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P. Eichelsbacher, Refined large deviations for von Mises statistics, *Теория вероятн. и ее примен.*, 2004, том 49, выпуск 1, 197–204

DOI: <https://doi.org/10.4213/tvp247>

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26 апреля 2023 г., 13:16:12

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Поступила в редакцию
1.IV.2003

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REFINED LARGE DEVIATIONS FOR VON MISES STATISTICS

Приводятся достаточные условия выполнения принципа больших уклонений для вещественнозначной статистики Мизеса, улучшающие предыдущие результаты. В качестве следствия получаются достаточные условия выполнения принципа больших уклонений для U -статистики со значениями в банаховом пространстве. Эти условия также улучшают полученные ранее результаты.

Доказательства основаны на результатах Арконеса о больших уклонениях для случайных процессов и спектральном разложении функции ядра для статистики Мизеса и U -статистики соответственно.

Ключевые слова и фразы: большие уклонения, статистика Мизеса, случайные процессы.

1. Introduction. Let $(B, \|\cdot\|)$ be a real separable Banach space with a norm $\|\cdot\|$ and let $\{X_i\}_{i \in \mathbb{N}}$ be a sequence of independent, identically distributed random variables taking values in a measurable space $(S, \mathcal{S})$. For simplicity we assume that they are defined on the product space $(\Omega, \mathcal{A}, \mathbf{P}) = (S^{\mathbb{N}}, \mathcal{S}^{\otimes \mathbb{N}}, \mu^{\otimes \mathbb{N}})$, where μ is the distribution of X_1 . Denote by $\mathcal{M}_1(S)$ the space of all probability measures on $(S, \mathcal{S})$. Consider for $n \geq m$, $n, m \in \mathbb{N}$,

$$U_n(h) := U_n^m(h, \mu) := \frac{1}{\binom{n}{m}} \sum_{1 \leq i_1 < \dots < i_m \leq n} h(X_{i_1}, \dots, X_{i_m}),$$

where $h: S^m \rightarrow B$ is a Bochner integrable symmetric function. $U_n(h)$ is called a B -valued U -statistic with kernel function h and degree m . Moreover we consider the corresponding von Mises statistic

$$V_n(h) := V_n^m(h, \mu) := \frac{1}{n^m} \sum_{i_1, \dots, i_m=1}^n h(X_{i_1}, \dots, X_{i_m}).$$

In this paper we analyze the large deviations principle for these statistics.

Let us recall the definition of the large deviations principle (LDP). A sequence of probability measures $\{\mu_n\}_{n \in \mathbb{N}}$ on a topological space $\mathcal{X}$ equipped with σ -field $\mathcal{B}$ is said to satisfy the LDP with speed $a_n \downarrow 0$ and good rate function $I: \mathcal{X} \rightarrow [0, \infty]$ if the level sets $\{x: I(x) \leq \alpha\}$ are compact for all $\alpha < \infty$ and for all $\Gamma \in \mathcal{B}$ the lower bound

$$\liminf_{n \rightarrow \infty} a_n \ln \mu_n(\Gamma) \geq - \inf_{x \in \text{int}(\Gamma)} I(x)$$

and the upper bound

$$\limsup_{n \rightarrow \infty} a_n \ln \mu_n(\Gamma) \leq - \inf_{x \in \text{cl}(\Gamma)} I(x)$$

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hold, where $\text{int}(\Gamma)$ and $\text{cl}(\Gamma)$ denote the interior and closure of Γ , respectively. We say that a sequence of random variables satisfies the LDP when the sequence of measures induced by these variables satisfies the LDP.

The LDP of $\{U_n(h)\}_{n \in \mathbb{N}}$ have been studied already. In [5] the real-valued case $B = \mathbf{R}^d$ was considered: If h satisfies the *strong Cramér condition*, that is,

$$\int_{S^m} \exp(\theta \|h\|) d\mu^{\otimes m} < \infty \quad \text{for all } \theta > 0,$$

then the sequence $\{U_n(h)\}_{n \in \mathbb{N}}$ satisfies the LDP with rate function

$$I(x) = \inf \left\{ H(\varrho \mid \mu) : \varrho \in K_\infty, F(\varrho) = x \right\}, \quad x \in \mathbf{R}^d.$$

Here $H(\varrho \mid \mu)$ is the relative entropy of $\varrho \in \mathcal{M}_1(S)$ with respect to $\mu \in \mathcal{M}_1(S)$ defined by

$$H(\varrho \mid \mu) = \begin{cases} \int_S f \ln f d\mu, & \text{if } \varrho \ll \mu \text{ and } f = \frac{d\varrho}{d\mu}, \\ \infty & \text{otherwise,} \end{cases}$$

$K_\infty = \bigcup_{L \geq 0} K_L$ with $K_L = \{\varrho \in \mathcal{M}_1(S) : H(\varrho \mid \mu) \leq L\}$ and $F: \mathcal{M}_1(S) \cap K_\infty \rightarrow \mathbf{R}^d$ is defined by $F(\varrho) = \int_{S^m} h d\varrho^{\otimes m}$ (note that the map is well defined on K_∞). If in addition h satisfies the condition that there exists at least one $\alpha_h > 0$ such that

$$\int_{S^m} \exp(\alpha_h \|h \circ \pi_\tau\|) d\mu^{\otimes m} < \infty \quad (1.1)$$

for every map $\tau: \{1, \dots, m\} \rightarrow \{1, \dots, m\}$, where $\pi_\tau: S^m \rightarrow S^m$ is defined by $\pi_\tau(s) = (s_{\tau(1)}, \dots, s_{\tau(m)})$ for every $s = (s_1, \dots, s_m) \in S^m$, then $\{U_n(h)\}_{n \in \mathbb{N}}$ satisfies the LDP with the same rate function I . This result is given in [6].

In the case of an arbitrary real separable Banach space valued kernel h , Eichelsbacher and Schmock [6] proved the LDP for U -statistics if h satisfies the strong Cramér condition, and for von Mises statistics if in addition h satisfies condition (1.1). As one can see from the definition of the rate function I , the proofs in the papers just mentioned apply a generalized contraction principle, see Theorem 4.2.23 in [3], and use a LDP for products of empirical measures in a topology which make integration with respect to certain unbounded functions a continuous operation.

This approach gives rise to the question if $F(\cdot)$ is continuous on the level sets K_L , $L \geq 0$, of the relative entropy (continuity on the level sets is the minimal condition needed to apply the generalized contraction principle). But we obtain from Theorem 3 in [9] that continuity of $F(\cdot)$ on the level sets of $H(\cdot \mid \mu)$ is *equivalent* to the strong Cramér condition holding for h . This result shows that in case a weaker assumption than the strong Cramér condition is assumed the LDP for U -statistics cannot be deduced by known contraction techniques from Sanov's theorem, even in the real-valued case. This seems to be surprising.

In the case $m = 1$ and $h(x) = x$ it is well known that the classical Cramér theorem states that the *weak Cramér condition* $\int_S \exp(\delta \|x\|) d\mu < \infty$ for some $\delta > 0$ is a necessary and sufficient condition to have a LDP with a good rate function.

Therefore natural questions are: Do we get a LDP for von Mises statistics and U -statistics, respectively, under weaker assumptions? Which method could replace the contraction technique?

In this paper we improve previous LDP results for real-valued von Mises statistics. Surprisingly the result looks like an exercise. Applying a spectral decomposition of the kernel function h as well as a new LDP-development for stochastic processes due to Arcones [1], we give new sufficient conditions for the LDP. Moreover we will weaken the assumptions for the LDP of U -statistics. If the diagonal terms fulfill the weak Cramér condition, we can give improved sufficient conditions for the LDP of U -statistics. This is presented in Section 2.

In Section 3 we improve the LDP result for a Banach space valued kernel h given in [6]. For the proof we apply results from [1] as well as an inequality due to Hoeffding (see [7]).

2. The real valued case. Along the lines of [2] and [1] we obtain new sufficient conditions for the LDP of a real valued von Mises statistic in the case $m = 2$. We state this result and afterwards give some comments for the formulation for an arbitrary $m \geq 2$. The result relies on a LDP of sums of Banach space valued i.i.d. random variables. Let us shortly summarize the state of the art. If $(B, \|\cdot\|)$ is a real separable Banach space and if $\{X_i\}$ is a sequence of B -valued i.i.d. random variables, Sethuraman [11] and Donsker and Varadhan [4] showed that if $\mathbf{E} \exp(\lambda \|X_1\|) < \infty$ for each $\lambda > 0$, then the LDP holds for $\{(1/n) \sum_{i=1}^n X_i\}$ with speed $1/n$ and with rate function

$$I(x) = \sup \left\{ f(x) - \ln \mathbf{E} \exp(f(X_1)) : f \in B^* \right\},$$

where B^* is the dual of B . Arcones [1, Corollary 2.8] proved that the LDP holds for $\{(1/n) \sum_{i=1}^n X_i\}$ with speed $1/n$ and a good rate function if and only if there exists a $\lambda > 0$ such that $\mathbf{E} \exp(\lambda \|X\|) < \infty$, and for each $\lambda > 0$ there exists an $\eta > 0$ such that $\mathbf{E} \exp(\lambda Y^\eta) < \infty$, where

$$Y^\eta := \sup \left\{ |f_1(X_1) - f_2(X_1)| : f_1, f_2 \in B_1^*, \mathbf{E} |f_1(X_1) - f_2(X_1)| \leq \eta \right\}, \quad (2.1)$$

where B_1^* is a unit ball of B^* . In [1, Theorem 2.7] we find some formula for the rate function in this case, but it seems not to be obvious that the formula is the same as in the case where the X_i satisfies the strong Cramér condition. The sequence $\{(1/n) \sum_{i=1}^n X_i\}$ satisfies the LDP in B with speed $1/n$ if and only if

$$\left\{ \frac{1}{n} \sum_{i=1}^n f(X_i) : f \in B_1^* \right\}$$

satisfies the LDP in $l_\infty(B_1^*)$ with speed $1/n$. Here $l_\infty(B_1^*)$ denotes the set of bounded functions in B_1^* with the norm $\|x\| := \sup_{t \in B_1^*} |x(t)|$, $x \in l_\infty(B_1^*)$. Arcones [1] gave a formula for the rate function on $l_\infty(B_1^*)$: for $x \in l_\infty(B_1^*)$ he obtained

$$I(x) = \sup \left\{ I_{f_1, \dots, f_m}(x(f_1), \dots, x(f_m)) : f_1, \dots, f_m \in B_1^*, m \geq 1 \right\},$$

where

$$I_{f_1, \dots, f_m}(u_1, \dots, u_m) = \sup \left\{ \sum_{j=1}^m \lambda_j u_j - \ln \left(\mathbf{E} \exp \left(\sum_{j=1}^m \lambda_j f_j(X_1) \right) \right) : \lambda_1, \dots, \lambda_m \in \mathbf{R} \right\}.$$

For notational reasons let us consider only the case $m = 2$ in detail. Using the notation $\pi_1 h(x) := \mathbf{E}_Y h(x, Y) - \mathbf{E} h(X, Y)$ and $\pi_2 h(x, y) := h(x, y) - \mathbf{E}_Y h(x, Y) - \mathbf{E}_X h(X, y) + \mathbf{E} h(X, Y)$, where $\mathbf{E}_X$ (respectively, $\mathbf{E}_Y$) indicates expectation with respect to X (respectively, Y) only, we recall *Hoeffding's decomposition*

$$\sum_{i,j=1}^n h(X_i, X_j) = n^2 \mathbf{E} h(X, Y) + 2n \sum_{i=1}^n \pi_1 h(X_i) + \sum_{i,j=1}^n \pi_2 h(X_i, X_j)$$

as well as

$$\sum_{1 \leq i < j \leq n} h(X_i, X_j) = \sum_{1 \leq i < j \leq n} \pi_2 h(X_i, X_j) + (n-1) \sum_{i=1}^n \pi_1 h(X_i) + \binom{n}{2} \mathbf{E} h(X, Y).$$

The identities hold a.s. for $\mu^{\otimes 2}$. The kernel $\pi_2 h$ is *canonical (degenerate)* for the law of X meaning that $\mathbf{E}_X \pi_2 h(X, Y) = \mathbf{E}_Y \pi_2 h(X, Y) = 0$ a.s. Note that $\pi_1 h(X)$ is centered. If $\pi_2 h \in L_2(\mu^{\otimes 2})$, then there exist an orthonormal sequence $\{\psi_r\}_{r=1}^\infty$ of centered square-integrable functions and a sequence of real constants $\{c_r\}_{r=1}^\infty$ such that $\pi_2 h(x, y) = \sum_{r=1}^\infty c_r \psi_r(x) \psi_r(y)$ with convergence in the sense of $L_2(\mu^{\otimes 2})$. Note that the sequence of eigenvalues $\{c_r\}_{r=1}^\infty$ is square-summable (see [8, Section 3.2.2, Theorem 1]). We take $c_0 := 1$ and $\psi_0 := 2\pi_1 h - \mathbf{E} h(X, Y)$. Then

$$V_n(h) = \frac{1}{n} \sum_{i=1}^n \psi_0(X_i) + \sum_{r=1}^\infty c_r \left(\frac{1}{n} \sum_{i=1}^n \psi_r(X_i) \right)^2 \quad (2.2)$$

and

$$U_n(h) = \frac{1}{n} \sum_{i=1}^n \psi_0(X_i) + \sum_{r=1}^{\infty} c_r \left(\frac{1}{n} \sum_{i=1}^n \psi_r(X_i) \right)^2 - \sum_{r=1}^{\infty} c_r \frac{1}{n^2} \sum_{i=1}^n \psi_r(X_i)^2. \quad (2.3)$$

In what follows we will assume that h has a representation such that $V_n(h)$ and $U_n(h)$, respectively, can be represented as in (2.2) and (2.3), respectively. Denote

$$l_2 = \left\{ x = (x_p)_{p=0}^{\infty}, x_p \in \mathbf{R}, \sum_{p=0}^{\infty} x_p^2 < \infty \right\}$$

and let l_2^* be the dual space, which is l_2 . Let us denote by $\|\cdot\|_2$ the norm in l_2 . Now we consider the following conditions.

Condition 1. There exists a $\lambda > 0$ such that

$$\mathbf{E} \exp \left\{ \lambda \left(\sum_{r=0}^{\infty} |c_r| (\psi_r(X_1))^2 \right)^{1/2} \right\} < \infty.$$

Condition 2. For each $\lambda > 0$ there exists an $\eta > 0$ such that $\mathbf{E} \exp(\lambda F^\eta) < \infty$ with

$$F^\eta := \sup_{f_1, f_2 \in l_2: \|f_i\|_2 \leq 1, d(f_1, f_2) \leq \eta} \left| (f_1 - f_2) \left(\left(|c_r|^{1/2} \psi_r(X_i) \right)_{r=0}^{\infty} \right) \right|$$

and $d(f_1, f_2) := \mathbf{E} \left| \sum_{r=0}^{\infty} (f_1 - f_2)(e_r) |c_r|^{1/2} \psi_r(X_i) \right|$.

Here $e_r = (0, \dots, 0, 1, 0, \dots) \in l_2$. Now we obtain the following results.

Theorem 2.1 (LDP for von Mises statistics). *The sequence $\{V_n(h)\}_{n \in \mathbf{N}}$ satisfies a LDP with a good rate function $I(\cdot)$ if $h: S^2 \rightarrow \mathbf{R}$ satisfies Conditions 1 and 2. The rate function has the form*

$$I(x) = \inf \left\{ J(y): y \in l_2, y_0 + \sum_{r=1}^{\infty} \text{sign}(c_r) y_r^2 = x \right\}, \quad (2.4)$$

where for every $y \in l_2$, $J(y)$ is the rate function of the sequence

$$\left(|c_r|^{1/2} \frac{1}{n} \sum_{i=1}^n \psi_r(X_i) \right)_{r=0}^{\infty} \in l_2.$$

Remark 2.1. If the strong Cramér condition is fulfilled for h , that is, $\mathbf{E} \exp \left\{ \lambda \left(\sum_{r=0}^{\infty} |c_r| \psi_r^2(X_i) \right)^{1/2} \right\} < \infty$ for every $\lambda > 0$, then one obtains the well-known representation

$$J(y) := \sup_{\xi \in l_2} \left\{ \sum_{r=0}^{\infty} \xi_r y_r - \ln \mathbf{E} \exp \left\{ \sum_{r=0}^{\infty} \xi_r c_r \psi_r(X_1) \right\} \right\},$$

see the discussion in [2].

Theorem 2.2 (LDP for U -statistics). *If h satisfies Condition 1, Condition 2 and (1.1), then the sequence $\{U_n(h)\}_{n \in \mathbf{N}}$ satisfies the LDP with good rate function $I(\cdot)$ given by (2.4).*

Our theorems are improvements of all LDP results for U -statistics and von Mises statistics, that is theorem 1 in [2], Theorem 2 and Corollary 3 in [5] and Theorems (1.7), (1.10) and (1.13) in [6].

Remark 2.2. If h satisfies Condition 1, its diagonal does not satisfy the weak Cramér condition (1.1) in any case. We will give a counterexample. Let Y_k be i.i.d. random variables with $\mathcal{N}(0, k^2)$ -distribution. Independently of this sequence let Z be integer-valued with $\mathbf{P}\{Z = k\} = ce^{-k^2}$, where c is chosen such that the distribution of Z will become a probability distribution. Let X be the l_2 -valued random variable with components $X_k = Y_k$, if $Z = k$, and 0 otherwise. Let $c_r = 1$ and $\psi_r(X) = X_r$, thus we

consider the case $S = l_2$. Now we obtain that $\sum_{r=0}^{\infty} \psi_r(X) \psi_r(X) = Y_Z^2$, and therefore we get

$$\mathbf{E} \exp \left\{ \lambda \left(\sum_{r=0}^{\infty} \psi_r(X) \psi_r(X) \right)^{1/2} \right\} = \sum_{k=0}^{\infty} \mathbf{E}[e^{\lambda Y_k}, Z = k].$$

After a little calculation we obtain $\sum_{k=0}^{\infty} \mathbf{E}[e^{\lambda Y_k}, Z = k] = c \sum_{k=0}^{\infty} e^{(\lambda^2 k^2 - k^2)/2} \leq c \sum_{k=0}^{\infty} e^{-k^2/2} < \infty$ for $\lambda \leq 1$. But for any given $\lambda > 0$ we can take m such that $\lambda > \frac{1}{2} m^2$ and find

$$\mathbf{E} e^{\lambda Y_Z^2} \geq \mathbf{E}[e^{\lambda Y_m^2}, Z = m] = \infty.$$

Thus we have found an example, where Condition 1 is fulfilled, but not condition (1.1).

P r o o f o f T h e o r e m 2.1. By the assumptions of the theorem we obtain that

$$X(n) := \left(|c_r|^{1/2} \frac{1}{n} \sum_{i=1}^n \psi_r(X_i) \right)_{r=0}^{\infty} \in l_2.$$

Since l_2 is a real separable Banach space, we obtain by Arcones' [1] result that the sequence $\{X(n)\}_{n \in \mathbf{N}}$ satisfies a LDP if and only if Conditions 1 and 2 are fulfilled. Given the continuous function $F_h(y): l_2 \rightarrow \mathbf{R}$, defined by $F_h(y) := y_0 + \sum_{r=1}^{\infty} \text{sign}(c_r) y_r^2$, we get the representation

$$V_n(h) = F_h(X(n)).$$

Using the contraction principle [3, Theorem 4.2.1], the theorem follows immediately.

P r o o f o f T h e o r e m 2.2. The only thing we have to prove is that condition (1.1) for h implies that the sequences $\{U_n(h)\}_{n \in \mathbf{N}}$ and $\{V_n(h)\}_{n \in \mathbf{N}}$ have identical large deviations behavior, that is, for every $\delta > 0$ we obtain

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \ln \mathbf{P}\{\|U_n(h) - V_n(h)\| \geq \delta\} = -\infty. \quad (2.5)$$

Notice that condition (1.1) implies that $\int_{S^2} \exp(\lambda \|h(x, x)\|) d\mu < \infty$ for some $\lambda > 0$. This condition suffices to get (2.5). This is proved more generally for kernel functions with $m \in \mathbf{N}$ variables in [6, proof of Theorem (1.10)], applying Baranyai's result on the factorization of complete uniform hypergraphs.

E x a m p l e 2.1. Let us consider the *sample variance* U_n^{var} , which is a U -statistic of degree 2 with kernel function $h(x, y) = \frac{1}{2} (x - y)^2$. A simple calculation shows that $\pi_1 h(x) = \frac{1}{2} ((x - \mathbf{E}X)^2 - \mathbf{D}X)$, where $\mathbf{D}X$ denotes the variance of X under μ . Moreover $\pi_2 h(x, y) = -(x - \mathbf{E}X)(y - \mathbf{E}X)$. Thus $\psi_0(x) = ((x - \mathbf{E}X)^2 - \mathbf{D}X) - \mathbf{E}h(X, Y)$, $c_1 = -1$ and $\psi_1(x) = x - \mathbf{E}X_1$; $\psi_i = 0$ for $i \geq 2$. Let us assume that X_i is $\mathcal{N}(0, 1)$ distributed. Then the strong Cramér condition does not hold for the variance-kernel. But it is easily seen that Condition 1 is fulfilled. Remark that Condition 2 is fulfilled since every $f \in l_2$ is a linear and continuous function and $f(x) \leq \|f(x)\| \leq \beta \|x\|$ for some $\beta > 0$. Using the decomposition of h we obtain Condition 2.

The theorems in this section can be formulated and proved for von Mises statistics and U -statistics, respectively, of arbitrary degree $m \geq 2$. Therefore we need to describe the spectral decomposition of a kernel function $h: S^m \rightarrow \mathbf{R}$ and its Hoeffding decomposition. Denote $\theta := \mathbf{E}h(X_1, \dots, X_m)$ and define the *canonical* functions $h_c: S^c \rightarrow \mathbf{R}$ by the formula

$$h_c(x_1, \dots, x_c) := \int_{S^c} h(y_1, \dots, y_m) \prod_{s=1}^c (\delta_{x_s} - \mu(dy_s)) \prod_{s=c+1}^m \mu(dy_s).$$

According to the canonical Hoeffding decomposition we have the representation

$$V_n(h) = \theta + \sum_{c=1}^m \binom{m}{c} V_n(h_c).$$

We only state the spectral decomposition in the case $m = 3$. It is straightforward to adapt the technique for an arbitrary degree m , but we omit this. We already have

seen that we can find a sequence $\{\psi_{r_1}^{(2)}\}_{r_1=0}^\infty$ of centered square integrable functions and a sequence of constants $c_{r_1}^{(2)}$ such that $h_2(x, y) = \sum_{r_1=1}^\infty c_{r_1}^{(2)} \psi_{r_1}^{(2)}(x) \psi_{r_1}^{(2)}(y)$. Now we proceed iteratively to write for $h_3(x_1, x_2, x_3)$ an expansion in L_2 like $h_3(x_1, x_2, x_3) = \sum_{r_1, r_2=1}^\infty c_{r_1}^{(3)} c_{r_1, r_2}^{(3)} \psi_{r_1}^{(3)}(x_1) \psi_{r_1, r_2}^{(3)}(x_2) \psi_{r_1, r_2}^{(3)}(x_3)$, where the functions are centered square integrable. The third term in the Hoeffding decomposition can be represented as

$$\sum_{r_1, r_2=1}^\infty \text{sign}(c_{r_1, r_2}^{(3)}) \Psi_{n, r_1}^{(3)} \left(\Psi_{n, r_1, r_2}^{(3)} \right)^2$$

with $\Psi_{n, r_1}^{(3)} = c_{r_1}^{(3)} n^{-1} \sum_{j=1}^n \psi_{r_1}^{(3)}(X_j)$ and $\Psi_{n, r_1, r_2}^{(3)} = |c_{r_1, r_2}^{(3)}|^{1/2} n^{-1} \sum_{j=1}^n \psi_{r_1, r_2}^{(3)}(X_j)$. Now we can put these sample-means in a vector

$$Y(n) = \left(c_{r_1}^{(3)} \frac{1}{n} \sum_{j=1}^n \psi_{r_1}^{(3)}(X_j), |c_{r_1, r_2}^{(3)}|^{1/2} \frac{1}{n} \sum_{j=1}^n \psi_{r_1, r_2}^{(3)}(X_j) \right)$$

and proceed as in the case $m = 2$ representing any event like $\{V_n(h) \in A\}$ in the form $\{Y(n) \in F_h(A)\}$ with an appropriate function F_h .

Since (2.5) is true for any degree $m \in \mathbb{N}$, we obtain Theorems 2.1 and 2.2 for an arbitrary degree $m \in \mathbb{N}$ using the spectral decomposition just described.

3. The Banach space case. Let $(B, \|\cdot\|)$ be an arbitrary real separable Banach space and let B^* be the dual space. For B -valued U -statistics we obtain the following result.

Theorem 3.1 (LDP for Banach space valued statistics). *Given a measurable, symmetric kernel function $h: S^2 \rightarrow B$, assume that for any $f \in B^*$ the real valued function $f \circ h$ satisfies Conditions 1 and 2 and condition (1.1) (considering a decomposition for $f \circ h$ as in Section 2 for every $f \in B^*$). Moreover, assume that for any $\lambda > 0$ there exists an $\eta > 0$ such that*

$$\mathbf{E} \exp(\lambda Y^\eta) < \infty, \quad (3.1)$$

where we define

$$Y^\eta := \sup \left\{ |f_1(h(X, Y)) - f_2(h(X, Y))| : f_1, f_2 \in B^*, \|f_i\| \leq 1, \right. \\ \left. \mathbf{E} |f_1(h(X, Y)) - f_2(h(X, Y))| \leq \eta \right\}.$$

Then the sequence $\{U_n(h)\}_{n \in \mathbb{N}}$ satisfies the LDP with a good rate function, that is, the sequence $\{U_n(f \circ h), f \in T\}_{n \in \mathbb{N}}$ satisfies the LDP in $l_\infty(T)$ with a good rate function, with $T := \{f \in B^* : \|f\| \leq 1\}$.

For the proof of this theorem we will use the following result of Arcones [1, Theorem 2.1]: Let $\{X_n(t) : t \in T\}$ be a sequence of stochastic processes, where T is an index set. Let $\{\varepsilon_n\}$ be a sequence of positive numbers that converges to zero. Let $I: l_\infty(T) \rightarrow [0, \infty]$ and let $I_{t_1, \dots, t_k}: \mathbf{R}^k \rightarrow [0, \infty]$ be a function for each $t_1, \dots, t_k \in T$. Let d be a pseudo-metric in T . Consider the following conditions:

- (1) (T, d) is totally bounded;
- (2) for each $t_1, \dots, t_k \in T$, the vector $(X_n(t_1), \dots, X_n(t_k))$ satisfies the LDP with speed ε_n and good rate function $I_{t_1, \dots, t_k}$;
- (3) for each $\tau > 0$

$$\lim_{\eta \rightarrow 0} \limsup_{n \rightarrow \infty} \varepsilon_n \ln \mathbf{P}^* \left\{ \sup_{d(s, t) \leq \eta} |X_n(t) - X_n(s)| \geq \tau \right\} = -\infty.$$

Then for each $0 < \alpha < \infty$ the set $\{z \in l_\infty(T) : I(z) \leq \alpha\}$ is a compact set in $l_\infty(T)$, where

$$I(z) = \sup \left\{ I_{t_1, \dots, t_k}(z(t_1), \dots, z(t_k)) : t_1, \dots, t_k \in T, k \geq 1 \right\}.$$

Moreover one gets the upper and lower bounds in the LDP with respect to outer and inner probabilities (because of the lack of measurability): for each $A \subset l_\infty(T)$,

$$-\inf_{z \in \text{int}(A)} I(z) \leq \liminf_{n \rightarrow \infty} \varepsilon_n \ln \mathbf{P}_* \{X_n \in A\} \leq \limsup_{n \rightarrow \infty} \varepsilon_n \ln \mathbf{P}^* \{X_n \in A\} \leq -\inf_{z \in (A)} I(z).$$

P r o o f o f T h e o r e m 3.1. We will check conditions (1)–(3) of the general LDP approach for stochastic processes due to Arcones [1]: The pseudo-metric on $T = \{f \in B^*: \|f\| \leq 1\}$ is chosen to be the L_1 -pseudo-metric $d(f_1, f_2) := \mathbf{E}|f_1(h)(X, Y) - f_2(h)(X, Y)|$. Condition 1 implies that $\mathbf{E}\|h(X, Y)\| < \infty$. Given $\varepsilon > 0$ there exists a simple (measurable, finitely valued) symmetric kernel $\tilde{h}$ such that $\mathbf{E}\|h(X, Y) - \tilde{h}(X, Y)\| < \varepsilon$. It follows that $\{f(\tilde{h}(X, Y)): f \in B^*, \|f\| \leq 1\}$ is totally bounded in L_1 and therefore $\{f(h(X, Y)): f \in B^*, \|f\| \leq 1\}$ is totally bounded in L_1 .

We will check condition (2): for each $f_1, \dots, f_k \in T$, the vector $(V_n(f_1(h)), \dots, V_n(f_k(h)))$ satisfies the LDP with speed $1/n$ and a good rate function. Notice that for every $k \in \mathbf{N}$ the product space $(l_2)^k$ is a real separable Banach space. Consider for each $f_1 \circ h, \dots, f_k \circ h$ the same expansion as in Section 2, that is, $\pi_2 f_l \circ h(x, y) = \sum_{r=1}^{\infty} c_{r, f_l} \psi_{r, f_l}(x) \psi_{r, f_l}(y)$, $c_{0, f_l} = 1$, $\psi_{0, f_l} = 2\pi_1 f_l \circ h - \mathbf{E} f_l h(X, Y)$, where we skip the dependence of h in the notation. With

$$X^k(n) := \left(\left(|c_{r, f_l}|^{1/2} \frac{1}{n} \sum_{i=1}^n \psi_{r, f_l}(X_i) \right)_{r=0}^{\infty} \right)_{l=1}^k \in (l_2)^k,$$

we obtain that the sequence $\{X^k(n)\}_{n \in \mathbf{N}}$ satisfies the LDP, if Conditions 1 and 2 are fulfilled for every $f_l \circ h$, $l = 1, \dots, k$. Therefore notice that every continuous, linear map on $(l_2)^k$ is described by an element $z = (z_1, \dots, z_k) \in (l_2)^k$ via $f(x_1, \dots, x_k) = \sum_{\nu=1}^{\infty} \sum_{l=1}^k (\sum_{r=1}^{\infty} x_{\nu l} z_{\nu l})$ for every $x = (x_1, \dots, x_k) \in (l_2)^k$ with the notation $x_{\nu} = (x_{\nu r})_{r=0}^{\infty} \in l_2$. Applying the result of Arcones we obtain the LDP for the sequence $\{X^k(n)\}_{n \in \mathbf{N}}$ if Conditions 1 and 2 are fulfilled for f in the dual of $(l_2)^k$. Using the representation of each such f , both conditions follow by applying Hölder's inequality and Conditions 1 and 2, given for each $f_l \circ h$, $l = 1, \dots, k$.

Let us define the continuous function $F_h^k: (l_2)^k \rightarrow \mathbf{R}^k$ by

$$F_h^k(y_1, \dots, y_k) := \left(y_{l0} + \sum_{r=1}^{\infty} \text{sign}(c_{r, f_l}) y_{lr}^2 \right)_{l=1}^k$$

with the notation $y_l = (y_{lr})_{r=0}^{\infty}$, we obtain the representation $(V_n(f_1(h)), \dots, V_n(f_k(h))) = F_h^k(X^k(n))$. Using the contraction principle, the LDP for the vector follows. We obtain that the vector $(U_n(f_1(h)), \dots, U_n(f_k(h)))$ satisfies the LDP with the same rate function: for $F_k := (f_1 \circ h, \dots, f_k \circ h)$ we get

$$\int_{S^2} \exp(\lambda \|F_k(x, x)\|) d\mu^{\otimes 2} < \infty$$

for some $\lambda > 0$, choosing the norm $\|\cdot\|$ on $\mathbf{R}^k$ defined by $\|(x_1, \dots, x_k)\| = |x_1| + \dots + |x_k|$ and applying Hölder's inequality and condition (1.1) for every $f_l \circ h$. With the proof of Theorem (1.10) in [6] we obtain the LDP for $(U_n(f_1(h)), \dots, U_n(f_k(h)))$ as in the proof of Theorem 2.2.

In the last step we will check that

$$\lim_{\eta \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \ln \mathbf{P} \left\{ \sup_{d(f_1, f_2) \leq \eta} |U_n(f_1(h)) - U_n(f_2(h))| \geq \tau \right\} = -\infty \quad (3.2)$$

for each $\tau > 0$. This is a bit involved.

Set $h_{\eta} := \sup_{d(f_1, f_2) \leq \eta} |f_1(h) - f_2(h)|$. Note that

$$\sup_{d(f_1, f_2) \leq \eta} |U_n(f_1(h)) - U_n(f_2(h))| \leq U_n(h_{\eta}). \quad (3.3)$$

Hence by assumption (3.1), for any $\lambda > 0$, $\Psi_{h_{\eta}}(\lambda) := \mathbf{E} \exp(\lambda h_{\eta}(X_1, X_2)) = \mathbf{E} \exp(\lambda Y^{\eta}) < \infty$, for all $\eta > 0$ sufficiently small. Moreover, with $k := \lfloor n/2 \rfloor$, we obtain that

$$\mathbf{E} \exp(\lambda k U_n(h_{\eta})) \leq \Psi_{h_{\eta}}^k(\lambda)$$

(Hoeffding's inequality, see, for example, [10, Lemma C, page 200]).

Further, since $h_\eta(X_1, X_2) \xrightarrow{P} 0$ as $\eta \rightarrow 0$, for any choice of $\tau > 0$, we have for all $\eta > 0$ sufficiently small both

$$\theta(\eta) := \mathbf{E}(h_\eta(X_1, X_2)) < \frac{\tau}{2}$$

and

$$\begin{aligned} \Psi_{h_\eta}(\lambda) e^{-\lambda\theta(\eta)} &= \mathbf{E} \exp \left\{ \lambda \left(h_\eta(X_1, X_2) - \theta(\eta) \right) \right\} \\ &\leq 1 + \frac{\lambda^2}{2} \mathbf{E} \left(\left(h_\eta(X_1, X_2) - \theta(\eta) \right)^2 \exp \left(\lambda |h_\eta(X_1, X_2) - \theta(\eta)| \right) \right) \leq 1 + \frac{\lambda\tau}{4}. \end{aligned}$$

Thus

$$\begin{aligned} \mathbf{P} \left\{ U_n(h_\eta) \geq \tau \right\} &\leq \mathbf{P} \left\{ U_n(h_\eta) - \theta(\eta) \geq \frac{\tau}{2} \right\} \leq e^{-k\lambda\tau/2} \mathbf{E} \exp \left(k\lambda \left[U_n(h_\eta) - \theta(\eta) \right] \right) \\ &\leq e^{-k\lambda\tau/2} \left(\Psi_{h_\eta}(\lambda) e^{-\lambda\theta(\eta)} \right)^k \leq e^{-k\lambda\tau/2} \left(1 + \frac{\lambda\tau}{4} \right)^k \leq e^{-k\lambda\tau/4}. \end{aligned}$$

From this bound we immediately get that for each choice of $\tau > 0$ and any $\lambda > 0$

$$\lim_{\eta \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \ln \mathbf{P} \left\{ U_n(h_\eta) > \tau \right\} \leq -\frac{\lambda\tau}{8},$$

which by the arbitrary choice of λ and the inequality in (3.3) implies (3.2).

Remark 3.1. Theorem 3.1 holds for any degree $m \geq 2$. This follows using the spectral decomposition introduced in Section 2 as well as the facts that (2.5) is proved for arbitrary degree in [6] and that the decomposition arguments using Baranyai's result are applicable for any degree $m \geq 2$, see [6].

Acknowledgment. I would like to thank Prof. D. Mason for fruitful discussions concerning an easier proof of (3.2).

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Поступила в редакцию
02.IX.2002