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W. Bryc, A. Dembo, A. Kagan, On the maximum correlation coefficient, *Теория вероятн. и ее примен.*, 2004, том 49, выпуск 1, 191–197

DOI: <https://doi.org/10.4213/tvp246>

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ON THE MAXIMUM CORRELATION COEFFICIENT¹⁾

Для произвольного случайного вектора (X, Y) и независимой случайной величины Z показывается, что максимальный коэффициент корреляции между X и $Y + \lambda Z$ как функция λ всюду полунепрерывен снизу и непрерывен в нуле, где и достигает своего максимума. Более того, если Z принадлежит классу саморазложимых случайных величин, то максимальный коэффициент корреляции непрерывен справа и не возрастает при $\lambda \geq 0$ и непрерывен слева и не убывает при $\lambda \leq 0$. Независимые случайные величины X и Z являются гауссовскими тогда и только тогда, когда максимальный коэффициент корреляции между X и $X + \lambda Z$ равен линейной корреляции между ними. Максимальный коэффициент корреляции между суммой n независимых одинаково распределенных случайных величин и суммой первых $m < n$ из них равен $\sqrt{m/n}$ (ранее этот результат был доказан только для случайных величин с конечными вторыми моментами, в этом случае максимальный коэффициент корреляции также совпадает с линейной корреляцией). Приводятся примеры, противоречащие интуитивным представлениям о поведении максимального коэффициента корреляции для Z более общего вида и в пределе при $\lambda \rightarrow \infty$.

Ключевые слова и фразы: зависимость, максимальная корреляция, саморазложимые случайные величины.

1. Introduction and statement of results. The maximum correlation coefficient between two random elements ξ, η introduced in [6] and [5] is

$$\rho(\xi, \eta) = \sup \left\{ \text{corr}(\varphi(\xi), \psi(\eta)) : 0 < \mathbf{E}|\varphi(\xi)|^2 < \infty, 0 < \mathbf{E}|\psi(\eta)|^2 < \infty \right\}, \quad (1)$$

where $\text{corr}(X, Y)$ is the classical (Pearson) correlation between random variables X and Y . The definition (1) is equivalent to

$$\rho(\xi, \eta) = \sup \mathbf{E}\{\varphi(\xi) \psi(\eta)\}, \quad (2)$$

where the supremum in (2) is taken over all φ, ψ with

$$\mathbf{E}\varphi(\xi) = \mathbf{E}\psi(\eta) = 0, \quad \mathbf{E}|\varphi(\xi)|^2 = \mathbf{E}|\psi(\eta)|^2 = 1. \quad (3)$$

Geometrically, $\rho(\xi, \eta)$ equals the cosine of the angle between the subspaces (of a larger Hilbert space $L^2(\xi, \eta)$) $L^2(\xi) = \{\varphi(\xi) : \mathbf{E}\varphi = 0, \mathbf{E}|\varphi|^2 < \infty\}$ and $L^2(\eta) = \{\psi(\eta) : \mathbf{E}\psi = 0, \mathbf{E}|\psi|^2 < \infty\}$. Another well-known interpretation of ρ is as the operator norm of the conditional expectation $\varphi \mapsto \mathbf{E}(\varphi(\xi) | \eta)$ acting on the closed subspace of L_2 consisting of functions orthogonal to constants. Thus

$$\rho^2(\xi, \eta) = \sup \left\{ \mathbf{E} \left\{ |\mathbf{E}(\varphi(\xi) | \eta)|^2 \right\} : \mathbf{E}\varphi(\xi) = 0, \mathbf{E}|\varphi(\xi)|^2 = 1 \right\}. \quad (4)$$

The main role of $\rho(\xi, \eta)$ is that of a convenient numerical measure of dependence between ξ and η . In particular, $\rho(\xi, \eta)$ vanishes if and only if ξ and η are independent.

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¹⁾ Research of the second author was partially supported by NSF (grant DMS- 0072331).

Explicit formulas for $\rho(\xi, \eta)$ are available in very few cases. If (X, Z) is a bivariate Gaussian vector, then

$$\rho(X, Z) = |\text{corr}(X, Z)| \quad (5)$$

(for a proof see, for example, [7]).

If $X = X_1 + \dots + X_m$, $Z = X_1 + \dots + X_n$, $m \leq n$, where $X_1, \dots, X_n$ are independent identically distributed, nondegenerate random variables with finite second moment, then the maximum correlation between X and Z

$$\rho(X, Z) = \sqrt{mn^{-1}} \quad (6)$$

(see [4]) and, thus, does not depend on the distribution of X_i . In Corollary 1 below, we show that (6) applies also for any independent identically distributed, nondegenerate random variables X_i . There are a few other isolated cases when $\rho(X, Y)$ is known in an explicit form. Different properties of the maximum correlation were studied in [11], [12], [9], [3], [1].

If X, Y, Z are Markov-dependent, then it is easy to see that

$$\rho(X, Y) \geq \rho(X, Y + Z). \quad (7)$$

Indeed, since $L^2(Y + Z) \subset L^2(Y, Z)$, one has

$$\rho(X, (Y, Z)) \geq \rho(X, Y + Z).$$

From Markov property $\mathbf{E}(\varphi(X) | Y, Z) = \mathbf{E}(\varphi(X) | Y)$ which by (4) implies the well-known formula

$$\rho(X, Y) = \rho(X, (Y, Z))$$

(cf. proof of Lemma 1 in [10, p. 207]). Thus (7) follows.

If bivariate random vectors (X_1, Y_1) and (X_2, Y_2) are independent, then

$$\max\{\rho(X_1, Y_1), \rho(X_2, Y_2)\} \geq \rho(X_1 + X_2, Y_1 + Y_2). \quad (8)$$

This follows from the fact that $\rho(X_1 + X_2, Y_1 + Y_2) \leq \rho((X_1, X_2), (Y_1, Y_2))$ and from the Csáki-Fisher identity

$$\rho((X_1, X_2), (Y_1, Y_2)) = \max\{\rho(X_1, Y_1), \rho(X_2, Y_2)\},$$

see Theorem 1 in [13]. Inequality (8) yields two implications in the same spirit as (7).

(i) If (X', Y') is an independent copy of (X, Y) , then

$$\rho(X, Y) \geq \rho(X + X', Y + Y'). \quad (9)$$

(ii) If (X, Y) is an arbitrary bivariate random vector and Z_1, Z_2 are independent of each other and of (X, Y) , then

$$\rho(X, Y) \geq \rho(X + Z_1, Y + Z_2).$$

We prove in this paper the following general properties of $\rho(X, Y + \lambda Z)$.

Theorem 1. *The function $\lambda \mapsto \rho(X, Y + \lambda Z)$ is lower semi-continuous in λ for any random variables X, Y, Z . In particular, if Z is independent of the pair (X, Y) , then $\rho(X, Y + \lambda Z)$ is continuous at $\lambda = 0$.*

A random variable Z is in $\mathcal{L}$ if for any c , $0 < c < 1$, there exists a random variable U_c independent of Z such that

$$Z \text{ is equidistributed with } cZ + U_c. \quad (10)$$

Equivalently, a real-valued random variable Z belongs to the class $\mathcal{L}$ if its characteristic function $f(t) = \mathbf{E}e^{itZ}$, $t \in \mathbf{R}$, possesses the following property: for any c , $0 < c < 1$, there exists a characteristic function $f_c(t)$ such that

$$f(t) = f(ct)f_c(t), \quad t \in \mathbf{R}. \quad (11)$$

The random variables in $\mathcal{L}$ are called self-decomposable. All random variables in $\mathcal{L}$ are infinitely divisible. Necessary and sufficient conditions (in terms of Lévy functions) are known for an infinitely divisible random variable to belong to $\mathcal{L}$ (see [8, Ch. 5]). In particular, all random variables having stable distributions are in $\mathcal{L}$.

We next detail additional properties of $\rho(X, Y + \lambda Z)$ in case Z is independent of (X, Y) and belongs to the class $\mathcal{L}$.

Theorem 2. *If a random variable $Z \in \mathcal{L}$ is independent of a bivariate random vector (X, Y) , then $\lambda \mapsto \rho(X, Y + \lambda Z)$ is a nonincreasing right continuous function on $[0, \infty)$ and a nondecreasing left continuous function on $(-\infty, 0]$.*

The above results hold for random elements X, Y, Z taking values in an arbitrary separable Banach space; the proofs remain the same.

The following theorem is a converse of (5) holding when X and Z are Gaussian.

Theorem 3. *If X and Z are independent, nondegenerate, square-integrable real-valued random variables such that for every real λ*

$$\rho(X, X + \lambda Z) = \text{corr}(X, X + \lambda Z), \quad (12)$$

then X and Z are Gaussian.

Dembo and co-authors [4] show that equality (12) may hold true for a fixed $\lambda \neq 0$ with independent, nondegenerate square-integrable *non-Gaussian* X and Z , see (6).

Our following result provides the value of $\rho(X, X + \lambda Z)$ in case both X and Z are symmetric α -stable random variables.

Theorem 4. *Suppose X and Z are independent copies of an α -stable random variable, $\alpha \in (0, 2]$. Then*

$$\rho(X, X + \lambda Z) = \frac{1}{\sqrt{1 + |\lambda|^\alpha}}, \quad (13)$$

for all $\lambda \geq 0$. If X and Z are symmetric, the equality (13) holds also for $\lambda < 0$.

The following lemma which is key to the proof of Theorem 4, is of independent interest.

Lemma 1. *Suppose X and Y are nondegenerate independent random variables with characteristic functions $\varphi_X(t)$ and $\varphi_Y(t)$ such that*

$$\liminf_{t \rightarrow 0} \frac{1 - |\varphi_Y(t)|^2}{1 - |\varphi_X(t)|^2} = c. \quad (14)$$

Then

$$\rho(X, X + Y) \geq \frac{1}{\sqrt{1 + c}}. \quad (15)$$

Suppose X_j are independent identically distributed, nondegenerate random variables with characteristic function $\varphi(t)$ (possibly with infinite second moment). The independent random variables $X = \sum_{j=1}^m X_j$ and $Y = \sum_{j=m+1}^n X_j$ have characteristic functions $\varphi_X(t) = \varphi^m(t)$ and $\varphi_Y(t) = \varphi^{n-m}(t)$. Applying Lemma 1 for the pair (X, Y) , where $c = (n - m)/m$ by the continuity of $|\varphi(t)|^2$ at $t = 0$, we get that

$$\rho(X, X + Y) \geq \sqrt{mn^{-1}}. \quad (16)$$

Combining this lower bound with the upper bound of inequality (19) of [4], we get the following corollary.

Corollary 1. *The equality (6) holds for any nondegenerate independent identically distributed $X_1, \dots, X_n$.*

2. Proofs.

P R O O F O F T H E O R E M 1. Note that if $\psi_n \rightarrow \psi$ in L^2 and $\varphi_n \rightarrow \varphi$ in L^2 , with both φ and ψ nonzero (in L^2), then $\text{corr}(\varphi_n, \psi_n) \rightarrow \text{corr}(\varphi, \psi)$. Consequently, in the definition (1) it suffices to consider φ and ψ in the dense subsets BL of $L^2(\xi)$ and $L^2(\eta)$, consisting of bounded Lipschitz functions, using the notation $\|\psi\|_{BL} = \|\psi\|_\infty + \|\psi\|_{\text{Lip}}$. Fix φ, ψ bounded and Lipschitz having positive variances at X and $Y + tZ$, respectively. Define $\Delta(r) = \min(1, |r|\|\psi\|)$, so

$$|\psi(Y + tZ) - \psi(Y + sZ)| \leq 2\|\psi\|_{BL} \Delta(t - s),$$

implying that

$$\begin{aligned} |\text{cov}(\varphi, \psi(Y + tZ)) - \text{cov}(\varphi, \psi(Y + sZ))| &\leq 4\|\varphi\|_\infty \|\psi\|_{BL} \mathbf{E}(\Delta(t - s)), \\ |\mathbf{D}(\psi(Y + tZ)) - \mathbf{D}(\psi(Y + sZ))| &\leq 8\|\psi\|_{BL}^2 \mathbf{E}(\Delta(t - s)). \end{aligned}$$

By the bounded convergence $\mathbf{E}(\Delta(t-s)) \rightarrow 0$ as $s \rightarrow t$, implying that $\mathbf{D}\{\psi(Y+sZ)\}$ is bounded away from 0 in neighborhood of t , so also

$$\text{corr}\{\varphi, \psi(Y+sZ)\} \rightarrow \text{corr}\{\varphi, \psi(Y+tZ)\} \quad \text{as } s \rightarrow t.$$

Thus

$$\lambda \mapsto \sup \left\{ \text{corr}(\varphi, \psi(Y+\lambda Z)): \mathbf{D}(\varphi(X)), \mathbf{D}(\psi(Y+\lambda Z)) \text{ are finite} \right\}$$

is lower semi-continuous. This ends the proof of the first part of the theorem.

Combining the first part of the theorem and inequality (7) for λZ , we have

$$\rho(X, Y) \geq \liminf_{\lambda \rightarrow 0} \rho(X, Y + \lambda Z) \geq \rho(X, Y),$$

proving the continuity of $\rho(X, Y + \lambda Z)$ at $\lambda = 0$ and completing the proof of Theorem 1.

P r o o f o f T h e o r e m 2. Let $\lambda_2 > \lambda_1 > 0$; write $\lambda_1 = c\lambda_2$, where $0 < c < 1$. Let U_c be a random variable independent of X, Y, Z such that $Z \cong cZ + U_c$. From (7),

$$\begin{aligned} \rho(X, Y + \lambda_1 Z) &= \rho(X, Y + \lambda_2 cZ) \geq \rho(X, Y + \lambda_2 cZ + \lambda_2 U_c) \\ &= \rho(X, Y + \lambda_2 (cZ + U_c)) = \rho(X, Y + \lambda_2 Z). \end{aligned}$$

If $\lambda_2 < \lambda_1 < 0$, then setting $\lambda'_2 = -\lambda_2$, $\lambda'_1 = -\lambda_1$ one has

$$\begin{aligned} \rho(X, Y + \lambda_1 Z) &= \rho(-X, -Y - \lambda_1 Z) = \rho(-X, -Y + \lambda'_1 Z) \\ &\geq \rho(-X, -Y + \lambda'_2 Z) = \rho(X, Y - \lambda'_2 Z) = \rho(X, Y + \lambda_2 Z). \end{aligned}$$

The inequality (7), applied for λZ , extends the above monotonicity properties of $\lambda \mapsto \rho(X, Y + \lambda Z)$ to $[0, \infty)$ and $(-\infty, 0]$, respectively. By Theorem 1 this function is lower semi-continuous in λ , hence it is right continuous wherever nonincreasing, and left continuous wherever nondecreasing.

P r o o f o f T h e o r e m 3. We may and shall assume without loss of generality that $\mathbf{E}X = \mathbf{E}Z = 0$, $\mathbf{E}X^2 = \mathbf{E}Z^2 = 1$. Using (12) with $\lambda = 1/s > 0$ we get $\rho(X, Z + sX) = \text{corr}(X, Z + sX)$ and hence

$$\mathbf{E}(X | Z + sX) = \frac{s}{1+s^2} (Z + sX) \quad (17)$$

(see [4, p. 344]). Replacing Z by $-Z$ in (12), it is easy to see that (17) holds also for $s \leq 0$. This implies

$$\mathbf{E}[X \exp\{it(Z + sX)\}] = \frac{s}{1+s^2} \mathbf{E}[(Z + sX) \exp\{it(Z + sX)\}].$$

Differentiating this relation with respect to s at $s = 0$ we get $it\mathbf{E}\{X^2 \exp(itZ)\} = \mathbf{E}\{Z \exp(itZ)\}$. Since X, Z are independent and $\mathbf{E}X^2 = 1$, this shows that the characteristic function $\varphi(t) = \mathbf{E} \exp(itZ)$ satisfies the differential equation $\varphi'(t) = -t\varphi(t)$, and hence Z is Gaussian. With $u = 1/s$, it follows from (17) that

$$\mathbf{E}(Z | X + uZ) = \mathbf{E}(Z | Z + sX) = (Z + sX) - s\mathbf{E}(X | Z + sX) = \frac{u}{1+u^2} (X + uZ).$$

Reversing the roles of X and Z , by the same argument as before X is also Gaussian.

P r o o f o f L e m m a 1. Recall that for characteristic function $|\varphi(t)|^2 \leq 1$. So, fixing $t \in \mathbf{R}$ such that $|\varphi_X(t)| \neq 1$, and considering separately the real and imaginary parts of $f(x) = e^{itx}$, it is easy to check that (4) implies

$$\rho^2(X, X + Y) \geq \frac{\mathbf{E}|\mathbf{E}(f(X+Y)|X)|^2 - |\mathbf{E}f(X+Y)|^2}{\mathbf{E}|f(X+Y)|^2 - |\mathbf{E}f(X+Y)|^2}.$$

We have $|\mathbf{E}f(X+Y)|^2 = |\varphi_X(t)\varphi_Y(t)|^2$,

$$|\mathbf{E}(f(X+Y)|X)|^2 = |\varphi_Y(t)|^2, \quad |f(X+Y)|^2 = 1.$$

Thus, if in addition $|\varphi_Y(t)| \neq 0$,

$$\rho^2(X, X + Y) \geq \frac{|\varphi_Y(t)|^2(1 - |\varphi_X(t)|^2)}{1 - |\varphi_X(t)|^2|\varphi_Y(t)|^2} = \frac{1}{1 + (|\varphi_Y(t)|^{-2} - 1)/(1 - |\varphi_X(t)|^2)}.$$

Taking now the limsup of the right-hand side as $t \rightarrow 0$, we get the conclusion (15) out of our assumption (14).

P r o o f o f T h e o r e m 4. Applying Lemma 1 to $Y = \lambda Z$ we get that

$$\rho(X, X + \lambda Z) \geq \frac{1}{\sqrt{1 + |\lambda|^\alpha}}. \quad (18)$$

If X and Z are symmetric, the pairs $(X, X + \lambda Z)$ and $(X, X - \lambda Z)$ have the same distribution. Hence, it suffices to prove the converse of (18) for $\lambda > 0$. To this end, fix $0 < \varepsilon < \lambda$ and let $m < n$ be positive integers such that $\lambda - \varepsilon < (n/m - 1)^{1/\alpha} < \lambda$. Then, by Theorem 2 (and by the invariance of ρ under nondegenerate linear transformations),

$$\begin{aligned} \rho(X, X + \lambda Z) &\leq \rho(X, X + (nm^{-1} - 1)^{1/\alpha} Z) \\ &= \rho(m^{1/\alpha} X, m^{1/\alpha} X + (n - m)^{1/\alpha} Z) = \rho(S_m, S_n), \end{aligned}$$

where S_n denotes the sum of n independent copies of the α -stable random variable X . Therefore, inequality (19) in [4] gives

$$\rho(X, X + \lambda Z) \leq \sqrt{\frac{m}{n}} \leq \frac{1}{\sqrt{1 + (\lambda - \varepsilon)^\alpha}}.$$

Since $\varepsilon > 0$ is arbitrary, this ends the proof.

3. Some counterintuitive examples. Here some examples are constructed that demonstrate counterintuitive features of the maximum correlation.

E x a m p l e 1. Let Z be a nondegenerate random variable independent of (X, Y) . As $|\lambda| \rightarrow \infty$, one may ask when X and $Y + \lambda Z$ become «asymptotically independent», i.e.,

$$\lim_{|\lambda| \rightarrow \infty} \rho(X, Y + \lambda Z) = 0. \quad (19)$$

From [2] it follows that for bounded φ, ψ

$$\lim_{|\lambda| \rightarrow \infty} \text{cov}\{\varphi(X), \psi(Y + \lambda Z)\} = 0$$

if Z has a density, and that

$$\lim_{|\lambda| \rightarrow \infty} \text{cov}\{\varphi(X), \psi(Y + \lambda Z)\} = \text{cov}\{\varphi(X), \psi(Y)\}$$

if Z is discrete. The latter shows that (19) does not hold in general. Here is a related explicit example. Let X be an arbitrary (nondegenerate) random variable, let the distribution of Y be concentrated on $[-\frac{1}{2}, \frac{1}{2}]$ and let Z be a binary random variable taking values -1 and $+1$. For any (known) λ with $|\lambda| > 1$, Y can be reconstructed from $Y + \lambda Z$ implying that for $|\lambda| > 1$, $\rho(X, Y + \lambda Z) = \rho(X, Y)$ and does not go to 0 as $|\lambda| \rightarrow \infty$.

E x a m p l e 2. These X, Y, Z would give an example of $\rho(X, Y + \lambda Z)$ that does not decrease monotonically in $\lambda \in (0, \infty)$ unless $\rho(X, Y + \lambda Z) \equiv \rho(X, Y)$.

We shall now show that if $X = Y$ with $\mathbf{P}\{X = -\frac{1}{2}\} = \mathbf{P}\{X = \frac{1}{2}\} = \frac{1}{2}$ and Z is independent of X with $\mathbf{P}\{Z = -1\} = \mathbf{P}\{Z = 1\} = \frac{1}{2}$, then $\rho(X, X + \lambda Z) = 1/\sqrt{2} < 1$ for $\lambda = \frac{1}{2}$.

The random variable $X + Z/2$ takes values $-1, 0, +1$ with probabilities $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$, respectively. In view of (3), one may always assume

$$\varphi\left(-\frac{1}{2}\right) = -1, \quad \varphi\left(\frac{1}{2}\right) = 1.$$

Then $\mathbf{E}(\varphi(X)|X + Z/2 = \pm 1) = \pm 1$ and $\mathbf{E}(\varphi(X)|X + Z/2 = 0) = 0$. Thus $\rho^2 = \sup_{\varphi} \mathbf{E}\{[\mathbf{E}(\varphi(X)|X + Z/2)]^2\} = \frac{1}{2}$ by (4).

E x a m p l e 3. The random variables X, Z from Example 2 are also an example of $\rho(X, X + \lambda Z)$ that, as a function of λ , is discontinuous at $\lambda = \frac{1}{2}$ and $-\frac{1}{2}$. Indeed, for any λ with $|\lambda| \neq \frac{1}{2}$, X can be reconstructed from $X + \lambda Z$ whence $\rho(X, X + \lambda Z) = 1$, $|\lambda| \neq \frac{1}{2}$, while $\rho(X, X \pm Z/2) < 1$ as shown above.

This construction can easily be generalized to X and Z with finite > 2 number of values such that the continuity of $\rho(X, X + \lambda Z)$ fails to hold at prescribed $\lambda_i > 0$, $i = 1, \dots, k$.

Taking X, Z as above and Y concentrated on $[-\frac{1}{6}, \frac{1}{6}]$, independent of Z , such that $1 > \rho(X, Y) > 1/\sqrt{2}$, we now see by Example 1 that for $\lambda = \frac{1}{2}$,

$$\rho(X, Y + \lambda Z) = \rho(X, Y) > \rho(X, X + \lambda Z),$$

that is, X and $Y + \lambda Z$ depend stronger than X and $X + \lambda Z$ at $\lambda = \frac{1}{2}$, whereas the opposite relationship holds at $\lambda = 0$.

Example 4. The asymptotic independence (19) may fail even when $X = Y$ and Z are both in $\mathcal{L}$. Indeed, let X be α -stable random variable and let Z be β -stable random variable independent of X , for some $0 < \alpha < \beta \leq 2$ ($\beta = 2$ in case Z is normal). With (X', Z') denoting an independent copy of (X, Z) , the distribution of $(X + X', Z + Z')$ equals up to a nonrandom constant, that of $(2^{1/\alpha}X, 2^{1/\beta}Z)$. Hence, by (9),

$$\rho(X, X + \lambda Z) \geq \rho(X + X', X + X' + \lambda(Z + Z')) = \rho(X, X + \lambda 2^{1/\beta-1/\alpha}Z).$$

Since $2^{1/\beta-1/\alpha} < 1$, Theorem 2 provides the reverse inequality, implying that $\rho(X, X + \lambda Z)$ is constant on $(0, \infty)$ and constant on $(-\infty, 0)$. By Theorem 1 this function of λ is continuous at $\lambda = 0$, hence $\rho(X, X + \lambda Z) = \rho(X, X) = 1$. Obviously, (19) fails to hold in this case.

4. Open problems. I. As shown in Theorem 2, $Z \in \mathcal{L}$ implies monotonicity of $\rho(X, Y + \lambda Z)$ in λ for any (X, Y) independent of Z .

It is interesting to find out whether the condition $Z \in \mathcal{L}$ is sufficient for continuity of $\rho(X, Y + \lambda Z)$ at $\lambda \neq 0$. More generally, what is the class of Z for which monotonicity or continuity of $\lambda \mapsto \rho(X, Y + \lambda Z)$ apply for all (X, Y) ?

II. More generally, one may consider the properties of

$$\rho_{p,q}(\xi, \eta) = \sup \left\{ \frac{\text{cov}(U, V)}{\|U\|_p \|V\|_q} : U \in L^p(\xi), V \in L^q(\eta), U, V \neq 0 \right\}$$

for $1/p + 1/q \leq 1$, where $\rho = \rho_{2,2}$. Of particular interest are $\rho_{\infty,\infty}$ corresponding to strong mixing, and $\rho_{\infty,1}$ and $\rho_{1,\infty}$ corresponding to uniform strong mixing.

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Поступила в редакцию
1.IV.2003

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REFINED LARGE DEVIATIONS FOR VON MISES STATISTICS

Приводятся достаточные условия выполнения принципа больших уклонений для вещественнозначной статистики Мизеса, улучшающие предыдущие результаты. В качестве следствия получаются достаточные условия выполнения принципа больших уклонений для U -статистики со значениями в банаховом пространстве. Эти условия также улучшают полученные ранее результаты.

Доказательства основаны на результатах Арконеса о больших уклонениях для случайных процессов и спектральном разложении функции ядра для статистики Мизеса и U -статистики соответственно.

Ключевые слова и фразы: большие уклонения, статистика Мизеса, случайные процессы.

1. Introduction. Let $(B, \|\cdot\|)$ be a real separable Banach space with a norm $\|\cdot\|$ and let $\{X_i\}_{i \in \mathbb{N}}$ be a sequence of independent, identically distributed random variables taking values in a measurable space $(S, \mathcal{S})$. For simplicity we assume that they are defined on the product space $(\Omega, \mathcal{A}, \mathbf{P}) = (S^{\mathbb{N}}, \mathcal{S}^{\otimes \mathbb{N}}, \mu^{\otimes \mathbb{N}})$, where μ is the distribution of X_1 . Denote by $\mathcal{M}_1(S)$ the space of all probability measures on $(S, \mathcal{S})$. Consider for $n \geq m$, $n, m \in \mathbb{N}$,

$$U_n(h) := U_n^m(h, \mu) := \frac{1}{\binom{n}{m}} \sum_{1 \leq i_1 < \dots < i_m \leq n} h(X_{i_1}, \dots, X_{i_m}),$$

where $h: S^m \rightarrow B$ is a Bochner integrable symmetric function. $U_n(h)$ is called a B -valued U -statistic with kernel function h and degree m . Moreover we consider the corresponding von Mises statistic

$$V_n(h) := V_n^m(h, \mu) := \frac{1}{n^m} \sum_{i_1, \dots, i_m=1}^n h(X_{i_1}, \dots, X_{i_m}).$$

In this paper we analyze the large deviations principle for these statistics.

Let us recall the definition of the large deviations principle (LDP). A sequence of probability measures $\{\mu_n\}_{n \in \mathbb{N}}$ on a topological space $\mathcal{X}$ equipped with σ -field $\mathcal{B}$ is said to satisfy the LDP with speed $a_n \downarrow 0$ and good rate function $I: \mathcal{X} \rightarrow [0, \infty]$ if the level sets $\{x: I(x) \leq \alpha\}$ are compact for all $\alpha < \infty$ and for all $\Gamma \in \mathcal{B}$ the lower bound

$$\liminf_{n \rightarrow \infty} a_n \ln \mu_n(\Gamma) \geq - \inf_{x \in \text{int}(\Gamma)} I(x)$$

and the upper bound

$$\limsup_{n \rightarrow \infty} a_n \ln \mu_n(\Gamma) \leq - \inf_{x \in \text{cl}(\Gamma)} I(x)$$

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