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М. В. Cwiklińska, Asymptotics in the law of the iterated logarithm, *Теория вероятн. и ее примен.*, 2008, том 53, выпуск 4, 823–826

DOI: <https://doi.org/10.4213/tvp2470>

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ASYMPTOTICS IN THE LAW OF THE ITERATED LOGARITHM

В статье рассматривается точная асимптотика в законе повторного логарифма. Результат работы [4] обобщается на случай неодинаково распределенных случайных величин.

Ключевые слова и фразы: закон повторного логарифма, неравенство типа Фука–Нараева.

1. Introduction. Let us consider a sequence $\{X_i, i \geq 1\}$ of independent random variables such that $\mathbf{E} X_i = 0$ and $\mathbf{E} X_i^2 = \sigma_i^2 < \infty$ for $i \geq 1$. Set $S_n = X_1 + \dots + X_n$ and assume throughout the paper that $B_n^2 = \mathbf{E} S_n^2 > 0$, $n \geq 1$. Our aim is to study the asymptotic behavior of the following expression:

$$\sum_{n=n_0}^{\infty} \frac{1}{B_n^2 \ln B_n^2} \mathbf{P} \left\{ |S_n| \geq \varepsilon \sqrt{B_n^2 \ln B_n^2} \right\} \quad \text{as } \varepsilon \rightarrow 0^+, \quad (1)$$

stipulating that $n_0 = \min\{n: B_n^2 > e\} \geq 1$. In particular, $B_n^2 = n\sigma^2$ when $\mathbf{E} X_n = 0$ and $\mathbf{E} X_n^2 = \sigma^2$ for any $n \geq 1$.

We use the following Nagaev–Fuk inequality providing the upper bound for $\mathbf{P}\{|S_n| \geq x\}$, where $x > 0$, and generalized for partial sums maxima by Borovkov.

Theorem 1 (see [3], [1]). *Let $\{X_i, i \geq 1\}$ be a sequence of random variables introduced above. Then for any $x > 0$ and $y > 0$ one has*

$$\begin{aligned} \mathbf{P}\{|S_n| \geq x\} &\leq \mathbf{P}\left\{\max_{1 \leq k \leq n} |S_k| \geq x\right\} \leq \sum_{i=1}^n \mathbf{P}\{|X_i| \geq y\} \\ &\quad + 2 \left(\frac{xy}{2B_n^2} + 1 \right)^{-x/(2y)} + 2 \exp \left(-\frac{x^2}{8e^2 B_n^2} \right). \end{aligned} \quad (2)$$

2. The main result. In this section we present the main result of the paper.

Theorem 2. *Set $b(\varepsilon, M) = \exp\{\exp(M/\varepsilon^2)\}$, where M and ε are any positive constants. Assume that $\{X_i, i \geq 1\}$ is a sequence of independent random variables with $\mathbf{E} X_i = 0$ and $\mathbf{E} X_i^2 = \sigma_i^2 < \infty$ and*

$$\lim_{n \rightarrow \infty} \sup_x |\mathbf{P}\{S_n \leq x B_n\} - \Phi(x)| = 0. \quad (3)$$

If

$$\lim_{M \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0^+} \varepsilon^2 \sum_{n > b(\varepsilon, M)} \frac{1}{n \ln n} \sum_{k=1}^n \mathbf{P}\{|X_k| \geq \varepsilon B_n \sqrt{\ln \ln n}\} = 0, \quad (4)$$

then

$$\lim_{\varepsilon \rightarrow 0^+} \varepsilon^2 \sum_{n=3}^{\infty} \frac{1}{n \ln n} \mathbf{P}\{|S_n| \geq \varepsilon B_n \sqrt{\ln \ln n}\} = 1. \quad (5)$$

P r o o f. Let us introduce the function $\psi(x) = \mathbf{P}\{|Y| \geq x\} = 1 - \Phi(x) + \Phi(-x)$ for any $x \geq 0$, where Φ is the cumulative distribution function of standardized normal distribution and Y denote a random variable having standardized normal distribution. Let $\psi_\varepsilon(x) := \psi(\varepsilon \sqrt{\ln \ln x})$, $\varepsilon > 0$, $x > e$.

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Step 1. Normal distribution. Let $X_i \sim \mathcal{N}(0, \sigma_i^2)$, $i \geq 1$. Then clearly $S_n \sim \mathcal{N}(0, B_n^2)$, $i \geq 1$. Therefore,

$$\sum_{n=3}^{\infty} \frac{1}{n \ln n} \mathbf{P} \left\{ |S_n| \geq \varepsilon B_n \sqrt{\ln \ln n} \right\} = \sum_{n=3}^{\infty} \frac{1}{n \ln n} \psi_{\varepsilon}(x).$$

Using the Euler-Maclaurin formula (see [2, Chap. 2, Section 2]) we have

$$\begin{aligned} \sum_{n=3}^{\infty} \frac{1}{n \ln n} \psi_{\varepsilon}(n) &= \int_3^{\infty} \frac{1}{x \ln x} \psi_{\varepsilon}(x) dx + \frac{1}{6 \ln 3} \psi_{\varepsilon}(3) \\ &\quad - \int_3^{\infty} P_1(x) d \left[\frac{1}{x \ln x} \psi_{\varepsilon}(x) \right], \end{aligned} \quad (6)$$

where $P_1(x) = [x] - x + \frac{1}{2}$ and $[x]$ denotes the largest integer $\leq x$.

Therefore, the formula in (5) has a form

$$\lim_{\varepsilon \rightarrow 0^+} \varepsilon^2 \left(\int_3^{\infty} \frac{1}{x \ln x} \psi_{\varepsilon}(x) dx + \frac{1}{6 \ln 3} \psi_{\varepsilon}(3) - \int_3^{\infty} P_1(x) d \frac{\psi_{\varepsilon}(x)}{x \ln x} \right) = 1. \quad (7)$$

By putting $y = \varepsilon \sqrt{\ln \ln x}$, the first term on the right-hand side in (6) can be described as follows:

$$\int_3^{\infty} \frac{1}{x \ln x} \psi_{\varepsilon}(x) dx = \int_{\psi_{\varepsilon}(3)}^{\infty} 2y \frac{1}{\varepsilon^2} \psi(y) dy.$$

Therefore (see [5, Chap 2, § 6, Subsection 11, Corollary 2]),

$$\lim_{\varepsilon \rightarrow 0^+} \varepsilon^2 \int_3^{\infty} \frac{1}{x \ln x} \psi_{\varepsilon}(x) dx = \lim_{\varepsilon \rightarrow 0^+} \int_{\psi_{\varepsilon}(3)}^{\infty} 2y \psi(y) dy = \int_0^{\infty} 2y \psi(y) dy = 1.$$

For any $x \geq 0$ and Y normally distributed, $\mathcal{N}(0, 1)$, the inequality $0 \leq \psi(x) \leq 1$ holds, so the second term in (7) is equal to

$$\lim_{\varepsilon \rightarrow 0^+} \varepsilon^2 \frac{1}{6 \ln 3} \psi_{\varepsilon}(3) = 0.$$

Further we prove that the last term in (7) is zero. Let us notice that

$$\begin{aligned} \left| \int_3^{\infty} P_1(x) d \left[\frac{1}{x \ln x} \psi_{\varepsilon}(x) \right] \right| &= \left| \int_3^{\infty} \left[P_1(x) \left(-\frac{1}{x^2 \ln x} - \frac{1}{x^2 (\ln x)^2} \right) \psi_{\varepsilon}(x) \right. \right. \\ &\quad \left. \left. + \frac{1}{x \ln x} \psi'_{\varepsilon}(x) \frac{\varepsilon}{2 \sqrt{\ln \ln x}} \frac{1}{x \ln x} \right] dx \right| \\ &\leq \int_3^{\infty} |P_1(x)| \left(\frac{1}{x^2 \ln x} + \frac{1}{x^2 (\ln x)^2} \right) \psi_{\varepsilon}(x) dx \\ &\quad + \frac{\varepsilon}{2} \int_3^{\infty} |P_1(x)| \frac{|\psi'_{\varepsilon}(x)|}{x^2 (\ln x)^2 \sqrt{\ln \ln x}} dx. \end{aligned} \quad (8)$$

Since $|P_1(x)| = |[x] - x + \frac{1}{2}| \leq \frac{1}{2}$ and for any $x \geq 3$, we have $1/(\ln x)^2 \leq 1$, $1/\sqrt{\ln \ln x} \leq 1/\sqrt{\ln \ln 3}$ and $\int_3^{\infty} x^{-2} dx = \frac{1}{3}$, the expression in (8) can be bounded by

$$C_1 \int_3^{\infty} \frac{dx}{x^2} + \varepsilon C_2 \int_3^{\infty} \frac{dx}{x^2 (\ln x)^2 \sqrt{\ln \ln x}} \leq C(1 + \varepsilon).$$

Therefore, we get

$$\lim_{\varepsilon \rightarrow 0^+} \varepsilon^2 \left| \int_3^{\infty} P_1(x) d \left[\frac{1}{x \ln x} \psi_{\varepsilon}(x) \right] \right| \leq \lim_{\varepsilon \rightarrow 0^+} \varepsilon^2 (1 + \varepsilon) C = 0.$$

Taking into account obtained relations we get

$$\lim_{\varepsilon \rightarrow 0^+} \varepsilon^2 \sum_{n=3}^{\infty} \frac{1}{n \ln n} \mathbf{P} \left\{ |S_n| \geq \varepsilon B_n \sqrt{\ln \ln n} \right\} = 1. \quad (9)$$

Thus, the theorem is proved for the sequence $\{X_i, i \geq 1\}$ of independent random variables having normal distribution.

Step 2. Arbitrary distribution. Assume that $\{X_i, i \geq 1\}$ is a sequence of independent random variables such that $\mathbf{E} X_i = 0$ and $\mathbf{E} X_i^2 = \sigma_i^2 < \infty$ for $i \geq 1$. Note that for any $\varepsilon > 0$ and all $n \geq 3$

$$\Delta_n := \left| \mathbf{P} \left\{ |S_n| \geq \varepsilon B_n \sqrt{\ln \ln n} \right\} - \psi_\varepsilon(n) \right| \leq 2 \sup_x |\mathbf{P} \{S_n \leq x B_n\} - \Phi(x)| \rightarrow 0$$

as $n \rightarrow \infty$. Therefore,

$$\lim_{n \rightarrow \infty} \frac{1}{\ln \ln n} \sum_{k=3}^n \frac{\Delta_k}{k \ln k} = 0.$$

For each $M > 1$, we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \varepsilon^2 \sum_{n \leq b(\varepsilon, M)} \frac{1}{n \ln n} \Delta_n &\leq \lim_{\varepsilon \rightarrow 0^+} \varepsilon^2 \ln \ln [b(\varepsilon, M)] \frac{1}{\ln \ln [b(\varepsilon, M)]} \sum_{n \leq b(\varepsilon, M)} \frac{1}{n \ln n} \Delta_n \\ &\leq \lim_{\varepsilon \rightarrow 0^+} \varepsilon^2 \left(\frac{M}{\varepsilon^2} \right) \frac{1}{\ln \ln [b(\varepsilon, M)]} \sum_{n \leq b(\varepsilon, M)} \frac{\Delta_n}{n \ln n} = 0. \end{aligned} \quad (10)$$

For sufficiently small $\varepsilon > 0$, say, $0 < \varepsilon \leq C_0$, where C_0 is a certain fixed number small enough, the following equality holds:

$$\lim_{M \rightarrow \infty} \varepsilon^2 \sum' \frac{1}{n \ln n} \psi_\varepsilon(n) = 0, \quad (11)$$

where $\sum' = \sum_{n > b(\varepsilon, M)}^\infty$. Since $\lim_{\varepsilon \rightarrow 0^+} b(\varepsilon, M) = \infty$, there exists a constant $C_0 < 1$ such that

$$b(\varepsilon, M) - 1 \geq \sqrt{b(\varepsilon, M)} \geq 1$$

for each $\varepsilon \leq C_0$. Therefore, we have

$$\sum' \frac{1}{n \ln n} \psi_\varepsilon(n) \leq \int_{b(\varepsilon, M)-1}^\infty \frac{1}{x \ln x} \psi_\varepsilon(x) dx \leq \int_{\sqrt{b(\varepsilon, M)}}^\infty \frac{1}{x \ln x} \psi_\varepsilon(x) dx.$$

Putting $y = \varepsilon \sqrt{\ln \ln n}$ and $z = z(\varepsilon, M) = \sqrt{\varepsilon^2 \ln \frac{1}{2} + M}$ we get

$$\lim_{M \rightarrow \infty} \sup_{0 < \varepsilon \leq C_0} \varepsilon^2 \sum' \frac{1}{n \ln n} \psi_\varepsilon(n) \leq \lim_{M \rightarrow \infty} \sup_{0 < \varepsilon \leq C_0} \int_z^\infty 2y \psi(y) dy.$$

Using the fact that $\mathbf{E} Y^2 = 1$ we get

$$\begin{aligned} \lim_{M \rightarrow \infty} \sup_{0 < \varepsilon \leq C_0} \int_z^\infty 2y \psi(y) dy &\leq \lim_{M \rightarrow \infty} \int_{\sqrt{M-1}}^\infty 2y \mathbf{P} \{|Y| \geq y\} dy \\ &= \lim_{M \rightarrow \infty} \mathbf{E} Y^2 I(|Y| \geq \sqrt{M-1}) = 0. \end{aligned}$$

Therefore, equality (11) is proved.

Now we prove that for $0 < \varepsilon \leq C$, where $C > 0$ is a sufficiently small number, the following equality holds:

$$\lim_{M \rightarrow \infty} \varepsilon^2 \sum' \frac{1}{n \ln n} \mathbf{P} \left\{ |S_n| \geq \varepsilon B_n \sqrt{\ln \ln n} \right\} = 0. \quad (12)$$

By putting $x = \varepsilon B_n \sqrt{\ln \ln n}$ and $y = \varepsilon B_n \sqrt{\ln \ln n}/4$ in (2) we get

$$\begin{aligned} &\sum' \frac{1}{n \ln n} \mathbf{P} \left\{ |S_n| \geq \varepsilon B_n \sqrt{\ln \ln n} \right\} \\ &\leq \sum' \frac{1}{n \ln n} \sum_{k=1}^n \mathbf{P} \left\{ |X_k| \geq \frac{1}{4} \varepsilon B_n \sqrt{\ln \ln n} \right\} \\ &\quad + 2 \sum' \frac{1}{n \ln n} \left(\frac{\varepsilon^2 \ln \ln n}{8} + 1 \right)^{-2} + 2 \sum' \frac{1}{n \ln n} \exp \left(-\frac{\varepsilon^2 \ln \ln n}{8e^2} \right) \end{aligned}$$

$$\leq \sum' \frac{1}{n \ln n} \sum_{k=1}^n \mathbf{P} \left\{ |X_k| \geq \frac{\varepsilon B_n}{4} \sqrt{\ln \ln n} \right\} + \frac{128}{\varepsilon^4} \sum' \frac{1}{n \ln n (\ln \ln n)^2} \\ + 2 \sum' \frac{1}{n (\ln n)^{1+\varepsilon^2/(8\varepsilon^2)}}.$$

Since

$$\sum' \frac{1}{n \ln n (\ln \ln n)^2} \leq \int_{b(\varepsilon, M)}^{\infty} \frac{dx}{x \ln x (\ln \ln x)^2} \leq \frac{1}{\ln \ln b(\varepsilon, M)} = \frac{\varepsilon^2}{M}$$

and

$$\sum' \frac{1}{n (\ln n)^{1+\varepsilon^2/(8\varepsilon^2)}} \leq \frac{8e^2}{\varepsilon^2 [\ln b(\varepsilon, M)]^{\varepsilon^2/(8\varepsilon^2)}} \\ = \frac{8e^2}{\varepsilon^2} \left(\exp \left\{ \frac{M}{\varepsilon^2} \right\} \right)^{-\varepsilon^2/(8\varepsilon^2)} = \frac{8e^2}{\varepsilon^2} \exp \left\{ -\frac{M}{8e^2} \right\},$$

we get

$$\sum' \frac{1}{n \ln n} \mathbf{P} \left\{ |S_n| \geq \varepsilon B_n \sqrt{\ln \ln n} \right\} \leq \sum' \frac{1}{n \ln n} \sum_{k=1}^n \mathbf{P} \left\{ |X_k| \geq \frac{1}{4} \varepsilon B_n \sqrt{\ln \ln n} \right\} \\ + \frac{128}{\varepsilon^2 M} + \frac{16e^2}{\varepsilon^2} \exp \left\{ -\frac{M}{8e^2} \right\}.$$

Thus, taking into account (4) we have

$$\lim_{M \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0^+} \varepsilon^2 \sum' \frac{1}{n \ln n} \mathbf{P} \left\{ |S_n| \geq \varepsilon B_n \sqrt{\ln \ln n} \right\} \\ \leq \lim_{M \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0^+} \varepsilon^2 \sum' \frac{1}{n \ln n} \sum_{k=1}^n \mathbf{P} \left\{ |X_k| \geq \frac{1}{4} \varepsilon B_n \sqrt{\ln \ln n} \right\} \\ + \lim_{M \rightarrow \infty} \frac{128}{M} + \lim_{M \rightarrow \infty} 16e^2 \exp \left\{ -\frac{M}{8e^2} \right\} = 0,$$

which proves (12).

Notice that in view of (10), (11), and (12) we get

$$\lim_{\varepsilon \rightarrow 0^+} \varepsilon^2 \sum_{n=3}^{\infty} \frac{1}{n \ln n} \left| \mathbf{P} \left\{ |S_n| \geq \varepsilon B_n \sqrt{\ln \ln n} \right\} - \psi_{\varepsilon}(n) \right| = 0. \quad (13)$$

Finally by (9) and (13) we get (5). The theorem is now proved.

Acknowledgments. The author thanks the referee for a lot of valuable remarks which improved the paper.

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Поступила в редакцию
31.III.2006

Исправленный вариант
25.V.2008