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BOUNDS FOR THE RATE OF STRONG APPROXIMATION IN
THE MULTIDIMENSIONAL INVARIANCE PRINCIPLE¹⁾

Цель работы — доказать некоторые следствия работы второго автора [34]. Мы устанавливаем оценки для скорости сильной гауссовской аппроксимации сумм независимых $\mathbf{R}^d$ -значных случайных векторов ξ_j , имеющих конечные моменты $\mathbf{E}\|\xi_j\|^\gamma$, $\gamma \geq 2$. Получены многомерные аналоги результатов работы А.И. Саханенко [27].

Ключевые слова и фразы: многомерный принцип инвариантности, сильная аппроксимация, суммы независимых случайных векторов.

1. Introduction. The aim of this paper is to derive consequences of a result of [34] providing optimal bounds for the strong Gaussian approximation of sums of independent $\mathbf{R}^d$ -valued random vectors with finite exponential moments (see Theorem 3 below). Theorem 3 may be considered as a multidimensional generalization of well-known results of Komlós, Major, and Tusnády (KMT) [19] and Sakhanenko [26], who generalized and essentially sharpened KMT results in the case of nonidentically distributed $\mathbf{R}^1$ -valued random variables. Then, Sakhanenko [27] used the main result of [26] to obtain optimal bounds for the strong Gaussian approximation of sums of independent random variables ξ_j having finite moments of the form $\mathbf{E}H(|\xi_j|)$, where $H(x)$ is a monotone function growing not slower than x^2 and not faster than e^{cx} . The most fine and complete results were obtained in the case $H(x) = x^\gamma$, $\gamma \geq 2$. In this paper, basing on the results of [34], we provide a new attempt to obtain a multidimensional generalization of the results of [27], restricting ourselves to the case $H(x) = x^\gamma$. The first attempt was realized in [36] (see Theorem 2 below), but the corresponding inequality was established not for all possible values of the argument z .

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The problem of strong approximation of sums of independent random vectors is more delicate than that of estimating the closeness of distributions. It is required to construct on a probability space a sequence of independent random vectors $X_1, \dots, X_n$ (with given distributions) and a corresponding sequence of independent Gaussian random vectors $Y_1, \dots, Y_n$ so that the quantity

$$\Delta(X, Y) = \max_{1 \leq k \leq n} \left\| \sum_{i=1}^k X_i - \sum_{i=1}^k Y_i \right\| \quad (1.1)$$

would be so small as possible with sufficiently large probability. The estimation of the rate of strong approximation in the invariance principle may be reduced to this problem.

The rate of strong approximation in the one-dimensional invariance principle was studied by many authors (see, for example, [23], [31], [3], [6], [19] and the bibliography in the monograph [7] and in the papers [5], [8], [20], [30]). One should mention the recent results of Sakhanenko [29], who sharpened the inequalities of [27] expressing the bounds in terms of truncated moments.

Multidimensional estimates in the strong invariance principle can be found in [17], [2], [22], [4], [1], [12], [13], [28]. The most of these papers contain the results for sufficiently small values of $\gamma \geq 2$ or bounds for the Prokhorov distance and for similar quantities. The Prokhorov distance estimation (coupled with the well-known Strassen–Dudley theorem) allows one to get the statements about the strong approximation for concrete values of the argument z (see (1.3), (1.7), and (1.16)–(1.18)). This means that the probability space involved in the results depends on this z .

Einmahl [14], obtained multidimensional results about the optimal rate of the strong Gaussian approximation of infinite sequences of sums of i.i.d. random vectors with $\mathbf{E}H(\|\xi_1\|) < \infty$. In particular, he established the optimal rate in the case $H(x) = x^\gamma$, $\gamma > 3$. Similarly to [19] and [27], the proof in [14], was based on the result for the vectors with finite exponential moments. The latter contained an extra logarithmic factor in the corresponding inequalities. Zaitsev [33], [34] removed this unnecessary logarithmic factor from the result of [14] and obtained multidimensional analogues of the KMT and Sakhanenko results for the strong Gaussian approximation of sums of independent $\mathbf{R}^d$ -valued random vectors with finite exponential moments.

Below we need some notation. The distribution of a random vector ξ will be denoted by $\mathcal{L}(\xi)$. The corresponding covariance operator will be denoted by $\text{cov } \xi$ or $\text{cov } V$, if $V = \mathcal{L}(\xi)$. We denote by c universal constants which might be different in different places of the text. We write $\ln^* b = \max\{1, \ln b\}$, for $b > 0$.

The aim of the present paper is to obtain multidimensional analogues of the following result of Sakhanenko.

Theorem 1 ([27]). *Let $\xi_1, \dots, \xi_n$ be independent random variables with $\mathbf{E}\xi_j = 0$, $j = 1, \dots, n$. Let $\gamma \geq 2$ and*

$$L_\gamma = \sum_{j=1}^n \mathbf{E}|\xi_j|^\gamma < \infty.$$

Then one can construct on a probability space a sequence of independent random variables $X_1, \dots, X_n$ and a corresponding sequence of independent Gaussian random variables $Y_1, \dots, Y_n$ so that $\mathcal{L}(X_j) = \mathcal{L}(\xi_j)$, $\mathbf{E}Y_j = 0$, $\mathbf{D}Y_j = \mathbf{D}X_j$, $j = 1, \dots, n$,

$$\mathbf{E}(\Delta(X, Y))^\gamma \leq c\gamma^{2\gamma}L_\gamma, \quad (1.2)$$

and, hence, for all $z > 0$,

$$\mathbf{P}\{\Delta(X, Y) \geq z\} \leq c\gamma^{2\gamma}L_\gamma z^{-\gamma}, \quad (1.3)$$

where c is a universal constant. Moreover, there exists an absolute positive constant C such that

$$\mathbf{P}\{\Delta(X, Y) \geq Cz\} \leq c\gamma^{2\gamma b}b^{\gamma b}(L_\gamma z^{-\gamma})^b + 2 \sum_{j=1}^n \mathbf{P}\left\{\|\xi_j\| \geq \frac{z}{b}\right\}, \quad (1.4)$$

for any $b \geq 1$ and $z \geq 0$.

It should be mentioned that, in [27], one can find more general results. In particular, there was considered the case, where the random variable ξ_j have finite moments of the form $\mathbf{E}H(|\xi_j|)$, where $H(x)$ is a monotone function growing not slower than x^2 and not faster than e^{cx} . The results of [27] are very precise and general. They imply a lot of other approximation results (see [30]). In particular, in [27] it is observed that inequality (1.2) implies the well-known Rosenthal [24], [25] inequality (see Lemma 3 below).

After the natural normalization, we see that (1.2) is equivalent to

$$\mathbf{E}(\Delta(X, Y)\sigma^{-1})^\gamma \leq c\gamma^{2\gamma}L_\gamma\sigma^{-\gamma},$$

where $\sigma^2 = \mathbf{D}(\sum_{j=1}^n \xi_j)$. It is clear that L_γ/σ^γ , $2 < \gamma \leq 3$, is the well-known Lyapunov fraction involved in the Lyapunov and Essén bounds for the Kolmogorov distance in the central limit theorem.

The following Theorem 2 was proved in a recent publication of Zaitsev (see [36]).

Theorem 2 ([36]). *Suppose that $\alpha > 0$ and $\xi_1, \dots, \xi_n$ are independent $\mathbf{R}^d$ -valued random vectors with $\mathbf{E}\xi_j = 0$, $j = 1, \dots, n$. Let $\gamma \geq 2$ and*

$$L_\gamma = \sum_{j=1}^n \mathbf{E}\|\xi_j\|^\gamma < \infty. \quad (1.5)$$

Assume that there exist a positive integer s and a strictly increasing sequence of nonnegative integers $m_0 = 0$, $m_1, \dots, m_s = n$ satisfying the following conditions. Let

$$\zeta_k = \xi_{m_{k-1}+1} + \dots + \xi_{m_k}, \quad \text{cov } \zeta_k = \mathbb{B}_k, \quad k = 1, \dots, s,$$

and assume that, for all $v \in \mathbf{R}^d$ and $k = 1, \dots, s$,

$$r^2 L_\gamma^{2/\gamma} \|v\|^2 \leq \langle \mathbb{B}_k v, v \rangle \leq C r^2 L_\gamma^{2/\gamma} \|v\|^2, \quad (1.6)$$

with some constants $r \geq 2e$ and $C \geq 1$. Then one can construct on a probability space a sequence of independent random vectors $X_1, \dots, X_n$ and a corresponding sequence of independent Gaussian random vectors $Y_1, \dots, Y_n$ so that $\mathcal{L}(X_j) = \mathcal{L}(\xi_j)$, $\mathbf{E}Y_j = 0$, $\text{cov } Y_j = \text{cov } X_j$, $j = 1, \dots, n$, and, for $z > a_1 d^{15/2+\alpha} \ln^* d \cdot r L_\gamma^{1/\gamma} \ln^* s$,

$$\mathbf{P}\{\Delta(X, Y) \geq 5z\} \leq 2L_\gamma z^{-\gamma} + \exp\left(-\frac{a_2 z}{r L_\gamma^{1/\gamma} d^{9/2} \ln^* d}\right), \quad (1.7)$$

where a_1, a_2 are positive quantities depending only on α and C .

Theorem 2 provides the multidimensional analogue of a weakened version of inequality (1.3). Roughly speaking, in inequality (1.3) of Theorem 1 the statement is the same as in Theorem 2 but without the exponential term in (1.7), without condition (1.6) and without any restrictions on $z > 0$. If $\mathbf{D}(\sum_{j=1}^n \xi_j) \geq 32 L_\gamma^{2/\gamma}$, then condition (1.6) is always satisfied in the one-dimensional situation since $\mathbf{D}\xi_j \leq (\mathbf{E}|\xi_j|^\gamma)^{2/\gamma} \leq L_\gamma^{2/\gamma}$, $j = 1, \dots, n$, and we can find $m_1, \dots, m_s$ so that (1.6) is valid with $e^{-1} r = C = 2$. Note that, for $z \leq L_\gamma^{1/\gamma}$, inequality (1.7) is satisfied trivially. It is evident that, in the case $d = 1$, $e^{-1} r = C = 2$, the exponential term may easily be estimated by $c\gamma^\gamma L_\gamma z^{-\gamma}$, for $z > L_\gamma^{1/\gamma}$, with an absolute constant c . In the degenerate case, where $\mathbf{D}(\sum_{j=1}^n \xi_j) \leq 32 L_\gamma^{2/\gamma}$, the required bounds are valid for any constructing the random vectors $\{X_j\}$ and $\{Y_j\}$ on the same probability space (see Proposition 1 below).

Thus, the only difference with Sakhanenko's inequality (1.3) is in the logarithmic factor $\ln^* s$ in the restriction on z .

It is clear that $\ln^* s$ behaves as $\ln^* n$ in the case of i.i.d. $\xi_1, \dots, \xi_n$ and as the logarithm of variances of coordinates of sums $\xi_1 + \dots + \xi_n$ in the general case. Therefore, it is natural to try to establish a multidimensional analogue of inequality (1.3) which is valid without any restrictions on z as well as those of inequalities (1.2) and (1.4). Under some additional conditions, this is realized in Theorem 4 below, which provides a partial solution of the problem.

Note that one can find in [36] the corresponding versions of inequality (1.7) for the vectors with finite moments of the form $\mathbf{E}H(\|\xi_j\|)$, where $H(x)$ is a monotone function growing not slower than x^2 and not faster than e^{cx} . Unfortunately, in these results there are similar logarithmic factors $\ln^* s$ in the restrictions on z . Such functions were also considered in [27].

The main tool for the proof of Theorems 2 and 4 is the following Theorem 3. It was proved in [34]. In this theorem, the most important case,

where the summands have finite exponential moments, was considered. The main result of [26] was successively generalized to the multidimensional case.

Let $\mathcal{A}_d(\tau)$, $\tau \geq 0$, $d \in \mathbf{N}$, denote classes of d -dimensional distributions, introduced in [32], see as well [33]–[35]. The class $\mathcal{A}_d(\tau)$ (with a fixed $\tau \geq 0$) consists of d -dimensional distributions V for which the function

$$\varphi(z) = \varphi(V, z) = \ln \int_{\mathbf{R}^d} e^{(z, x)} V\{dx\} \quad (\varphi(0) = 0) \quad (1.8)$$

is defined and analytic for $\|z\| \tau < 1$, $z \in \mathbf{C}^d$, and

$$|d_u d_v^2 \varphi(z)| \leq \|u\| \tau \langle \mathbb{D}v, v \rangle, \quad (1.9)$$

for all $u, v \in \mathbf{R}^d$ and $\|z\| \tau < 1$, where $\mathbb{D} = \text{cov } V$ and $d_u \varphi$ is the derivative of the function φ in direction u . Below we shall use simplest properties of the classes $\mathcal{A}_d(\tau)$ which are listed in [35].

Theorem 3 ([34]). *Suppose that $\alpha > 0$, $\tau \geq 1$, and $\xi_1, \dots, \xi_n$ are independent $\mathbf{R}^d$ -valued random vectors with $\mathbf{E}\xi_j = 0$, $j = 1, \dots, n$. Assume that there exist a positive integer s and a strictly increasing sequence of nonnegative integers $m_0 = 0, m_1, \dots, m_{s-1}, m_s = n$ satisfying the following conditions. Let*

$$\zeta_k = \xi_{m_{k-1}+1} + \dots + \xi_{m_k}, \quad \text{cov } \zeta_k = \mathbb{B}_k, \quad \mathcal{L}(\zeta_k) \in \mathcal{A}_d(\tau), \quad k = 1, \dots, s,$$

and suppose that, for all $v \in \mathbf{R}^d$,

$$C_1 \|v\|^2 \leq \langle \mathbb{B}_k v, v \rangle \leq C_2 \|v\|^2, \quad k = 1, \dots, s, \quad (1.10)$$

with some positive constants C_1 and C_2 . Then one can construct on a probability space a sequence of independent random vectors $X_1, \dots, X_n$ and a corresponding sequence of independent Gaussian random vectors $Y_1, \dots, Y_n$ so that $\mathcal{L}(X_j) = \mathcal{L}(\xi_j)$, $\mathbf{E}Y_j = 0$, $\text{cov } Y_j = \text{cov } X_j$, $j = 1, \dots, n$, and

$$\mathbf{E} \exp \left(\frac{a_3 \Delta(X, Y)}{\tau d^{9/2} \ln^* d} \right) \leq \exp \left(a_4 d^{3+\alpha} \ln^* \frac{s}{\tau^2} \right),$$

where a_3, a_4 are positive quantities depending on α, C_1, C_2 only.

The i.i.d. version of Theorem 3 was proved by Zaitsev in [33], where he removed an additional logarithmic factor from a result of [14] and obtained a multidimensional analogue of the KMT result. Theorem 3 provides an adequate multidimensional generalization of the main result of [26]. In Theorem 3, the random vectors are, generally speaking, nonidentically distributed. However, they have to satisfy condition (1.10) on the covariance operators. Condition (1.10) generates conditions (1.6) and (1.12). A plausible conjecture is that condition (1.10) could be weakened in the statement of Theorem 3. The same may be said about conditions (1.6) and (1.12) in our results. At least, no similar restrictions are imposed in [32] (which, however, is not related to the invariance principle). But the proof of this conjecture would require some new ideas and methods.

The main result of this paper is the following theorem.

Theorem 4. Suppose that $\alpha > 0$ and $\xi_1, \dots, \xi_n$ are independent $\mathbf{R}^d$ -valued random vectors with $\mathbf{E}\xi_j = 0$, $j = 1, \dots, n$. Let $\gamma \geq 2$ and let the quantity L_γ be defined by (1.5). Assume that there exist a positive integer s and a strictly increasing sequence of nonnegative integers $m_0 = 0$, $m_1, \dots, m_{s-1}$, $m_s = n$ satisfying the following conditions. Let

$$\zeta_k = \xi_{m_{k-1}+1} + \dots + \xi_{m_k}, \quad \text{cov } \zeta_k = \mathbb{B}_k, \quad k = 1, \dots, s, \quad (1.11)$$

and assume that, for all $v \in \mathbf{R}^d$ and $k = 1, \dots, s$,

$$w^2 \|v\|^2 \leq \langle \mathbb{B}_k v, v \rangle \leq C_1 w^2 \|v\|^2, \quad (1.12)$$

where

$$w = C_2 \frac{L_\gamma^{1/\gamma}}{\ln^* s}, \quad (1.13)$$

with some positive constants C_1 and C_2 . Suppose that the quantities

$$\lambda_{k,\gamma} = \sum_{j=m_{k-1}+1}^{m_k} \mathbf{E} \|\xi_j\|^\gamma, \quad k = 1, \dots, s, \quad (1.14)$$

satisfy, for some $0 < \varepsilon < 1$,

$$C_3 d^{\gamma/2} s^\varepsilon (\ln^* s)^{\gamma+3} \max_{1 \leq k \leq s} \lambda_{k,\gamma} \leq L_\gamma, \quad (1.15)$$

with a positive constant C_3 . Then one can construct on a probability space a sequence of independent random vectors $X_1, \dots, X_n$ and a corresponding sequence of independent Gaussian random vectors $Y_1, \dots, Y_n$ so that $\mathcal{L}(X_j) = \mathcal{L}(\xi_j)$, $\mathbf{E}Y_j = 0$, $\text{cov } Y_j = \text{cov } X_j$, $j = 1, \dots, n$, and, for any $z > a_5 \varepsilon^{-1} L_\gamma^{1/\gamma} d^{21/2+\alpha} \ln^* d$ and any $b \geq 1$,

$$\begin{aligned} \mathbf{P}\{\Delta(X, Y) \geq 9z\} &\leq 10 d^{1+\gamma b/2} (2b)^{b(\gamma-1)} z^{-\gamma b} \sum_{k=1}^s \lambda_{k,\gamma}^b + \exp\left(-\frac{a_6 z^2}{d L_\gamma^{2/\gamma}}\right) \\ &\quad + \exp\left(-\frac{a_7 \varepsilon z \ln^* s}{L_\gamma^{1/\gamma} d^{15/2} \ln^* d}\right) \\ &\quad + 5d \sum_{j=1}^n \mathbf{P}\left\{\|\xi_j\| \geq \frac{z}{2b\sqrt{d}}\right\}, \end{aligned} \quad (1.16)$$

$$\begin{aligned} \mathbf{P}\{\Delta(X, Y) \geq 9z\} &\leq d^{1+\gamma/2} (a_8 b^4 \varepsilon^{-1})^{\gamma b} (L_\gamma z^{-\gamma})^b \\ &\quad + 5d \sum_{j=1}^n \mathbf{P}\left\{\|\xi_j\| \geq \frac{z}{2b\sqrt{d}}\right\}, \end{aligned} \quad (1.17)$$

$$\mathbf{P}\{\Delta(X, Y) \geq z\} \leq a_9 d^{1+\gamma/2} L_\gamma (\varepsilon z)^{-\gamma}, \quad (1.18)$$

and

$$\mathbf{E}(\Delta(X, Y))^\gamma \leq a_{10} (\varepsilon^{-1} d^{21/2+\alpha} \ln^* d)^\gamma L_\gamma, \quad (1.19)$$

where $a_5, \dots, a_{10}$ are positive quantities depending on α, γ, C_1, C_2 , and C_3 only.

Note that, for $z \leq a_5 \varepsilon^{-1} L_\gamma^{1/\gamma} d^{21/2+\alpha} \ln^* d$, we have, under conditions of Theorem 4,

$$\mathbf{P}\{\Delta(X, Y) \geq z\} \leq 1 \leq (a_5 \varepsilon^{-1} d^{21/2+\alpha} \ln^* d)^\gamma L_\gamma z^{-\gamma}.$$

On the other hand, inequality (1.19) implies that

$$\mathbf{P}\{\Delta(X, Y) \geq z\} \leq a_{10} (\varepsilon^{-1} d^{21/2+\alpha} \ln^* d)^\gamma L_\gamma z^{-\gamma}, \quad (1.20)$$

for all $z > 0$. Thus, in Theorem 4, the inequality of type (1.3) is established, in fact, for all possible values of z . It is impossible, however, to derive from (1.19) inequality (1.18) which is sharper with respect to the dependence of constants on the dimension d . Note that in this paper we do not try to optimize the dependence of constants on the dimension. It is important here that it is a power-type dependence. It is also clear that (1.19) does not follow from (1.20) or from (1.18). Moreover, inequality (1.17) is sometimes stronger than (1.18) (for example, if the last sum in (1.17) is equal to zero). Inequality (1.16) is sharper than (1.17) and (1.18). On the other hand, (1.17) and (1.18) are formulated simpler than (1.16).

Condition (1.15) is rather cumbersome, but it is trivially satisfied if $\xi_1, \dots, \xi_n$ are i.i.d. random vectors and n is sufficiently large. Indeed, it is clear that in this case $m_k \asymp k n^{1/\gamma} / \ln^* n$, $\lambda_{k,\gamma} \asymp n^{1/\gamma} / \ln^* n$, $k = 1, \dots, s$, $s \asymp n^{1-1/\gamma} \ln^* n$, and condition (1.15) turns into

$$C_3 d^{\gamma/2} n^{\varepsilon(1-1/\gamma)} (\ln^* n)^{\gamma+2+\varepsilon} \leq A n^{1-1/\gamma}$$

with some A which does not depend on n . Moreover, in the general case

$$L_\gamma = \sum_{k=1}^s \lambda_{k,\gamma} \quad (1.21)$$

and (1.15) has to be satisfied if the quantities $\lambda_{k,\gamma}$ have a regular behavior.

Choosing, for example, $C_3 = d^{-\gamma/2}$, we could remove the dependence on d in condition (1.15). But, of course, this would imply that $a_5, \dots, a_{10}$ depend on d too.

Let $2 \leq \beta < \gamma$. Then, under conditions of Theorem 4,

$$(\mathbf{E}(\Delta(X, Y))^\beta)^{1/\beta} \leq (\mathbf{E}(\Delta(X, Y))^\gamma)^{1/\gamma} \leq a_{10}^{1/\gamma} \varepsilon^{-1} d^{21/2+\alpha} \ln^* d \cdot L_\gamma^{1/\gamma}.$$

This bound could be better than the bound which could be obtained if we apply Theorem 4 with $\gamma = \beta$. For example, in the i.i.d. case, $L_\gamma^{1/\gamma} = (n \mathbf{E} \|\xi_1\|^\gamma)^{1/\gamma} \asymp n^{1/\gamma}$, while $L_\beta^{1/\beta} = (n \mathbf{E} \|\xi_1\|^\beta)^{1/\beta} \asymp n^{1/\beta}$. In the general case, for a fixed β , one should try to minimize $L_\gamma^{1/\gamma}$, $\gamma \geq \beta$. Note, however, that conditions (1.12) and (1.15) are different for different γ .

Conditions (1.6) and (1.12) imply that the minimal eigenvalue of $\text{cov}(\sum_{j=1}^n \xi_j)$ is bounded from below by $r^2 L_\gamma^{2/\gamma}$ and w^2 respectively. Nevertheless, if the eigenvalues of $\text{cov}(\sum_{j=1}^n \xi_j)$ are small, less than $C_4^2 L_\gamma^{2/\gamma}$, the required inequalities are valid for any constructing the random vectors $\{X_j\}$ and $\{Y_j\}$ on the same probability space.

Proposition 1. Suppose that $\xi_1, \dots, \xi_n$ are independent $\mathbf{R}^d$ -valued random vectors with $\mathbf{E}\xi_j = 0$, $j = 1, \dots, n$. Let $\gamma \geq 2$ and let the quantity L_γ be defined by (1.5). Let σ^2 be the maximal eigenvalue of $\text{cov}(\sum_{j=1}^n \xi_j)$. Assume that

$$\sigma \leq C_4 L_\gamma^{1/\gamma} \quad (1.22)$$

with some positive constant C_4 . Then, for any construction on a probability space of a sequence of independent random vectors $X_1, \dots, X_n$ and a corresponding sequence of independent Gaussian random vectors $Y_1, \dots, Y_n$ such that $\mathcal{L}(X_j) = \mathcal{L}(\xi_j)$, $\mathbf{E}Y_j = 0$, $\text{cov } Y_j = X_j$, $j = 1, \dots, n$, the following inequalities are valid:

$$\begin{aligned} \mathbf{P}\{\Delta(X, Y) \geq 4z\} &\leq d^{1+\gamma/2} (a_{11}b)^{b(\gamma-1)} (L_\gamma z^{-\gamma})^b \\ &\quad + 3d \sum_{j=1}^n \mathbf{P}\left\{\|\xi_j\| \geq \frac{z}{2b\sqrt{d}}\right\}, \end{aligned} \quad (1.23)$$

for all $z > 0$, $b \geq 1$,

$$\mathbf{E}(\Delta(X, Y))^\gamma \leq a_{12} d^{1+\gamma/2} L_\gamma, \quad (1.24)$$

and, hence,

$$\mathbf{P}\{\Delta(X, Y) \geq z\} \leq a_{12} d^{1+\gamma/2} L_\gamma z^{-\gamma}, \quad (1.25)$$

where a_{11} and a_{12} are positive quantities depending on γ and C_4 only.

The proof of Proposition 1 is based, in fact, on the trivial inequality $\Delta(X, Y) \leq \Delta(X, 0) + \Delta(0, Y)$. It remains to estimate $\Delta(X, 0)$ and $\Delta(0, Y)$.

The proof of Theorem 4 is a modified version of the corresponding proofs from [27] and [36]. The main difference is that we truncate not the initial summands ξ_j but the sums ζ_k over blocks of summands ξ_j and then show that the distributions of truncated vectors belong to the class $\mathcal{A}_d(\tau)$ with τ which is essentially less than the level of truncation u . A similar idea was used in [14]. The complete proof in [27] is yet not generalized due to technical difficulties occurring in the multidimensional case.

Below we shall need the following Lemmas 1–3.

Lemma 1. Let $X_1, \dots, X_n$ be independent random vectors and $S_k = X_1 + \dots + X_k$, $k = 1, \dots, n$. Then

$$\mathbf{P}\left\{\max_{1 \leq k \leq n} \|S_k\| \geq 3t\right\} \leq 3 \max_{1 \leq k \leq n} \mathbf{P}\{\|S_k\| \geq t\}, \quad t \geq 0. \quad (1.26)$$

Lemma 1 is a version of the Ottaviani inequality, see [10, Proposition 1.1.2]. In the form (1.26), this inequality can be found in [18] and (with 4 instead of 3 (twice)) in [15].

Lemma 2. Let $\gamma \geq 2$, $b \geq 1$, let $S_n = \psi_1 + \dots + \psi_n$ be the sum of independent random variables $\psi_j \in \mathbf{R}^1$ with $\mathbf{E}\psi_j = 0$, $\mathbf{E}|\psi_j|^\gamma < \infty$,

$j = 1, \dots, n$, and let $\mathbf{D} S_n = \sigma^2$. Then

$$\begin{aligned} \mathbf{P}\{|S_n| \geq x\} &\leq 2 \exp\left(-\frac{a_{13}x^2}{\sigma^2}\right) + 2(2b)^{b(\gamma-1)} \left(\sum_{j=1}^n \frac{\mathbf{E}|\psi_j|^\gamma}{x^\gamma}\right)^b \\ &\quad + \sum_{j=1}^n \mathbf{P}\left\{|\psi_j| \geq \frac{x}{2b}\right\}, \end{aligned} \quad (1.27)$$

for all $x > 0$, where a_{13} is a positive quantity depending on γ only.

Lemma 2 follows easily from the main result of [16], see also [21, Corollary 1.7].

It is easy to see that, by the Chebyshev inequality, (1.27) with $b = 1$ implies that

$$\mathbf{P}\{|S_n| \geq x\} \leq 2 \exp\left(-\frac{a_{13}x^2}{\sigma^2}\right) + 2^{\gamma+1} \sum_{j=1}^n \frac{\mathbf{E}|\psi_j|^\gamma}{x^\gamma}, \quad (1.28)$$

for all $x > 0$, under conditions of Lemma 2.

Lemma 3. Let $\xi_1, \dots, \xi_n$ denote independent random vectors which have mean zero and assume values in $\mathbf{R}^d$. Then

$$\mathbf{E} \left\| \sum_{j=1}^n \xi_j \right\|^\gamma \leq a_{14} \left(\sum_{j=1}^n \mathbf{E} \|\xi_j\|^\gamma + \left(\sum_{j=1}^n \mathbf{E} \|\xi_j\|^2 \right)^{\gamma/2} \right), \quad 0 < \gamma < \infty,$$

with a_{14} depending on γ only.

This Rosenthal type inequality easily follows from a result of [9] and the fact that the $\mathbf{R}^d$ is a type 2 space, that is, $\mathbf{E} \|\sum_{j=1}^n \xi_j\|^2 \leq c \sum_{j=1}^n \mathbf{E} \|\xi_j\|^2$.

2. Proofs. For any random vector ξ we shall denote by $\xi^{[u]}$ its truncation on the level $u > 0$:

$$\xi^{[u]} = \begin{cases} \xi, & \text{if } \|\xi\| \leq u, \\ 0, & \text{otherwise.} \end{cases} \quad (2.1)$$

Set

$$\xi^{(u)} = \xi - \xi^{[u]}, \quad \xi^{\{u\}} = \xi^{[u]} - \mathbf{E}\xi^{[u]}. \quad (2.2)$$

Assuming $\mathbf{E} \|\xi\|^2 < \infty$ and $\mathbf{E}\xi = 0$, we have $\xi = \xi^{(u)} + \xi^{[u]}$,

$$\mathbf{E}\xi^{(u)} + \mathbf{E}\xi^{[u]} = \mathbf{E}\xi = 0,$$

and, hence,

$$\|\mathbf{E}\xi^{(u)}\| = \|\mathbf{E}\xi^{[u]}\| \leq \mathbf{E}\|\xi^{(u)}\|.$$

Moreover, for any $v \in \mathbf{R}^d$,

$$|\mathbf{E}\langle \xi^{(u)}, v \rangle| = |\mathbf{E}\langle \xi^{[u]}, v \rangle| \leq \|v\| \mathbf{E}\|\xi^{(u)}\|. \quad (2.3)$$

Furthermore,

$$\mathbf{D}\langle \xi, v \rangle = \mathbf{E}\langle \xi, v \rangle^2 = \mathbf{E}\langle \xi^{[u]}, v \rangle^2 + \mathbf{E}\langle \xi^{(u)}, v \rangle^2 \geq \mathbf{D}\langle \xi^{[u]}, v \rangle + \mathbf{E}\langle \xi^{(u)}, v \rangle^2. \quad (2.4)$$

By (2.3), (2.4) and by the Cauchy–Bunyakovskii inequality, we have

$$\begin{aligned} \mathbf{D}\langle \xi^{[u]}, v \rangle &= \mathbf{E}\langle \xi^{[u]}, v \rangle^2 - (\mathbf{E}\langle \xi^{[u]}, v \rangle)^2 = \mathbf{D}\langle \xi, v \rangle - \mathbf{E}\langle \xi^{(u)}, v \rangle^2 - (\mathbf{E}\langle \xi^{(u)}, v \rangle)^2 \\ &\geq \mathbf{D}\langle \xi, v \rangle - 2\mathbf{E}\langle \xi^{(u)}, v \rangle^2. \end{aligned} \quad (2.5)$$

Inequalities (2.4) and (2.5) imply

$$2\mathbf{E}\langle \xi^{(u)}, v \rangle^2 \geq \mathbf{D}\langle \xi, v \rangle - \mathbf{D}\langle \xi^{[u]}, v \rangle \geq \mathbf{E}\langle \xi^{(u)}, v \rangle^2. \quad (2.6)$$

P r o o f o f T h e o r e m 4. The symbols $c, c_2, c_1, \dots$ will be used for absolute positive constants. The letter c can denote different constants when we do not need to fix their numerical values. The same notation will be used for positive quantities which depend only on α, γ, C_1, C_2 , and C_3 involved in the conditions of Theorem 4. This means, in fact, that we assume α, γ, C_1, C_2 , and C_3 to be absolute positive constants. We shall write $A \ll B$, if there exists a c such that $A \leq cB$.

Let $\xi_1, \dots, \xi_n$ and $\zeta_1, \dots, \zeta_s$ be random vectors satisfying the conditions of Theorem 4. For $u > 0$, consider independent Gaussian random vectors $\eta_1^{\{u\}}, \dots, \eta_n^{\{u\}}$ with

$$\mathbf{E}\eta_k^{\{u\}} = \mathbf{E}\zeta_k^{\{u\}} = 0, \quad \text{cov } \eta_k^{\{u\}} = \text{cov } \zeta_k^{\{u\}} = \text{cov } \zeta_k^{[u]} \stackrel{\text{def}}{=} \mathbb{B}_k^{\{u\}}, \quad k = 1, \dots, s.$$

According to (1.11) and (2.6), for any $v \in \mathbf{R}^d$, we have

$$2\mathbf{E}\langle \zeta_k^{(u)}, v \rangle^2 \geq \langle \mathbb{B}_k v, v \rangle - \langle \mathbb{B}_k^{\{u\}} v, v \rangle \geq \mathbf{E}\langle \zeta_k^{(u)}, v \rangle^2, \quad k = 1, \dots, s. \quad (2.7)$$

Thus, the operators

$$\mathbb{B}_k - \mathbb{B}_k^{\{u\}} \stackrel{\text{def}}{=} \mathbb{B}_k^{(u)} \quad (2.8)$$

are positive definite and may be treated as covariance operators. Therefore, we can construct the Gaussian vectors $\eta_k^{(u)}$ with $\mathbf{E}\eta_k^{(u)} = 0$, $\text{cov } \eta_k^{(u)} = \mathbb{B}_k^{(u)}$, such that they are jointly independent and independent of the vectors $\eta_k^{\{u\}}$. Denote $\eta_k = \eta_k^{(u)} + \eta_k^{\{u\}}$. Clearly,

$$\mathbf{E}\eta_k = 0, \quad \text{cov } \eta_k = \mathbb{B}_k.$$

Using the standard tool of strong approximation, namely, Lemma A of [2], we can take ξ_j as X_j in the proof of Theorem 4, choosing below the joint distribution of $\{\zeta_k^{\{u\}}\}$ and $\{\eta_k^{\{u\}}\}$ in a special way. Moreover, we can assume that

$$\eta_k = \sum_{j=m_{k-1}+1}^{m_k} Y_j, \quad k = 1, \dots, s, \quad (2.9)$$

where $Y_1, \dots, Y_n$ are the independent random vectors satisfying the conditions of Theorem 4.

Denoting by $\mathbb{B}^{(u)}$ the covariance operator of the sum $\sum_{k=1}^s \eta_k^{(u)}$, we have

$$\mathbb{B}^{(u)} = \sum_{k=1}^s \mathbb{B}_k^{(u)}. \quad (2.10)$$

Let the quantity $B^2(u)$ be defined by

$$B^2(u) \stackrel{\text{def}}{=} \sum_{k=1}^s \mathbf{E} \|\zeta_k^{(u)}\|^2. \quad (2.11)$$

By (2.7)–(2.11), for any $v \in \mathbf{R}^d$, the inequalities

$$\sum_{k=1}^s \mathbf{E} \langle \zeta_k^{(u)}, v \rangle^2 \leq \langle \mathbb{B}^{(u)} v, v \rangle \leq 2 \sum_{k=1}^s \mathbf{E} \langle \zeta_k^{(u)}, v \rangle^2 \leq 2B^2(u) \|v\|^2. \quad (2.12)$$

are valid. Let σ_u^2 be the maximal eigenvalue of the operator $\mathbb{B}^{(u)}$. By (2.12),

$$\sigma_u^2 = \max_{\|v\|=1} \langle \mathbb{B}^{(u)} v, v \rangle \leq 2B^2(u). \quad (2.13)$$

Let $Z, Z_1, \dots, Z_d$ be i.i.d. standard normal random variables. Using standard tools to obtain exponential inequalities, it is easy to show that there exists an absolute constant c such that, for any $x > c\sigma_u \sqrt{d}$,

$$\mathbf{P} \left\{ \left\| \sum_{k=1}^s \eta_k^{(u)} \right\| \geq x \right\} \leq \mathbf{P} \left\{ \sigma_u^2 \sum_{k=1}^d Z_k^2 \geq x^2 \right\} \leq \exp \left(- \frac{x^2}{8\sigma_u^2} \right). \quad (2.14)$$

It suffices to note that $\mathbf{E} e^{hZ^2} = (1 - 2h)^{-1/2}$, for $0 < h < \frac{1}{2}$, and to apply the exponential Chebyshev inequality, choosing $h = \frac{1}{6}$.

Replacing n by s in the definition (1.1) of $\Delta(\cdot, \cdot)$, introduce the notation

$$\Delta = \Delta(\zeta, \eta), \quad \Delta^{\{u\}} = \Delta(\zeta^{\{u\}}, \eta^{\{u\}}), \quad \Delta_u = \Delta(\zeta - \mathbf{E}\zeta^{\{u\}}, \eta^{\{u\}}). \quad (2.15)$$

It is easy to see that

$$\Delta \leq \Delta_u + \delta_u + A(u), \quad (2.16)$$

where the quantity $A(u)$ is defined by

$$A(u) \stackrel{\text{def}}{=} \max_{1 \leq m \leq s} \left\| \sum_{k=1}^m \mathbf{E} \zeta_k^{[u]} \right\| = \max_{1 \leq m \leq s} \left\| \sum_{k=1}^m \mathbf{E} \zeta_k^{(u)} \right\|, \quad (2.17)$$

and

$$\delta_u = \Delta(\eta, \eta^{\{u\}}). \quad (2.18)$$

Following [27], introduce the class $\mathcal{G}(u)$, $u \geq 0$, of positive function $G(\cdot)$ such that the functions $G(x)$ and $x/\ln G(x)$ are nondecreasing for $x > u$. For example, the functions $G(x) = x^\gamma$, $\gamma > 0$, and $\exp(\lambda x^\beta)$, $0 \leq \beta \leq 1$, $\lambda > 0$, belong to $\mathcal{G}(e)$ and $\mathcal{G}(0)$, respectively.

The formulation and proof of the following lemma almost coincide with the formulation and proof of Theorem 3 in [27]. One can find it also in [36] (with $\{\zeta_k\}$ replaced by $\{\xi_j\}$).

Lemma 4. Let $u > 0$, $G(\cdot) \in \mathcal{G}(u)$, and

$$F(x) \stackrel{\text{def}}{=} \sum_{k=1}^s \mathbf{P}\{\|\zeta_k\| \geq x\} \leq \frac{1}{G(x)}, \quad \text{for all } x > u. \quad (2.19)$$

Then, for all $x_0 \geq 0$ and $x \geq z \geq u$,

$$\mathbf{P}\{\Delta_u \geq x_0 + x\} \leq \mathbf{P}\{\Delta^{\{u\}} \geq x_0\} + \mathbf{P}\left\{\max_{1 \leq k \leq s} \|\zeta_k\| \geq z\right\} + \frac{eG^{1-x/z}(z)}{G(u)}. \quad (2.20)$$

It is easy to see that, for $k = 1, \dots, s$, $x \geq 0$,

$$F_k(x) \stackrel{\text{def}}{=} \mathbf{P}\{\|\zeta_k\| \geq x\} \leq T_k(x) \stackrel{\text{def}}{=} \max_{m_{k-1}+1 \leq m \leq m_k} \mathbf{P}\left\{\left\|\sum_{j=m_{k-1}+1}^m \xi_j\right\| \geq x\right\}. \quad (2.21)$$

It is clear that

$$F(x) = \sum_{k=1}^s F_k(x) \leq \sum_{k=1}^s T_k(x) \quad (2.22)$$

(see (2.19) and (2.21)).

We shall use the following Fuk–Nagaev type inequalities:

$$\begin{aligned} F_k(x) &\leq T_k(x) \leq 2d \exp\left(-\frac{c_1 x^2}{dw^2}\right) \\ &\quad + 2d(2b)^{b(\gamma-1)} \left(d^{\gamma/2} \sum_{j=m_{k-1}+1}^{m_k} \frac{\mathbf{E}\|\xi_j\|^\gamma}{x^\gamma}\right)^b \\ &\quad + d \sum_{j=m_{k-1}+1}^{m_k} \mathbf{P}\left(\|\xi_j\| \geq \frac{x}{2b\sqrt{d}}\right), \quad k = 1, \dots, s, \end{aligned} \quad (2.23)$$

$$\begin{aligned} F_k(x) &\leq T_k(x) \ll d \exp\left(-\frac{c_1 x^2}{dw^2}\right) \\ &\quad + d^{1+\gamma/2} \sum_{j=m_{k-1}+1}^{m_k} \frac{\mathbf{E}\|\xi_j\|^\gamma}{x^\gamma}, \quad k = 1, \dots, s, \end{aligned} \quad (2.24)$$

which hold for any $x > 0$ and $b \geq 1$. They follow from (2.21) and from the corresponding one-dimensional inequalities (1.27) and (1.28). Constants depend here on γ and C_1 , but recall that we treat γ and C_1 as universal constants. Inequality (2.24) is weaker than (2.23), but it is more convenient in some applications below.

It should be mentioned that inequality (2.24) with constants which do not depend on the dimension d was proved for $\gamma = 3$ in [11], see [21, Theorem 4.3]. Therefore, for $\gamma = 3$, the dependence of constants on the dimension in inequality (1.18) of Theorem 4 might be sharpened.

We shall choose

$$u = c_2 d L_\gamma^{1/\gamma}, \quad (2.25)$$

where the constant c_2 will be so large as it will be required below.

Using (1.14) and (2.23)–(2.25), we get that, for $x \geq u$, $b \geq 1$,

$$\exp\left(-\frac{c_1 x^2}{2dw^2}\right) \ll \frac{1}{2ds}, \quad (2.26)$$

$$\begin{aligned} T_k(x) &\leq \frac{1}{s} \exp\left(-\frac{c_1 x^2}{2dw^2}\right) + 2d(2b)^{b(\gamma-1)} \left(d^{\gamma/2} \frac{\lambda_{k,\gamma}}{x^\gamma}\right)^b \\ &\quad + d \sum_{j=m_{k-1}+1}^{m_k} \mathbf{P}\left\{\|\xi_j\| \geq \frac{x}{2b\sqrt{d}}\right\}, \quad k = 1, \dots, s, \end{aligned} \quad (2.27)$$

and

$$F_k(x) \leq T_k(x) \ll \frac{1}{s} \exp\left(-\frac{c_1 x^2}{2dw^2}\right) + d^{1+\gamma/2} \frac{\lambda_{k,\gamma}}{x^\gamma}, \quad k = 1, \dots, s, \quad (2.28)$$

if c_2 is large enough.

According to (1.13), (1.21), (2.19), (2.22), (2.25), and (2.28), we have

$$F(x) \ll \exp\left(-\frac{c_1 x^2}{2dw^2}\right) + d^{1+\gamma/2} \sum_{j=1}^n \frac{\mathbf{E}\|\xi_j\|^\gamma}{x^\gamma} \ll d^{1+\gamma/2} \frac{L_\gamma}{x^\gamma}, \quad (2.29)$$

for $x \geq u$, that is, there exists c_3 such that

$$F(x) \leq \frac{1}{G(x)}, \quad \text{where} \quad G(x) = \frac{x^\gamma}{c_3 d^{1+\gamma/2} L_\gamma}, \quad (2.30)$$

for $x \geq u$, and

$$G(u) \geq e^\gamma, \quad (2.31)$$

if c_2 is large enough. We used here that $\gamma \geq 2$ and, hence, $1 + \gamma/2 \leq \gamma$. In order to apply Lemma 4 we have to verify that $G(\cdot) \in \mathcal{G}(u)$. Let us show that the function $x/\ln G(x)$ is nondecreasing for $x > u$. Indeed, for $x \geq y \geq u$, we have $G(y) \geq G(u) \geq e^\gamma$ and $G(x)/x^\gamma = G(y)/y^\gamma$. Denoting $t = x/y \geq 1$, $q = G(y)$, we get $q \geq e^\gamma$, $t^\gamma \leq e^{\gamma(t-1)} \leq q^{t-1}$ and $G(x) = qt^\gamma \leq q^t$. Hence, $\ln G(x) \leq t \ln q = (x/y) \ln G(y)$. Thus, $G(\cdot) \in \mathcal{G}(u)$.

By (2.16) and (2.20), for all $x_0, x_1 \geq 0$ and $x \geq z \geq u$,

$$\begin{aligned} \mathbf{P}\{\Delta \geq x_0 + x_1 + x + A(u)\} &\leq \mathbf{P}\{\Delta_u \geq x_0 + x\} + \mathbf{P}\{\delta_u \geq x_1\} \\ &\leq \mathbf{P}\{\Delta^{\{u\}} \geq x_0\} + \mathbf{P}\left\{\max_{1 \leq k \leq s} \|\zeta_k\| \geq z\right\} \\ &\quad + \frac{eG^{1-x/z}(z)}{G(u)} + \mathbf{P}\{\delta_u \geq x_1\}. \end{aligned} \quad (2.32)$$

Using (2.18) and the Lévy inequality (see [10, Theorem 1.1.1]) and taking into account that $\eta_k - \eta_k^{\{u\}} = \eta_k^{(u)}$, we obtain

$$\begin{aligned} \mathbf{P}\{\delta_u \geq x_1\} &= \mathbf{P}\left\{\max_{1 \leq m \leq s} \left\| \sum_{k=1}^m (\eta_k - \eta_k^{\{u\}}) \right\| \geq x_1\right\} \\ &\leq 2\mathbf{P}\left\{\left\| \sum_{k=1}^s \eta_k^{(u)} \right\| \geq x_1\right\}. \end{aligned} \quad (2.33)$$

By (2.13), (2.14), and (2.33), we have

$$\mathbf{P}(\delta_u \geq x_1) \leq 2 \exp\left(-\frac{x_1^2}{16B^2(u)}\right), \quad (2.34)$$

for any $x_1 > cB(u)\sqrt{d}$, where c is an absolute constant.

Integrating by parts and using relations (1.14), (1.15), (1.21), (2.25), (2.26), and (2.28)–(2.31), we get rough bounds

$$\begin{aligned} \mathbf{E}\|\zeta_k^{(u)}\|^2 &= -\int_{\{x>u\}} x^2 dF_k(x) = u^2 F_k(u) + 2 \int_{\{x>u\}} x F_k(x) dx \\ &\leq cu^2 \left(\frac{1}{s} \exp\left(-\frac{c_1 u^2}{2dw^2}\right) + d^{1+\gamma/2} \frac{\lambda_{k,\gamma}}{u^\gamma} \right) \\ &\quad + c \int_{\{x \geq u\}} x \left(\frac{1}{s} \exp\left(-\frac{c_1 x^2}{2dw^2}\right) + d^{1+\gamma/2} \frac{\lambda_{k,\gamma}}{x^\gamma} \right) dx \\ &\leq \frac{w^2}{8s} + \frac{cd^{1+\gamma/2}\lambda_{k,\gamma}}{u^{\gamma-2}} \leq \frac{w^2}{4} \leq \frac{u^2}{4}, \quad k = 1, \dots, s, \end{aligned} \quad (2.35)$$

and

$$\begin{aligned} B^2(u) &= -\int_{\{x>u\}} x^2 dF(x) = u^2 F(u) + 2 \int_{\{x>u\}} x F(x) dx \\ &\leq \frac{u^2}{2} + \frac{cd^{1+\gamma/2} L_\gamma}{u^{\gamma-2}} \leq u^2, \end{aligned} \quad (2.36)$$

if c_2 is large enough.

By (1.13), (2.1), (2.2), (2.35) and by the Chebyshev inequality,

$$\|\mathbf{E}\zeta_k^{(u)}\| \leq \frac{\mathbf{E}\|\zeta_k^{(u)}\|^2}{u} \leq \frac{w^2}{u} \leq w, \quad k = 1, \dots, s. \quad (2.37)$$

By (2.7) and (2.35), for any $v \in \mathbf{R}^d$,

$$\langle \mathbb{B}_k v, v \rangle - \langle \mathbb{B}_k^{\{u\}} v, v \rangle \leq 2 \mathbf{E}\langle \zeta_k^{(u)}, v \rangle^2 \leq 2 \mathbf{E}\|\zeta_k^{\{u\}}\|^2 \|v\|^2 \leq \frac{w^2 \|v\|^2}{2}. \quad (2.38)$$

Using (1.12), (2.7), and (2.38), it is not difficult to make sure that, for any $v \in \mathbf{R}^d$,

$$\frac{w^2 \|v\|^2}{2} \leq \langle \mathbb{B}_k^{\{u\}} v, v \rangle \leq C_1 w^2 \|v\|^2, \quad k = 1, \dots, s. \quad (2.39)$$

Arguing similarly to the proof of (2.14) and using (1.12), we obtain that there exists an absolute constant c such that, for any $x > cw\sqrt{dC_1}$, $k = 1, \dots, s$,

$$\mathbf{P}\left\{\left\|\sum_{j=m_{k-1}+1}^m Y_j\right\| \geq x\right\} \leq \exp\left(-\frac{x^2}{8C_1w^2}\right), \quad m_{k-1}+1 \leq m \leq m_k. \quad (2.40)$$

Therefore, for any $x > u$, $k = 1, \dots, s$,

$$\mathbf{P}\left\{\left\|\sum_{j=m_{k-1}+1}^m Y_j\right\| \geq x\right\} \leq \frac{1}{s} \exp\left(-\frac{x^2}{16C_1w^2}\right), \quad m_{k-1}+1 \leq m \leq m_k, \quad (2.41)$$

if c_2 in (2.25) is large enough.

Let

$$w_0 = c_4 dw \varepsilon^{-1}, \quad (2.42)$$

where $0 < \varepsilon < 1$ and c_4 is large enough. Then, choosing $c_4 \geq 1$, we can ensure that $w_0 \geq dw \geq w$.

Below we shall estimate the expectation

$$\mathbf{E}\|\zeta_k^{\{u\}}\|^3 \exp(\|\zeta_k^{\{u\}}\|w_0^{-1}) = I_1 + I_2, \quad k = 1, \dots, s, \quad (2.43)$$

where

$$I_1 = \mathbf{E}\|\zeta_k^{\{u\}}\|^3 \exp(\|\zeta_k^{\{u\}}\|w_0^{-1}) \mathbf{1}\{\|\zeta_k^{\{u\}}\| < w\} \quad (2.44)$$

and

$$I_2 = \mathbf{E}\|\zeta_k^{\{u\}}\|^3 \exp(\|\zeta_k^{\{u\}}\|w_0^{-1}) \mathbf{1}\{\|\zeta_k^{\{u\}}\| \geq w\}. \quad (2.45)$$

Notation $\mathbf{1}\{A\}$ is used here for the indicator function of the event A . Taking into account relations (2.2), (2.37), (2.44), and $w_0 \geq w$, we obtain $\zeta_k^{\{u\}} = \zeta_k^{[u]} - \mathbf{E}\zeta_k^{[u]} = \zeta_k^{[u]} + \mathbf{E}\zeta_k^{\{u\}}$ and

$$I_1 \ll w^3. \quad (2.46)$$

Moreover, by (2.1), (2.37), and (2.45),

$$\begin{aligned} I_2 &\leq \mathbf{E}\|\zeta_k + \mathbf{E}\zeta_k^{\{u\}}\|^3 \exp(\|\zeta_k + \mathbf{E}\zeta_k^{\{u\}}\|w_0^{-1}) \mathbf{1}\{w \leq \|\zeta_k\| \leq u\} \\ &\ll \mathbf{E}(\|\zeta_k\|^3 + w^3) \exp(\|\zeta_k\|w_0^{-1}) \mathbf{1}\{w \leq \|\zeta_k\| \leq u\}. \end{aligned} \quad (2.47)$$

Integrating by parts and using relations (2.21) and $w_0 \geq w$, we obtain

$$\begin{aligned} &\mathbf{E}(\|\zeta_k\|^3 + w^3) \exp(\|\zeta_k\|w_0^{-1}) \mathbf{1}\{w \leq \|\zeta_k\| \leq u\} \\ &= - \int_{\{w \leq x \leq u\}} (x^3 + w^3) e^{x/w_0} dF_k(x) \\ &\leq 2ew^3 F_k(w) + \int_{\{w \leq x \leq u\}} \left(3x^2 + \frac{x^3 + w^3}{w_0}\right) e^{x/w_0} F_k(x) dx. \end{aligned} \quad (2.48)$$

Furthermore, by (1.14) and (2.24), we have

$$\int_{\{w \leq x \leq u\}} \left(3x^2 + \frac{x^3 + w^3}{w_0} \right) e^{x/w_0} F_k(x) dx \ll I_3 + I_4, \quad (2.49)$$

where

$$I_3 = d \int_{\{w \leq x \leq u\}} \left(3x^2 + \frac{x^3 + w^3}{w_0} \right) e^{x/w_0} \exp \left(-\frac{c_1 x^2}{2dw^2} \right) dx \quad (2.50)$$

and

$$I_4 = \int_{\{w \leq x \leq u\}} \left(3x^2 + \frac{x^3 + w^3}{w_0} \right) e^{x/w_0} d^{1+\gamma/2} \frac{\lambda_{k,\gamma}}{x^\gamma} dx. \quad (2.51)$$

It is easy to see that

$$I_3 \ll \int_{\{w \leq x\}} \left(3x^2 + \frac{x^3 + w^3}{w_0} \right) \exp \left(-\frac{c_1 x^2}{4dw^2} \right) dx \ll d^2 w^3 \ll d w_0 w^2, \quad (2.52)$$

if c_4 in (2.42) is sufficiently large. Furthermore, by (1.13), (1.15), (2.25), (2.42), and (2.51),

$$\begin{aligned} I_4 &\ll d^{1+\gamma/2} \left(3u^2 + \frac{u^3 + w^3}{w_0} \right) e^{u/w_0} \lambda_{k,\gamma} \int_{\{w \leq x\}} \frac{dx}{x^\gamma} \\ &\ll \frac{d^{2+\gamma/2} \lambda_{k,\gamma} s^\varepsilon (\ln^* s)^{\gamma+3}}{L_\gamma} w_0 w^2 \ll d^2 w_0 w^2, \end{aligned} \quad (2.53)$$

if c_4 is large enough. Collecting bounds (2.43) and (2.46)–(2.53), we obtain

$$\mathbf{E} \|\zeta_k^{\{u\}}\|^3 \exp(\|\zeta_k^{\{u\}}\| w_0^{-1}) \ll d^2 w_0 w^2. \quad (2.54)$$

Using (2.54), we get, for $m = 3, 4, \dots$,

$$\mathbf{E} \|\zeta_k^{\{u\}}\|^m \leq (m-3)! w_0^{m-3} \mathbf{E} \|\zeta_k^{\{u\}}\|^3 \exp(\|\zeta_k^{\{u\}}\| w_0^{-1}) \ll d^2 m! w_0^{m-2} w^2. \quad (2.55)$$

By (2.39), inequality (2.55) is valid for $m = 2, 3, \dots$.

Let us prove that $\mathcal{L}(\zeta_k^{\{u\}}) \in \mathcal{A}_d(c_5 d^2 w_0)$, where the constant c_5 will be chosen to be so large as it will be necessary for the arguments below. Note that, in [36], the trivial relation $\mathcal{L}(\zeta_k^{\{u\}}) \in \mathcal{A}_d(cu)$ with a universal constant c was used. To simplify the formulas, denote $\xi = \zeta_k^{\{u\}}$. Let $z \in \mathbb{C}^d$, $\|z\| c_5 w_0 \leq 1$, $x = \operatorname{Re} z$, $y = \operatorname{Im} z$. Recall that $dw \leq w_0$. Choosing c_5 to be sufficiently large, and using (2.39), (2.42), and (2.55), we ensure the validity of the relations

$$\langle \mathbb{B}_k^{\{u\}} x, x \rangle + \langle \mathbb{B}_k^{\{u\}} y, y \rangle \leq C_1 w^2 (\|x\|^2 + \|y\|^2) = C_1 w^2 \|z\|^2 \leq 1, \quad (2.56)$$

$$\begin{aligned} |\mathbf{E} e^{\langle z, \xi \rangle} - 1| &= \left| \sum_{m=2}^{\infty} \frac{1}{m!} \mathbf{E} \langle z, \xi \rangle^m \right| \\ &\leq \sum_{m=2}^{\infty} \frac{2^m}{m!} \left(\mathbf{E} |\langle x, \xi \rangle|^m + \mathbf{E} |\langle y, \xi \rangle|^m \right) \end{aligned}$$

$$\begin{aligned}
&\leq c \sum_{m=2}^{\infty} 2^m \left((\|x\| w_0)^{m-2} + (\|y\| w_0)^{m-2} \right) d^2 w^2 \|z\|^2 \\
&\leq 16c w_0^2 \|z\|^2 \leq \frac{1}{2},
\end{aligned} \tag{2.57}$$

and, consequently,

$$\frac{1}{2} \leq |\mathbf{E} e^{\langle z, \xi \rangle}| \leq \frac{3}{2}. \tag{2.58}$$

Choosing c_5 to be sufficiently large, expanding the exponent and using (2.55), we see that

$$\mathbf{E} \langle \xi, v \rangle^2 e^{2|\langle \xi, x \rangle|} \ll d^2 w^2 \|v\|^2, \quad \text{for all } v \in \mathbf{R}^d. \tag{2.59}$$

Differentiating the function $\varphi(V, z)$ with $V = \mathcal{L}(\zeta_k^{\{u\}})$ (see (1.8)), we obtain

$$\begin{aligned}
d_u d_v^2 \varphi(V, z) &= \frac{\mathbf{E} \langle \xi, u \rangle \langle \xi, v \rangle^2 e^{\langle z, \xi \rangle} \mathbf{E} e^{\langle z, \xi \rangle} - \mathbf{E} \langle \xi, v \rangle^2 e^{\langle z, \xi \rangle} \mathbf{E} \langle \xi, u \rangle e^{\langle z, \xi \rangle}}{(\mathbf{E} e^{\langle z, \xi \rangle})^2} \\
&\quad - \frac{2\mathbf{E} \langle \xi, v \rangle e^{\langle z, \xi \rangle}}{\mathbf{E} e^{\langle z, \xi \rangle}} \frac{1}{(\mathbf{E} e^{\langle z, \xi \rangle})^2} \\
&\quad \times \left(\mathbf{E} \langle \xi, u \rangle \langle \xi, v \rangle e^{\langle z, \xi \rangle} \mathbf{E} e^{\langle z, \xi \rangle} - \mathbf{E} \langle \xi, v \rangle e^{\langle z, \xi \rangle} \mathbf{E} \langle \xi, u \rangle e^{\langle z, \xi \rangle} \right).
\end{aligned} \tag{2.60}$$

Using (2.39), (2.42), (2.55), (2.58)–(2.60), and the Hölder inequality, we have

$$\begin{aligned}
|d_u d_v^2 \varphi(V, z)| &\leq c \left((\mathbf{E} \langle \xi, u \rangle^2 \langle \xi, v \rangle^2 \mathbf{E} \langle \xi, v \rangle^2 e^{2\langle \xi, x \rangle})^{1/2} \right. \\
&\quad \left. + (\mathbf{E} \langle \xi, v \rangle^2 + \mathbf{E} \langle \xi, v \rangle^2 e^{2\langle \xi, x \rangle}) (\mathbf{E} \langle \xi, u \rangle^2 e^{2\langle \xi, x \rangle})^{1/2} \right) \\
&\leq c \|u\| w_0 d^2 w^2 \|v\|^2 \leq c_5 d^2 w_0 \|u\| \langle \mathbb{B}_k^{\{u\}} v, v \rangle,
\end{aligned} \tag{2.61}$$

if c_5 is chosen to be large enough. This implies that

$$\mathcal{L}(\zeta_k^{\{u\}}) \in \mathcal{A}_d(c_5 d^2 w_0)$$

(see (1.9)). Using (2.42) and the properties of the classes $\mathcal{A}_d(\tau)$, we see that

$$\mathcal{L}(\zeta_k^{\{u\}} w^{-1}) \in \mathcal{A}_d(c_5 d^2 w_0 w^{-1}) \subset \mathcal{A}_d(c_6 d^3),$$

with some $c_6 \geq 1$.

By Theorem 3 applied (with $\tau = c_6 d^3$) to the vectors $\zeta_k^{\{u\}}/w$, there exists a construction such that the joint distribution of $\{\zeta_k^{\{u\}}\}$ and $\{\eta_k^{\{u\}}\}$ satisfies

$$\mathbf{E} \exp \left(\frac{c_7 \Delta^{\{u\}}}{w d^{15/2} \ln^* d} \right) = \mathbf{E} \exp \left(\frac{c_8 \varepsilon \Delta^{\{u\}} \ln^* s}{L_\gamma^{1/\gamma} d^{15/2} \ln^* d} \right) \leq \exp(c_9 d^{3+\alpha} \ln^* s), \tag{2.62}$$

where c_7, c_8, c_9 are positive quantities depending on α, γ, C_1, C_2 , and C_3 only. Recall that we consider them as universal constants. Condition (1.10)

of Theorem 3 is satisfied due to (2.39). By (2.62) and by the Chebyshev inequality,

$$\begin{aligned} \mathbf{P}\{\Delta^{\{u\}} \geq x_0\} &\leq \exp\left(c_9 d^{3+\alpha} \ln^* s - \frac{c_8 \varepsilon x_0 \ln^* s}{L_\gamma^{1/\gamma} d^{15/2} \ln^* d}\right) \\ &\leq \exp\left(-\frac{c_8 \varepsilon x_0 \ln^* s}{2 L_\gamma^{1/\gamma} d^{15/2} \ln^* d}\right), \end{aligned} \quad (2.63)$$

provided that $c_9 d^{3+\alpha} < c_8 \varepsilon x_0 / (2 L_\gamma^{1/\gamma} d^{15/2} \ln^* d)$.

Using (2.32), (2.34), and (2.63), we get

$$\begin{aligned} \mathbf{P}\{\Delta \geq x_0 + x_1 + x + A(u)\} \\ \leq 2 \exp\left(-\frac{x_1^2}{16 B^2(u)}\right) + \exp\left(-\frac{c_8 \varepsilon x_0 \ln^* s}{2 L_\gamma^{1/\gamma} d^{15/2} \ln^* d}\right) \\ + \mathbf{P}\left\{\max_{1 \leq k \leq s} \|\zeta_k\| \geq z\right\} + \frac{e G^{1-x/z}(z)}{G(u)}, \end{aligned} \quad (2.64)$$

for $x \geq z \geq u$ and $x_0 > 2 L_\gamma^{1/\gamma} c_9 c_8^{-1} \varepsilon^{-1} d^{21/2+\alpha} \ln^* d$, $x_1 > c B(u) \sqrt{d}$.

We shall apply this inequality in the case, where $G(x)$ is defined in (2.30) and

$$u \leq z = x_0 = x_1 \leq x = 2z. \quad (2.65)$$

By (2.1), (2.2), (2.11), (2.17), (2.30), (2.36), and (2.65), we have

$$A(u) u^{-1} \leq B^2(u) u^{-2} \leq 1, \quad (2.66)$$

$$\mathbf{P}\left\{\max_{1 \leq k \leq s} \|\zeta_k\| \geq z\right\} \leq \sum_{k=1}^s \mathbf{P}\{\|\zeta_k\| \geq z\} = F(z) \ll d^{1+\gamma/2} \frac{L_\gamma}{z^\gamma}. \quad (2.67)$$

Recall that $G(u) \geq e^\gamma$ (see (2.31)). Finally, by (2.64)–(2.67),

$$\begin{aligned} \mathbf{P}\{\Delta \geq 5z\} &\leq \mathbf{P}\{\Delta \geq 4z + A(u)\} = \mathbf{P}\{\Delta \geq x_0 + x_1 + x + A(u)\} \\ &\leq 2 \exp\left(-\frac{z^2}{16 u^2}\right) + \exp\left(-\frac{c_{10} \varepsilon z \ln^* s}{L_\gamma^{1/\gamma} d^{15/2} \ln^* d}\right) + 2 F(z), \end{aligned} \quad (2.68)$$

provided that

$$z > z_0 \stackrel{\text{def}}{=} c_{11} \varepsilon^{-1} L_\gamma^{1/\gamma} d^{21/2+\alpha} \ln^* d \geq d^9 u \geq u, \quad (2.69)$$

where c_{10} , c_{11} are positive quantities depending on α , γ , C_1 , C_2 , and C_3 only. The c_{11} will be chosen to be sufficiently large. Define

$$\Delta^* = \max_{1 \leq k \leq s} \max_{m_{k-1}+1 \leq m \leq m_k} \left\| \sum_{j=m_{k-1}+1}^m \xi_j \right\|, \quad (2.70)$$

$$\Delta^{**} = \max_{1 \leq k \leq s} \max_{m_{k-1}+1 \leq m \leq m_k} \left\| \sum_{j=m_{k-1}+1}^m Y_j \right\|. \quad (2.71)$$

Then, according to (1.11), (2.9), (2.15), (2.70), and (2.71), we have, for any $z \geq 0$,

$$\mathbf{P}\{\Delta(\xi, Y) \geq 9z\} \leq \mathbf{P}\{\Delta \geq 5z\} + \mathbf{P}\{\Delta^* \geq 3z\} + \mathbf{P}\{\Delta^{**} \geq z\}. \quad (2.72)$$

To bound the second summand in (2.72), we apply Lemma 1. By this lemma, we derive

$$\begin{aligned} \mathbf{P}\{\Delta^* \geq 3z\} &\leq \sum_{k=1}^s \mathbf{P}\left\{ \max_{m_{k-1}+1 \leq m \leq m_k} \left\| \sum_{j=m_{k-1}+1}^m \xi_j \right\| \geq 3z \right\} \\ &\leq 3 \sum_{k=1}^s \max_{m_{k-1}+1 \leq m \leq m_k} \mathbf{P}\left\{ \left\| \sum_{j=m_{k-1}+1}^m \xi_j \right\| \geq z \right\} \\ &= 3 \sum_{k=1}^s T_k(z) \end{aligned} \quad (2.73)$$

(see (2.21)). By (2.41) and by the Lévy inequality,

$$\begin{aligned} \mathbf{P}\{\Delta^{**} \geq z\} &\leq \sum_{k=1}^s \mathbf{P}\left\{ \max_{m_{k-1}+1 \leq m \leq m_k} \left\| \sum_{j=m_{k-1}+1}^m Y_j \right\| \geq z \right\} \\ &\leq 2 \sum_{k=1}^s \mathbf{P}\left\{ \left\| \sum_{j=m_{k-1}+1}^{m_k} Y_j \right\| \geq z \right\} \\ &\leq 2 \exp\left(-\frac{z^2}{16C_1 w^2}\right), \end{aligned} \quad (2.74)$$

for $z > u$. Collecting bounds (2.22), (2.67), (2.68), and (2.72)–(2.74), we obtain

$$\begin{aligned} \mathbf{P}\{\Delta(\xi, Y) \geq 9z\} &\leq 5 \sum_{k=1}^s T_k(z) + 2 \exp\left(-\frac{z^2}{16C_1 w^2}\right) \\ &\quad + 2 \exp\left(-\frac{z^2}{16u^2}\right) + \exp\left(-\frac{c_{10}\varepsilon z \ln^* s}{L_\gamma^{1/\gamma} d^{15/2} \ln^* d}\right), \end{aligned} \quad (2.75)$$

provided that $z > z_0$. Relations (1.13), (1.21), (2.25), (2.27), and (2.75) imply inequality (1.16).

Choosing c_{11} to be large enough and using (2.69) and the inequality $r^\beta \beta^{-\beta} \leq e^r$, we obtain the rough bound

$$\begin{aligned} \exp\left(-\frac{c_{10}\varepsilon z}{L_\gamma^{1/\gamma} d^{15/2} \ln^* d}\right) &\leq \left(\frac{2 L_\gamma^{1/\gamma} d^{15/2} \ln^* d \cdot \gamma b}{c_{10}\varepsilon z}\right)^{\gamma b} \\ &\quad \times \exp\left(-\frac{c_{10}\varepsilon z}{2 L_\gamma^{1/\gamma} d^{15/2} \ln^* d}\right) \\ &\leq (2 c_{10}^{-1} \gamma b \varepsilon^{-1} d^{15/2} \ln^* d)^{\gamma b} (L_\gamma z^{-\gamma})^b e^{-d^3}, \end{aligned}$$

for any $b \geq 1$ and $z > z_0$. Moreover, $(d^9)^{\gamma b} \leq (3\gamma b)^{3\gamma b} e^{d^3}$, and, hence, there exists c_{13} such that

$$\exp\left(-\frac{c_{10} \varepsilon z}{L_\gamma^{1/\gamma} d^{15/2} \ln^* d}\right) \leq (c_{13} \gamma^4 b^4 \varepsilon^{-1})^{\gamma b} (L_\gamma z^{-\gamma})^b, \quad (2.76)$$

for any $b \geq 1$ and $z > z_0$. In a similar way one can estimate the other exponential terms in (2.75). Using (1.14), (1.15), and (1.21), we get

$$\sum_{k=1}^s \lambda_{k,\gamma}^b \leq L_\gamma \left(\max_{1 \leq k \leq s} \lambda_{k,\gamma} \right)^{b-1} \leq \frac{L_\gamma^b}{(C_3 d^{\gamma/2} s^\varepsilon (\ln^* s)^{\gamma+3})^{b-1}}, \quad (2.77)$$

for $b \geq 1$. Relations (1.13), (2.25), (2.27), and (2.75)–(2.77) imply inequality (1.17). Inequality (1.18) may be easily deduced from (1.17) with $b = 1$ estimating the last sum by the Chebyshev inequality.

Integrating by parts, we get

$$\mathbf{E} \left(\frac{\Delta(\xi, Y)}{9} \right)^\gamma = \gamma \int_0^\infty z^{\gamma-1} \mathbf{P}\{\Delta(\xi, Y) \geq 9z\} dz = \gamma(K_1 + K_2), \quad (2.78)$$

where

$$K_1 = \int_0^{z_0} z^{\gamma-1} \mathbf{P}\{\Delta(\xi, Y) \geq 9z\} dz, \quad (2.79)$$

and

$$K_2 = \int_{z_0}^\infty z^{\gamma-1} \mathbf{P}\{\Delta(\xi, Y) \geq 9z\} dz. \quad (2.80)$$

It is easy to see that, by (2.69) and (2.79),

$$K_1 \leq \int_0^{z_0} z^{\gamma-1} dz = \frac{z_0^\gamma}{\gamma} \ll L_\gamma (\varepsilon^{-1} d^{21/2+\alpha} \ln^* d)^\gamma. \quad (2.81)$$

Applying inequality (1.17) with $b = 2$, we see that

$$K_2 \ll J_1 + J_2, \quad (2.82)$$

where

$$J_1 = d^{1+\gamma/2} \varepsilon^{-2\gamma} L_\gamma^2 \int_{z_0}^\infty \frac{dz}{z^{\gamma+1}}, \quad (2.83)$$

$$J_2 = d \int_{z_0}^\infty z^{\gamma-1} \sum_{j=1}^n \mathbf{P} \left\{ \|\xi_j\| \geq \frac{z}{4\sqrt{d}} \right\} dz. \quad (2.84)$$

Using (2.69) and (2.83), we see that

$$J_1 \ll d^{1+\gamma/2} \varepsilon^{-2\gamma} L_\gamma^2 z_0^{-\gamma} \ll \varepsilon^{-\gamma} L_\gamma. \quad (2.85)$$

Moreover,

$$\int_{z_0}^\infty z^{\gamma-1} \mathbf{P} \left\{ \|\xi_j\| \geq \frac{z}{4\sqrt{d}} \right\} dz \ll \mathbf{E} \left(4\sqrt{d} \|\xi_j\| \right)^\gamma \ll d^{\gamma/2} \mathbf{E} \|\xi_j\|^\gamma, \quad (2.86)$$

Using (1.5), (2.84), and (2.86), we obtain

$$J_2 \ll d^{1+\gamma/2} L_\gamma. \quad (2.87)$$

Collecting relations (2.78)–(2.82), (2.85), and (2.87), we obtain

$$\mathbf{E}(\Delta(\xi, Y))^\gamma \ll L_\gamma (\varepsilon^{-1} d^{21/2+\alpha} \ln^* d)^\gamma, \quad (2.88)$$

proving (1.19). The proof of Theorem 4 is thus completed.

P r o o f o f P r o p o s i t i o n 1. Similarly to the proof of Theorem 4, we shall treat γ and C_4 as absolute positive constants.

Define

$$\Delta_1 = \max_{1 \leq m \leq n} \left\| \sum_{j=1}^m X_j \right\|, \quad (2.89)$$

$$\Delta_2 = \max_{1 \leq m \leq n} \left\| \sum_{j=1}^m Y_j \right\|, \quad (2.90)$$

Then, according to (1.1), (2.89), and (2.90), we have, for any $z \geq 0$,

$$\mathbf{P}\{\Delta(X, Y) \geq 4z\} \leq \mathbf{P}\{\Delta_1 \geq 3z\} + \mathbf{P}\{\Delta_2 \geq z\}. \quad (2.91)$$

To bound the first summand, we apply Lemma 1. By this lemma, we derive

$$\mathbf{P}\{\Delta_1 \geq 3z\} \leq 3 \max_{1 \leq m \leq n} \mathbf{P}\left\{\left\| \sum_{j=1}^m X_j \right\| \geq z\right\} \quad (2.92)$$

(see (2.89)). By (2.90) and by the Lévy inequality,

$$\mathbf{P}\{\Delta_2 \geq z\} \leq 2 \mathbf{P}\left\{\left\| \sum_{j=1}^n Y_j \right\| \geq z\right\}. \quad (2.93)$$

Applying Lemma 2 and using (1.5), we get, similarly to (2.23), that, for any $z > 0$ and $b \geq 1$,

$$\begin{aligned} \mathbf{P}\left\{\left\| \sum_{j=1}^m X_j \right\| \geq z\right\} &\leq 2d \exp\left(-\frac{c_{12}z^2}{d\sigma^2}\right) + 2(2b)^{b(\gamma-1)} d^{1+\gamma b/2} \frac{L_\gamma^b}{z^{b\gamma}} \\ &\quad + d \sum_{j=1}^n \mathbf{P}\left\{\|X_j\| \geq \frac{z}{2b\sqrt{d}}\right\}, \quad m = 1, \dots, n. \end{aligned} \quad (2.94)$$

Integrating by parts and using (2.91), we get

$$\mathbf{E}\left(\frac{\Delta(X, Y)}{4}\right)^\gamma = \gamma \int_0^\infty z^{\gamma-1} \mathbf{P}\{\Delta(X, Y) \geq 4z\} dz = \gamma (M_1 + M_2 + M_3), \quad (2.95)$$

where

$$M_1 = \int_0^y z^{\gamma-1} \mathbf{P}\{\Delta(X, Y) \geq 4z\} dz, \quad (2.96)$$

$$M_2 = \int_y^\infty z^{\gamma-1} \mathbf{P}\{\Delta_1 \geq 3z\} dz, \quad (2.97)$$

$$M_3 = \int_y^\infty z^{\gamma-1} \mathbf{P}\{\Delta_2 \geq z\} dz \quad (2.98)$$

with $y = c_{13}\sqrt{d} L_\gamma^{1/\gamma}$ and sufficiently large constant c_{13} . It is easy to see that

$$M_1 \leq \int_0^y z^{\gamma-1} dz = \frac{y^\gamma}{\gamma} \ll d^{\gamma/2} L_\gamma. \quad (2.99)$$

To estimate M_2 , we shall apply inequality (2.94) with $b = 2$. It is clear that

$$\int_y^\infty z^{\gamma-1} \exp\left(-\frac{c_{12}z^2}{d\sigma^2}\right) dz \ll d^{\gamma/2} \sigma^\gamma \leq d^{\gamma/2} L_\gamma \quad (2.100)$$

(see (1.22)) and

$$\int_y^\infty z^{\gamma-1} \frac{L_\gamma^2}{z^{2\gamma}} dz \ll L_\gamma^2 y^{-\gamma} \ll d^{-\gamma/2} L_\gamma. \quad (2.101)$$

Moreover,

$$\int_y^\infty z^{\gamma-1} \mathbf{P}\left\{\|\xi_j\| \geq \frac{z}{4\sqrt{d}}\right\} dz \ll \mathbf{E}(4\sqrt{d}\|\xi_j\|)^\gamma = 4^\gamma d^{\gamma/2} \mathbf{E}\|\xi_j\|^\gamma, \quad (2.102)$$

and, by (1.5), (2.92), (2.94), (2.97), and (2.100)–(2.102),

$$M_2 \ll d^{1+\gamma/2} L_\gamma. \quad (2.103)$$

Arguing similarly to the proof of (2.14) and using the definition of σ^2 , we obtain that there exists an absolute constant c_{14} such that, for any $z > c_{14}\sigma\sqrt{d}$,

$$\mathbf{P}\left\{\left\|\sum_{j=1}^n Y_j\right\| \geq z\right\} \leq \exp\left(-\frac{z^2}{8\sigma^2}\right). \quad (2.104)$$

Choosing $c_{13} \geq c_{14} C_4$ and using (1.22), we have $y \geq c_{14}\sigma\sqrt{d}$. By (1.22), (2.93), (2.98), and (2.104), we get

$$M_3 \ll \sigma^\gamma \leq L_\gamma. \quad (2.105)$$

Collecting relations (2.95), (2.99), (2.103), and (2.105), we obtain

$$\mathbf{E}\left(\frac{\Delta(X, Y)}{4}\right)^\gamma \ll d^{1+\gamma/2} L_\gamma \quad (2.106)$$

proving (1.24). Note that a similar inequality for $\|\sum_{j=1}^n X_j - \sum_{j=1}^n Y_j\|$ instead of $\Delta(X, Y)$ might easily be derived from Lemma 3. Inequality (1.25) follows from (1.24) by the Chebyshev inequality.

By (1.22), there exists c_{15} such that

$$\exp\left(-\frac{c_{12}z^2}{d\sigma^2}\right) + \exp\left(-\frac{z^2}{8\sigma^2}\right) \leq (c_{15}\gamma bd)^{b\gamma/2} \frac{L_\gamma^b}{z^{b\gamma}}. \quad (2.107)$$

Inequality (1.23) follows from (2.91)–(2.94), (2.104), and (2.107). The proof of Proposition 1 is thus completed.

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