{0, 1\}^{n-1} : \sum_{i=1}^{n-1} \varepsilon_i a_i \in \bigcup_{j=1}^m C_j \right\}, \\ A_2 &:= \left\{ \varepsilon \in \{0, 1\}^{n-1} : \sum_{i=1}^{n-1} \varepsilon_i a_i \in C_r - a_n \right\}, \\ A_3 &:= \left\{ \varepsilon \in \{0, 1\}^{n-1} : \sum_{i=1}^{n-1} \varepsilon_i a_i \in \bigcup_{j \neq r} (C_j - a_n) \right\} \end{aligned}$$

with $A_1 \cap A_2 = \emptyset$ and thus

$$\sharp A \leq \sharp A_1 + \sharp A_2 + \sharp A_3 = \sharp(A_1 \cup A_2) + \sharp A_3 \leq G_{n-1}(m+1) + G_{n-1}(m-1).$$

Hence we have

$$G_n(m) \leq G_{n-1}(m-1) + G_{n-1}(m+1) \quad (n, m \in \mathbf{N}). \quad (11)$$

Now (6), (9), (10), and (11) together imply $G_n(k) \leq F_n(m)$ for all $n, m \in \mathbf{N}_0$, as was to be shown. The claim about equality is obvious. Theorem 2.1 is proved.

We call a random vector X symmetric if it has the same law as $-X$. We recall definitions (2) and (4) and put

$$Q_p := \sum_{i=1}^n \left((1 - p_i) \delta_0 + \frac{p_i}{2} (\delta_{-1} + \delta_1) \right) \quad (p \in [0, 1]^n).$$

Theorem 2.2 (generalized Lemma 4.2 of [8]). Let $h \in]0, \infty[$, $n, m \in \mathbf{N}$, and $p \in [0, 1]^n$. Then the supremum of

$$\mathbf{P}\left\{\sum_{i=1}^n X_i \in \bigcup_{j=1}^m C_j\right\}$$

taken over all measurable $\mathbf{R}$ -vector spaces E , measurable seminorms $\|\cdot\|$ on E , measurable sets $C_1, \dots, C_m \subset E$ with

$$x, y \in C_j \Rightarrow \|x - y\| < 2h$$

for every $j \in \{1, \dots, m\}$, and all independent and symmetric E -valued random vectors X_i with

$$\mathbf{P}\{\|X_i\| < h\} \leq 1 - p_i \quad (i = 1, \dots, n)$$

is attained for $E = \mathbf{R}$, $\|\cdot\| = |\cdot|$, $C_j = \{0, h\} + (2j - m - 1)h$, and the X_i symmetric $\mathbf{R}$ -valued with $\mathbf{P}\{X_i = 0\} = 1 - p_i = 1 - \mathbf{P}\{|X_i| = h\}$. The value of the supremum is

$$Q_p([-m + 1, m]) = \sum_{k=0}^n 2^{-k} F_k(m) B_p(\{k\}). \quad (12)$$

R e m a r k. Analytically convenient and sharp upper bounds for the quantity in (12) in the special case $m = 1$ are provided by [8, Lemma 4.3]) and in [10]. It is an open problem to prove analogous bounds for $m \geq 2$.

P r o o f. We may assume $h = 1$. Let n etc. up to the X_i be as stated, and let us put $\pi_i := 1 - \mathbf{P}\{\|X_i\| < 1\}$. We may assume that $X_i = (1 - B_i)Y_i + B_i R_i Z_i$ with $B_1, \dots, B_n, Y_1, \dots, Y_n, Z_1, \dots, Z_n, R_1, \dots, R_n$ independent, $B_i \sim B_{p_i}$, $Y_i \sim \mathbf{P}\{X_i \in \cdot \mid \|X_i\| < 1\}$:= the conditional distribution of X_i given $\|X_i\| < 1$, $Z_i \sim \mathbf{P}\{X_i \in \cdot \mid \|X_i\| \geq 1\}$, and $\mathbf{P}\{R_i = -1\} = \mathbf{P}\{R_i = 1\} = \frac{1}{2}$. Then, with $B := (B_1, \dots, B_n)$, with Q denoting the law of $(Y, Z) := (Y_1, \dots, Y_n, Z_1, \dots, Z_n)$, and with $|b| := \sum_{i=1}^n b_i$, we have

$$\begin{aligned} \mathbf{P}\left\{\sum_{i=1}^n X_i \in \bigcup_{j=1}^m C_j\right\} &= \sum_{b \in \{0,1\}^n} \mathbf{P}\{B = b\} \\ &\times \underbrace{\int \mathbf{P}\left\{\sum_{i=1}^n b_i \frac{R_i + 1}{2} z_i \in \bigcup_{j=1}^m \frac{1}{2} \left(C_j + \sum_{i=1}^n (b_i z_i - (1 - b_i) y_i)\right)\right\}}_{\leq 2^{-|b|} F_{|b|}(m)} dQ(y, z) \end{aligned} \quad (13)$$

$$\leq \text{R.H.S. of (12) with } \pi \text{ instead of } p \quad (14)$$

$$\leq \text{R.H.S. of (12)}. \quad (15)$$

Here the inequality in (13), and hence (14), follow from Theorem 2.1, with those z_i with $b_i = 1$ playing the role of the a_i , and with $\frac{1}{2}(C_j + \sum_{i=1}^n (b_i z_i - (1 - b_i) y_i))$ in place of C_j . Inequality (15) is true, since $N_0 \ni k \mapsto 2^{-k} F_k(m)$ is decreasing by Lemma 2.2, and $[0, 1]^n \ni p \mapsto B_p$ is increasing with respect to the coordinatewise order on $[0, 1]^n$ and the usual stochastic order. In the special case $E = \mathbf{R}$ etc. as stated, we have $Y_i \sim \delta_0$ and may replace the distribution of Z_i by δ_1 in deriving (13), and hence get equality everywhere. Theorem 2.2 is proved.

Corollary 2.1. Let $0 < h \leq H < \infty$ with $m := \lceil H/h \rceil < H/h + \frac{1}{2}$, $n \in \mathbf{N}$, and $p \in [0, 1]^n$. Then the supremum of

$$\mathbf{P}\left\{\sum_{i=1}^n X_i \in]-H, H] + a\right\} \quad (16)$$

taken over all independent and symmetric $\mathbf{R}$ -valued random variables X_i with

$$\mathbf{P}\{|X_i| < h\} \leq 1 - p_i \quad (i = 1, \dots, n) \quad (17)$$

and all $a \in \mathbf{R}$, is attained for $\mathbf{P}\{X_i = 0\} = 1 - p_i = 1 - \mathbf{P}\{|X_i| = h\}$ and $a = mh - H$. The value of the supremum is given in (12).

P r o o f. Given h, H, m, n, p, X_i , and a as above, we have that the probability in (16) is

$$\leq \mathbf{P}\left\{\sum_{i=1}^n X_i \in]-mh, mh] + b\right\} = \mathbf{P}\left\{\sum_{i=1}^n X_i \in \bigcup_{j=1}^m C_j\right\} \leq \text{R.H.S. of (12)}$$

with $b := a$ and $C_j :=]-h, h] + (2j - m - 1)h + b$, using Theorem 2.2 with $E = \mathbf{R}$ and $\|\cdot\| = |\cdot|$. On the other hand, if $\mathbf{P}\{X_i = 0\} = 1 - p_i = 1 - \mathbf{P}\{|X_i| = h\}$ and $a = mh - H$, and if we set $b := 0$ instead of $b := a$, then we can replace the two inequalities in the above calculation by equalities, since the assumption $m < H/h + \frac{1}{2}$ yields $-mh \leq -H + a < -(m - 1)h$.

Corollary 2.2. Let $S = \sum_{i=1}^n X_i$ with independent and symmetric $\mathbf{R}$ -valued random variables X_i , and let $h \in]0, \infty[$. Then (3) holds with $p_i := \mathbf{P}\{|X_i| \geq h\}$ for $i = 1, \dots, n$.

P r o o f. For $t > 0$, we apply Corollary 2.1 with $a = 0$ and $H = mh$ with $m := \lfloor t/h \rfloor + 1$ to get

$$\mathbf{P}\{|S| \leq t\} \leq \mathbf{P}\{S \in]-mh, mh]\} \leq \text{R.H.S. of (12)}.$$

Inequality (3) follows by taking complements, since $F_k(m) = 2^k$ for $k \leq m - 1$.

3. Comparison theorems. For $\mathbf{R}$ -valued random variables U and V , we write $U \geq_{\text{st}} V$, if U is stochastically larger than V , that is, if $\mathbf{P}\{U \geq t\} \geq \mathbf{P}\{V \geq t\}$ for every $t \in \mathbf{R}$. A specialization of Corollary 2.1 can be viewed as one of three results yielding at least almost a stochastic ordering $|S| \geq_{\text{st}} |T|$ for sums S, T of independent symmetric random variables assuming a corresponding ordering of their terms, the other two results being theorems of Pruss [12] and Birnbaum [3]. It therefore appears natural to summarize these results here, and to use this opportunity to extend Birnbaum's theorem to the lattice case.

Let us agree on the following unimodality definitions for laws P on $\mathbf{R}$. We call P *unimodal on $\mathbf{R}$* , if P is unimodal in the usual sense that, for some $x_0 \in \mathbf{R}$, the distribution function of P is convex on $]-\infty, x_0[$ and concave on $]x_0, \infty[$. For $a \in \mathbf{R}$ and $h \in]0, \infty[$, we call P *unimodal on $h\mathbf{Z} + a$* , if $P(h\mathbf{Z} + a) = 1$ and if there is a $k_0 \in \mathbf{Z}$ such that $k \mapsto P(\{hk + a\})$ is increasing on $\{k \in \mathbf{Z} : k \leq k_0\}$ and decreasing on $\{k \in \mathbf{Z} : k \geq k_0\}$. For $h \in [0, \infty[$, we call P *unimodal with span h* , if either $h = 0$ and P is unimodal on $\mathbf{R}$, or $h > 0$ and P is unimodal on $h\mathbf{Z} + a$ for some $a \in \mathbf{R}$. As usual, we attribute any property just defined to a random variable X , if its distribution enjoys it.

Theorem 3.1. Let $n \in \mathbf{N}$, and let $X_1, \dots, X_n$ as well as $Y_1, \dots, Y_n$ be independent and symmetrically distributed $\mathbf{R}$ -valued random variables with sums $S = \sum_{i=1}^n X_i$ and $T = \sum_{i=1}^n Y_i$ and with

$$|X_i| \geq_{\text{st}} |Y_i| \quad (i = 1, \dots, n). \quad (18)$$

(a) (Pruss, [12]) Then

$$\mathbf{P}\{|S| \geq t\} \geq \frac{1}{2} \mathbf{P}\{|T| \geq t\} \quad (t > 0).$$

(b) If $h \in]0, \infty[$ and $\mathbf{P}\{Y_i \in \{-h, 0, h\}\} = 1$ for $i = 1, \dots, n$, then

$$\mathbf{P}\{|S| > mh\} + \frac{1}{2} \mathbf{P}\{|S| = mh\} \geq \mathbf{P}\{|T| > mh\} + \frac{1}{2} \mathbf{P}\{|T| = mh\} \quad (m \in \mathbf{N}) \quad (19)$$

(c) (Generalization of the result of Birnbaum [3]) Let $h \in [0, \infty[$, and let $X_1, \dots, X_n, Y_1, \dots, Y_n$ be unimodal with span h . In case of $h > 0$ assume further for each $i \in \{1, \dots, n\}$ that X_i, Y_i are both $h\mathbf{Z}$ -valued or both $h(\mathbf{Z} + \frac{1}{2})$ -valued. Then $|S| \geq_{\text{st}} |T|$.

See [1, Theorem 1.1] for a further related comparison theorem.

E x a m p l e. Let $n = 2$, $X_1, X_2, Y_1 \sim (\delta_{-1} + \delta_1)/2$, and $Y_2 = 0$. Then $|X_i| \geq_{\text{st}} |Y_i|$ for $i = 1, 2$. Since $\mathbf{P}\{|S| \geq 1\} = \frac{1}{2}$ and $\mathbf{P}\{|T| \geq 1\} = 1$, it follows that the constant $\frac{1}{2}$ in Pruss' theorem is best possible. As each of the four random variables is unimodal with span 2, it also follows that the second sentence in part (c) cannot be omitted. Further, in this example, $\mathbf{P}\{Y_i \in \{-1, 0, 1\}\} = 1$ for $i = 1, 2$ but $\mathbf{P}\{S > 0\} + \mathbf{P}\{S = 0\}/2 = \frac{3}{4} \neq 1 = \mathbf{P}\{T > 0\} + \mathbf{P}\{T = 0\}/2$, showing that in (19) we may not replace $\mathbf{N}$ by $\mathbf{N}_0$.

P r o o f. (a) See [12].

(b) Here (18) is equivalent to (17) with $p_i = \mathbf{P}\{|Y_i| = h\}$, so that Corollary 2.1 with $H = mh$ and $a = 0$ yields (19).

(c) Induction based on Lemmas 3.1 and 3.2 given below. In the step from $n-1$ to n , we may assume $X_1, \dots, X_n, Y_1, \dots, Y_n$ to be independent, and conclude that

$$\left| \sum_{i=1}^n X_i \right| \geq_{\text{st}} \left| \sum_{i=1}^{n-1} X_i + Y_n \right| \geq_{\text{st}} \left| \sum_{i=1}^n Y_i \right|$$

by applying Lemma 3.1 first to $U_1 := \sum_{i=1}^{n-1} X_i$, $V_1 := X_n$, $W_1 := Y_n$ and then to $U_2 := Y_n$, $V_2 := \sum_{i=1}^{n-1} X_i$, $W_2 := \sum_{i=1}^{n-1} Y_i$, observing that by Lemma 3.2 the sum U_1 is symmetric and unimodal with span h , and that in case of $h > 0$ the sums V_2, W_2 are both $h\mathbf{Z}$ -valued or both $h(\mathbf{Z} + \frac{1}{2})$ -valued. Theorem 3.1 is proved.

Lemma 3.1. *Let U, V, W be symmetrically distributed $\mathbf{R}$ -valued random variables with U, V independent, U, W independent, and $|V| \geq_{\text{st}} |W|$. Let $h \in [0, \infty[$ with U unimodal with span h . In case of $h > 0$ let further V, W be both $h\mathbf{Z}$ -valued or both $h(\mathbf{Z} + \frac{1}{2})$ -valued. Then $|U + V| \geq_{\text{st}} |U + W|$.*

P r o o f. We may assume that $h \in \{0, 1\}$. In case of $h = 0$ we put $A := B := [0, \infty[$, while for $h = 1$ we let $A, B \in \{\mathbf{N}_0, \mathbf{N}_0 + \frac{1}{2}\}$ with $\mathbf{P}\{|U| \in A\} = \mathbf{P}\{|V| \in B\} = 1$. Then for $t \in A + B := \{a + b\} : a \in A, b \in B$ and denoting by P_U etc. the laws of the random variables occurring as subscripts, we have

$$\begin{aligned} \mathbf{P}\{|U + V| \leq t\} &= \int_B P_U([v - t, v + t]) dP_{|V|}(v) \\ &\leq \int_B P_U([v - t, v + t]) dP_{|W|}(v) = \mathbf{P}\{|U + W| \leq t\}, \end{aligned}$$

since in each case the function $B \ni v \mapsto P_U([v - t, v + t])$ is decreasing. As $\mathbf{P}\{|U + V| \in A + B\} = 1$, this proves $|U + V| \geq_{\text{st}} |U + W|$.

Lemma 3.2 (Wintner). *Let X and Y be independent $\mathbf{R}$ -valued random variables, and let $h \in [0, \infty[$. If X and Y are symmetric and unimodal with span h , then so is $X + Y$.*

P r o o f. Obvious by writing the laws of X and Y as mixtures of uniform distributions on symmetric intervals in $\mathbf{R}$ or $h\mathbf{Z}$ or $h(\mathbf{Z} + \frac{1}{2})$. See [4, p. 13, 109] for the cases where $h = 0$ or X and Y are both symmetric unimodal on $h\mathbf{Z}$. The remaining three cases are analogous.

4. Historical notes. Theorem 2.1 in the Hilbert space case, and assuming the sets C_j to be slightly smaller than necessary, was proved by Kleitman [9], generalizing several earlier results and in particular the one-dimensional case due to Erdős [5, Theorems 1 and 3]. Jones [6, p. 4, footnote 7] observed that Kleitman's result and proof extends to general (semi-)normed spaces. Meanwhile, Kanter [8, Lemma 4.1] proved a weaker result, assuming in particular symmetry of the sets C_j . The present proof of Theorem 2.1 is just a slightly refined rewrite of Kleitman's proof and Jones' footnote.

Kanter [8] essentially stated and proved Theorem 2.2 for $m = 1$ and C_1 symmetric. Le Cam [7, p. 408–409] adopted Kanter's approach.

Theorem 3.1 (c) in the case of $h = 0$ and without atoms at zero is due to Birnbaum [3]. Bickel and Lehmann [2] and Shaked and Shantikumar [14, p. 78] allowed atoms at zero in their statements, but apparently not in their proofs. Sherman [15] extended Birnbaum's result to the absolutely continuous multivariate case. Dharmadhikari and Joag-Dev [4, p. 164] gave an elegant development of Sherman's theorem, dispensing with unnecessary continuity assumptions. They also essentially stated without proof Theorem 3.1(c) for $h > 0$ in the case where all random variables are $h\mathbf{Z}$ -valued.

СПИСОК ЛИТЕРАТУРЫ

1. Berger E. Comparing sums of independent bounded random variables and sums of Bernoulli random variables. — Statist. Probab., 1997, v. 34, № 3, p. 251–258.

2. *Bickel P. J., Lehmann E. L.* Descriptive statistics for nonparametric models. III. Dispersion. — *Ann. Statist.*, 1976, v. 4, № 6, p. 1139–1158.
3. *Birnbaum Z. W.* On random variables with comparable peakedness. — *Ann. Math. Statist.*, 1948, v. 19, p. 76–81.
4. *Dharmadhikari S., Joag-Dev K.* Unimodality, Convexity, and Applications. Boston: Academic Press, 1988, 278 p.
5. *Erdős P.* On a lemma of Littlewood and Offord. — *Bull. Amer. Math. Soc.*, 1945, v. 51, p. 898–902.
6. *Jones L.* On the distribution of sums of vectors. — *SIAM J. Appl. Math.*, 1978, v. 34, № 1, p. 1–6.
7. *Le Cam.* Asymptotic Methods in Statistical Decision Theory. New York: Springer-Verlag, 1986, 742 p.
8. *Kanter M.* Probability inequalities for convex sets and multidimensional concentration functions. — *J. Multivariate Anal.*, 1976, v. 6, № 2, p. 222–236.
9. *Kleitman D. J.* On a lemma of Littlewood and Offord on the distributions of linear combinations of vectors. — *Adv. in Math.*, 1970, v. 5, p. 155–157.
10. *Mattner L., Roos B.* A shorter proof of Kanter's Bessel function concentration bound. *Probab. Theory Related Fields*, 2007, v. 139, № 1–2, p. 191–205.
11. *Нагаев С. В.* Нижние границы для вероятностей больших отклонений сумм независимых случайных величин. — *Теория вероятн. и ее примен.*, 2001, т. 46, в. 4, с. 785–792.
12. *Pruss A. R.* Comparisons between tail probabilities of sums of independent symmetric random variables. — *Ann. Inst. Henri Poincaré*, 1997, v. 33, № 5, p. 651–671.
13. *Rudin W.* Functional Analysis. 2nd ed. New York: McGraw-Hill, 1991, 424 p.
14. *Shaked M., Shantikumar J. G.* Stochastic Orders and their Applications. Boston: Academic Press, 1994, 545 p.
15. *Sherman S.* A theorem on convex sets with applications. — *Ann. Math. Statist.*, 1955, v. 26, p. 763–767.

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ON THE SECOND ORDER PROPERTIES OF THE MULTIDIMENSIONAL PERIODOGRAM FOR REGULARLY SPACED DATA¹⁾

В последние годы наметился возрастающий интерес к расширению наших знаний о пространственных и пространственно-временных процессах с помощью спектральной техники. Свойства оценки спектральной плотности — периодограммы — были широко изучены при различных асимптотических предположениях, которые влекут существенную потерю информации о поведении исходного процесса, зачастую наблюдаемого на решетке малого размера и с разреженными

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