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сходится по вариации, и его сумма является мерой $d_{D_H F}(\zeta\mu)$, причем $\rho_\zeta = d_{D_H F}(\zeta\mu)/\mu \in L^2(\mu)$. Следовательно, функция F , локализирующее семейство функций $\mathcal{L}$ и мера μ удовлетворяют условиям теоремы 5.2, и при всех $a > 0$ существуют σ -конечные поверхностные меры $\nu^{(a)}$ и $\mu_\sigma^{(a)}$; очевидно, что при $a \leq 0$ они обращаются в нуль.

Поскольку $C_{1,p}(X \setminus K_t) \rightarrow 0$ при любом $p \geq 1$, справедлива формула (8) для $C_{1,4}$ -квазинепрерывных $g \in W_{\text{Loc}}^{1,r}(\mu)$ при $r > 4$.

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INTRINSICALLY STATIONARY VARIOGRAMS IN SPACE AND TIME

В статье обсуждается вопрос о том, когда произведение двух внутренне стационарных вариограмм структурных функций, а также степень внутренне стационарной вариограммы от пространственной и временной переменных остается вариограммой. Указывается несколько новых классов пространственно-временных вариограмм, а также приводятся степенные модели и модели с медленным затуханием ковариации.

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В частности, доказано, что произведение двух изотропных вариограмм, построенных с помощью функции Бернштейна, является изотропной вариограммой, и указана наибольшая допустимая степень такая, что возведенная в эту степень вариограмма, построенная с помощью функции Бернштейна, остается вариограммой.

Ключевые слова и фразы: функция Бернштейна, ковариационная функция, преобразование Фурье, изотропия, долговременная зависимость, отрицательная определенность, степенная модель, стационарность.

1. Introduction. The variogram, or the structure function, was first used by Kolmogorov [13], [14] to describe statistical properties of stochastic processes [29]. The term «variogram» is widely adopted in spatial or spatio-temporal statistics literature [5], [7], [8], while the term «structure function» is popular in physics. The square root of the variogram is usually employed as a pseudo-metric in the study of stochastic processes [3], [23]. Like the covariance function, the variogram is an important dependence measure for stochastic processes or random fields that have second-order increments, and is extensively employed in practice as a useful structural tool for analyzing space and/or time data; see, for instance, [4]–[8], [10]–[12], [15], [16], [18], [19], [21], [22], [24], [25], [27], [28].

Consider a real-valued stochastic process or random field $\{Z(\mathbf{x}), \mathbf{x} \in \mathcal{D}\}$, where $\mathcal{D}$ is a spatial, or temporal, or spatio-temporal index domain. When the increments of the random field have finite second-order moments, the variogram of $\{Z(\mathbf{x}), \mathbf{x} \in \mathcal{D}\}$ is defined as half of the variance of the increment $Z(\mathbf{x}_1) - Z(\mathbf{x}_2)$, that is,

$$\gamma(\mathbf{x}_1, \mathbf{x}_2) = \frac{1}{2} \mathbf{D}(Z(\mathbf{x}_1) - Z(\mathbf{x}_2)), \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{D}.$$

When the random field itself has the finite second-order moment, its covariance function is defined by

$$C(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{E}\{(Z(\mathbf{x}_1) - \mathbf{E}Z(\mathbf{x}_1))(Z(\mathbf{x}_2) - \mathbf{E}Z(\mathbf{x}_2))\}, \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{D}.$$

In case $\gamma(\mathbf{x}_1, \mathbf{x}_2)$ depends only on $\mathbf{x}_1 - \mathbf{x}_2$, we simply write $\gamma(\mathbf{x}_1, \mathbf{x}_2)$ as $\gamma(\mathbf{x}_1 - \mathbf{x}_2)$ and call it an intrinsically stationary variogram on $\mathcal{D}$. Similarly, a stationary covariance function, denoted by $C(\mathbf{x}_1 - \mathbf{x}_2)$, depends only on $\mathbf{x}_1 - \mathbf{x}_2$.

This paper is motivated by looking for appropriate answers to the following two interesting and challenging questions:

- (i) When is the product of two variograms still a variogram?
- (ii) What is the largest power such that the power of a variogram is a variogram?

It leads us to develop new classes of intrinsically stationary variograms, as well as power-law and other long-range dependent covariance models in space and time, which will benefit the application of space-time random fields in practice.

A typical answer to the first question is «no», because the product of two variograms is not necessarily a variogram. For example, $|x|$ and x^2 are variograms on the real line, but their product $|x|^3$ ($x \in \mathbf{R}$) is not a variogram. Neither is $\{1 - \exp(-\alpha_1 x^2)\}\{1 - \exp(-\alpha_2 x^2)\}$, $x \in \mathbf{R}$, the product of two variograms $1 - \exp(-\alpha_1 x^2)$ and $1 - \exp(-\alpha_2 x^2)$, where α_1 and α_2 are positive constants. Somehow these examples, combining those facts in [20], give us a hint that we should restrict our search in certain particular classes. In Section 3 we investigate the permissibility of the product of two isotropic variograms that are of the form $B(\|\mathbf{x}\|)$, where $B(x)$ is a Bernstein function on $[0, \infty)$. Here, as usual, by an isotropic variogram in $\mathbf{R}^d$ we mean a variogram that is only a function of the Euclidean norm $\|\mathbf{x}\| = (\sum_{k=1}^d |x_k|^2)^{1/2}$, $\mathbf{x} \in \mathbf{R}^d$.

The second question we ask above is essentially required to identify the domain of the parameter λ that ensures the validity of $\gamma^\lambda(\mathbf{x})$ on $\mathcal{D}$, for a given intrinsically stationary variogram $\gamma(\mathbf{x})$. Of course, λ cannot be negative, and $\lambda = 0$ corresponds to a degenerate case that will be excluded from our investigation. It is well known that $\gamma^\lambda(\mathbf{x})$ is an intrinsically stationary variogram whenever λ is a positive constant between 0 and 1 (see, e.g., [1, Corollary 2.10, p. 78]). In some cases, λ has to be restricted in the interval $(0, 1]$,

while in other cases the domain of the parameter λ is much wider than $(0, 1]$. A classic example is $|x|^{\lambda_0}$, $x \in \mathbf{R}$, the variogram associated with a fractional Brownian motion whose covariance is $|x_1|^{\lambda_0} + |x_2|^{\lambda_0} - |x_1 - x_2|^{\lambda_0}$, where $\lambda_0 \in (0, 2]$. In such a case, $(|x|^{\lambda_0})^\lambda = |x|^{\lambda_0 \lambda}$ is a variogram on $\mathbf{R}$ provided that $\lambda \in (0, 2/\lambda_0]$, and particularly, the interval $(0, 2/\lambda_0]$ coincides with $(0, 1]$ only if $\lambda_0 = 2$. In general, for a continuous variogram $\gamma(\mathbf{x})$ in $\mathbf{R}^d$ that is not identical to zero, there exists a finite upper bound λ^* such that $\gamma^\lambda(\mathbf{x})$ is a variogram in $\mathbf{R}^d$ if $0 < \lambda \leq \lambda^*$, and is not if $\lambda > \lambda^*$ (see Lemma 2.3 below).

The domain of the positive parameter λ is generally related to properties of the underlying variogram $\gamma(\mathbf{x})$, although it seems difficult to describe it precisely. It is shown in Section 3 that $B^\lambda(\|\mathbf{x}\|)$ is an isotropic variogram in $\mathbf{R}^d$ provided that $\lambda \in (0, 2]$, where $B(x)$ is a Bernstein function on $[0, \infty)$. Two other extreme cases, where $\gamma^\lambda(\mathbf{s}; t)$ is a space-time variogram only if $\lambda \in (0, 1]$, are presented in Section 4.

A fundamental characterization of an intrinsically stationary variogram $\gamma: \mathcal{D} \rightarrow [0, \infty)$ is that it is symmetric (i.e., $\gamma(-\mathbf{x}) = \gamma(\mathbf{x})$, $\mathbf{x} \in \mathcal{D}$) with $\gamma(\mathbf{0}) = 0$, and (conditionally) negative definite. The latter means that for every integer $n \geq 2$, the inequality

$$\sum_{i=1}^n \sum_{j=1}^n \gamma(\mathbf{x}_i - \mathbf{x}_j) a_i a_j \leq 0$$

holds for all $\mathbf{x}_k \in \mathcal{D}$ and all real numbers a_k ($k = 1, \dots, n$) subject to $\sum_{k=1}^n a_k = 0$. However, direct verification of negative definiteness is often not an easy job. Neither is to check nonnegativity of the Fourier transform for a covariance function. For instance, $x^2(1+|x|)^{-2}$ ($x \in \mathbf{R}$) is a variogram (see Example 3.3 below), and $1 - x^2(1+|x|)^{-2}$ ($x \in \mathbf{R}$) is a stationary correlation function, but the Fourier transform of the latter involves $\text{si}(x)$ and $\text{ci}(x)$ and is not so easy to see whether it is nonnegative directly. Other interesting examples can be found in [30]. To derive our results in this paper, some preliminary results are given in Section 2. Proofs of Lemmas 2.3 and 2.4 and theorems are presented in Section 5.

Throughout this paper, $\lambda, \alpha_1, \alpha_2$, and α are assumed to be positive constants. The terms *intrinsically stationary variogram* and *variogram* will be used interchangeably. A spatio-temporal index domain will be denoted by $\mathcal{S} \times \mathcal{T}$, where $\mathcal{S}$ is typically $\mathbf{R}^d$ or $\mathbf{Z}^d$, and $\mathcal{T}$ equals $\mathbf{R}$ or $\mathbf{Z}$, and in such a case a variogram will be written by $\gamma(\mathbf{s}; t)$ instead of $\gamma(\mathbf{x})$. Mathematically speaking, a result in $\mathbf{R}^d$ ($d > 1$) can be of course treated as a space-time statement on $\mathbf{R}^{d-1} \times \mathbf{R}$.

2. Preliminary results. Since it is often not easy to check whether a function is negative definite, in order to derive our results in this paper, we adopt the following simple and effective procedure: first, express the given function as a nonnegative mixture of another function, and then, show that the latter is negative definite. This procedure is stated as Lemma 2.1 without a proof.

Lemma 2.1. *The integral $\int_{\mathbf{u} \in \mathbf{R}^{d_0}} \gamma_{\mathbf{u}}(\mathbf{x}) p(\mathbf{u}) d\mu(\mathbf{u})$ is an intrinsically stationary variogram on $\mathcal{D}$ if the integral exists and the following conditions hold:*

- (i) $\gamma_{\mathbf{u}}(\mathbf{x})$ is an intrinsically stationary variogram for every fixed $\mathbf{u} \in \mathbf{R}^{d_0}$,
- (ii) $p(\mathbf{u})$ is nonnegative, and
- (iii) $\mu(\mathbf{u})$ is a nonnegative measure on $\mathbf{R}^{d_0}$.

The following lemma allows us always to look at the largest power. Its proof is straightforward and is omitted.

Lemma 2.2. *Let λ_0 be a positive constant. Then $\gamma^\lambda(\mathbf{x})$ is an intrinsically stationary variogram on $\mathcal{D}$ for every constant $\lambda \in (0, \lambda_0]$ if and only if $\gamma^{\lambda_0}(\mathbf{x})$ is an intrinsically stationary variogram on $\mathcal{D}$.*

The following lemma tells us that $\gamma(\mathbf{x})$, $\mathbf{x} \in \mathcal{D}$, must be bounded if all of its powers are variograms, and that in the particular case, where $\gamma(\mathbf{x})$, $\mathbf{x} \in \mathbf{R}^d$, is not identical to zero and is continuous at the origin, there exists a finite power λ^* such that $\gamma^\lambda(\mathbf{x})$ is a variogram on $\mathbf{R}^d$ if $0 < \lambda \leq \lambda^*$, and is not if $\lambda > \lambda^*$.

Lemma 2.3. (i) If $\gamma^\lambda(\mathbf{x})$, $\mathbf{x} \in \mathcal{D}$, is an intrinsically stationary variogram for all positive constants λ , then for every integer k ,

$$\gamma(k\mathbf{x}) \leq \gamma(\mathbf{x}), \quad \mathbf{x} \in \mathcal{D}. \quad (1)$$

(ii) If, in addition, $\gamma(\mathbf{x})$, $\mathbf{x} \in \mathbf{R}^d$, is continuous at the origin, then it is identical to zero.

Lemma 2.3, (ii) may not hold if $\gamma(\mathbf{x})$ is discontinuous at the origin. For example, the function

$$\gamma(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} = \mathbf{0}, \\ c_0 & \text{otherwise,} \end{cases} \quad (2)$$

and all of its positive powers, $\gamma^\lambda(\mathbf{x})$, are variograms on $\mathcal{D}$, whenever $c_0 \in [0, 1]$ is a constant. The function (2) is not the only case, where $\gamma^\lambda(\mathbf{x})$ is a variogram for every positive constant λ . Another one on the real line [9] is

$$\gamma(x) = \begin{cases} 0 & \text{if } x \text{ is rational,} \\ 1 & \text{if } x \text{ is irrational,} \end{cases} \quad x \in \mathbf{R}.$$

The domain of the positive parameter λ is generally related to the smoothness of the underlying variogram $\gamma: \mathbf{R}^d \rightarrow [0, \infty)$ near the origin. The following lemma provides a maximum possibility of λ for $\gamma^\lambda(\mathbf{x})$ to be a variogram in $\mathbf{R}^d$, in case one of the margins of $\gamma(\mathbf{x})$ has a nonzero first- or second-order right-hand derivative at the origin.

Lemma 2.4. Assume that $\gamma: \mathbf{R}^d \rightarrow [0, \infty)$ is an intrinsically stationary variogram.

(i) If one of the margins of $\gamma(\mathbf{x})$, $\gamma(x_1, 0, \dots, 0), \dots, \gamma(0, \dots, 0, x_d)$, has a positive right-hand derivative at the origin, and $\gamma^\lambda(\mathbf{x})$ is an intrinsically stationary variogram in $\mathbf{R}^d$, then λ is at most 2.

(ii) When one of the margins of $\gamma(\mathbf{x})$, $\gamma(x_1, 0, \dots, 0), \dots, \gamma(0, \dots, 0, x_d)$, has a zero right-hand derivative and a positive second-order right-hand derivative at the origin, a necessary and sufficient condition for $\gamma^\lambda(\mathbf{x})$ to be an intrinsically stationary variogram in $\mathbf{R}^d$ is $\lambda \in (0, 1]$.

3. The product and power of isotropic variograms. In this section we examine the validity of the product of two isotropic variograms and the power of an isotropic variogram, which are formed through the Bernstein function. In particular, Theorem 3.1, (ii) below provides a good reason for why we just deal with the permissibility of the product of two isotropic or geometrically anisotropic variograms.

Recall that a continuous function $\ell: [0, \infty) \rightarrow [0, \infty)$ is completely monotone on $[0, \infty)$, if it possesses derivatives of all orders on $(0, \infty)$ and for every positive integer n ,

$$(-1)^n \ell^{(n)}(x) \geq 0, \quad x > 0,$$

or, equivalently, by Bernstein's theorem, if it is the Laplace transform of a bounded and nondecreasing function $F(x)$ on $[0, \infty)$,

$$\ell(x) = \int_0^\infty \exp(-xu) dF(u), \quad x \geq 0.$$

A continuous function $B: [0, \infty) \rightarrow [0, \infty)$ with $B(0) = 0$ is called a Bernstein function [2], if it possesses derivatives of all orders on $(0, \infty)$ and for every positive integer n ,

$$(-1)^{n-1} B^{(n)}(x) \geq 0, \quad x > 0,$$

or, equivalently, if

$$B(x) = \int_0^\infty \{1 - \exp(-xu)\} u^{-1} dF(u), \quad x \geq 0, \quad (3)$$

where $F(x)$, $x > 0$, is nondecreasing and $\int_0^1 u^{-1} dF(u)$ exists.

Two useful approaches to generate Bernstein functions are: (i) to integrate a completely monotone function defined on $[0, \infty)$, and (ii) to compose two Bernstein functions defined on $[0, \infty)$. See [26] for a recent expository survey of properties of completely monotone functions as well as many examples.

Theorem 3.1. (i) If $B_1: [0, \infty) \rightarrow [0, \infty)$ and $B_2: [0, \infty) \rightarrow [0, \infty)$ are Bernstein functions, then

$$\gamma(\mathbf{x}) = B_1(\|\mathbf{x}\|)B_2(\|\mathbf{x}\|), \quad \mathbf{x} \in \mathbf{R}^d,$$

is an intrinsically stationary variogram in $\mathbf{R}^d$.

(ii) Suppose that, in addition, $\lim_{x \rightarrow 0+} B_k(x)/x$ exists and is positive ($k = 1, 2$). If $g: \mathbf{R}^d \rightarrow [0, \infty)$ is a homogeneous function, i.e., $g(\alpha\mathbf{x}) = |\alpha|g(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{R}^d$ and $\alpha \in \mathbf{R}$, and

$$\gamma(\mathbf{x}) = B_1(g(\mathbf{x}))B_2(g(\mathbf{x})), \quad \mathbf{x} \in \mathbf{R}^d, \quad (4)$$

is an intrinsically stationary variogram in $\mathbf{R}^d$, then there exists a $d \times d$ matrix A such that

$$g(\mathbf{x}) = \|A\mathbf{x}\|, \quad \mathbf{x} \in \mathbf{R}^d. \quad (5)$$

As a particular case, Theorem 3 in [20] is obtained from Theorem 3.1 by choosing $B_1(x) = 1 - \exp(-\alpha_1 x)$ and $B_2(x) = 1 - \exp(-\alpha_2 x)$, $x \geq 0$, where α_1 and α_2 are positive constants.

Clearly, $\ell(0) - \ell(x)$, $x \geq 0$, is a Bernstein function whenever $\ell(x)$ is completely monotone on $[0, \infty)$.

Corollary 3.1. If $\ell_1(x)$ and $\ell_2(x)$ are completely monotone functions on $[0, \infty)$, then

$$\gamma(\mathbf{x}) = \{\ell_1(0) - \ell_1(\|\mathbf{x}\|)\}\{\ell_2(0) - \ell_2(\|\mathbf{x}\|)\}, \quad \mathbf{x} \in \mathbf{R}^d,$$

is a variogram in $\mathbf{R}^d$, and

$$C(\mathbf{x}) = \ell_2(0)\ell_1(\|\mathbf{x}\|) + \ell_1(0)\ell_2(\|\mathbf{x}\|) - \ell_1(\|\mathbf{x}\|)\ell_2(\|\mathbf{x}\|), \quad \mathbf{x} \in \mathbf{R}^d, \quad (6)$$

is a stationary covariance function in $\mathbf{R}^d$.

A linear combination of covariance functions with a negative coefficient is not always a covariance function, unlike the sum of covariances. This makes (6) exceptional. Other such examples can be found in [18], [19], and [21].

Corollary 3.2. If $B: [0, \infty) \rightarrow [0, \infty)$ is a Bernstein function, and $\lambda_0 \in (0, 1]$ is a constant, then

$$\gamma(\mathbf{x}) = B^2(\|\mathbf{x}\|^{\lambda_0}), \quad \mathbf{x} \in \mathbf{R}^d,$$

is a variogram in $\mathbf{R}^d$.

Corollary 3.3. If $\ell(x)$ is a completely monotone function on $[0, \infty)$, and $\lambda_0 \in (0, 1]$ is a constant, then

$$\gamma(\mathbf{x}) = \{\ell(0) - \ell(\|\mathbf{x}\|^{\lambda_0})\}^2, \quad \mathbf{x} \in \mathbf{R}^d,$$

is a variogram in $\mathbf{R}^d$ and

$$C(\mathbf{x}) = 2\ell(0)\ell(\|\mathbf{x}\|^{\lambda_0}) - \ell^2(\|\mathbf{x}\|^{\lambda_0}), \quad \mathbf{x} \in \mathbf{R}^d,$$

is a stationary covariance function in $\mathbf{R}^d$.

Example 3.1. Below are three simple Bernstein functions on $[0, \infty)$:

$$B_1(x) = x^\lambda, \quad B_2(x) = \ln(1 + x^\lambda), \quad \text{and} \quad B_3(x) = x^\lambda(1 + x^\lambda)^{-1}, \quad x \geq 0,$$

where λ is a positive constant between 0 and 1. By Theorem 3.1, we obtain the following isotropic variograms in $\mathbf{R}^d$:

- (i) $\|\mathbf{x}\|^{\lambda_1} \ln(1 + \alpha\|\mathbf{x}\|^{\lambda_2}),$
- (ii) $\|\mathbf{x}\|^{\lambda_1} (1 + \alpha_1\|\mathbf{x}\|^{\lambda_1})^{-1} \ln(1 + \alpha_2\|\mathbf{x}\|^{\lambda_2}),$ and
- (iii) $\|\mathbf{x}\|^{\lambda_1 + \lambda_2} (1 + \alpha\|\mathbf{x}\|^{\lambda_2})^{-1}, \quad \mathbf{x} \in \mathbf{R}^d,$

where α, α_k , and $\lambda_k \in (0, 1]$ ($k = 1, 2$) are positive constants.

Example 3.2. Let $\alpha_1 > \alpha_2 > 0$. The function $\ln^2(1 + \alpha_1 x) - \ln^2(1 + \alpha_2 x)$ is the product of two Bernstein functions, $\ln(1 + \alpha_1 x) + \ln(1 + \alpha_2 x)$ and

$$\ln(1 + \alpha_1 x) - \ln(1 + \alpha_2 x) = \int_0^x \frac{\alpha_1 - \alpha_2}{(1 + \alpha_1 u)(1 + \alpha_2 u)} du, \quad x \geq 0.$$

Thus, by Theorem 3.1, $\ln^2(1+\alpha_1\|\mathbf{x}\|^{\lambda_0})-\ln^2(1+\alpha_2\|\mathbf{x}\|^{\lambda_0})$ is an isotropic variogram in $\mathbf{R}^d$, where $\lambda_0 \in (0, 1]$. Moreover, by Lemma 2.4, $\lambda_0 \in (0, 1]$ is also a necessary restriction.

The next theorem is about the permissibility of the power of an isotropic variogram in terms of the Bernstein function.

Theorem 3.2. *Let $B: [0, \infty) \rightarrow [0, \infty)$ be a Bernstein function with a positive right-hand derivative at the origin, and let λ_0 and λ be positive constants.*

(i) *A necessary condition for the function*

$$\gamma(\mathbf{x}) = B^\lambda(\|\mathbf{x}\|^{\lambda_0}), \quad \mathbf{x} \in \mathbf{R}^d, \tag{7}$$

to be an intrinsically stationary variogram in $\mathbf{R}^d$ is $0 < \lambda_0 \lambda \leq 2$.

(ii) *If $\lambda_0 \in (0, 1]$ and $\lambda \in (0, 2]$, or if $\lambda_0 \in (0, 2]$ and $\lambda \in (0, 1]$, then (7) is an intrinsically stationary variogram in $\mathbf{R}^d$.*

Corollary 3.4. (i) *When $\lambda_0 = 1$, (7) is a variogram in $\mathbf{R}^d$ if and only if $\lambda \in (0, 2]$.*

(ii) *When $\lambda_0 = 2$, (7) is a variogram in $\mathbf{R}^d$ if and only if $\lambda \in (0, 1]$.*

(iii) *When $\lambda = 1$, (7) is a variogram in $\mathbf{R}^d$ if and only if $\lambda_0 \in (0, 2]$.*

(iv) *When $\lambda = 2$, (7) is a variogram in $\mathbf{R}^d$ if and only if $\lambda_0 \in (0, 1]$.*

It seems reasonable to conjecture that a necessary and sufficient condition for (7) to be a variogram is that $\lambda_0 \in (0, 2]$, $\lambda \in (0, 2]$, and $0 < \lambda_0 \lambda \leq 2$, as is seen from Corollary 3.4 and the following example.

Example 3.3. Taking in (7) the Bernstein function $B(x) = x(1 + \alpha x)^{-1}$, $x \geq 0$, yields

$$\gamma(\mathbf{x}) = \left(\frac{\|\mathbf{x}\|^{\lambda_0}}{1 + \alpha \|\mathbf{x}\|^{\lambda_0}} \right)^\lambda, \quad \mathbf{x} \in \mathbf{R}^d, \tag{8}$$

where α is a positive constant. A necessary condition for (8) being a variogram in $\mathbf{R}^d$ is $0 < \lambda_0 \lambda \leq 2$.

Next we show that $0 < \lambda_0 \lambda \leq 2$ is also a sufficient condition for (8) to be a variogram in $\mathbf{R}^d$ if $\lambda_0 \in (0, 1]$ or $\lambda_0 = 2$. The case $\lambda_0 = 2$ is obviously covered by Corollary 3.4 (ii). By Lemma 2.2, we just need to check the permissibility of (8) when $\lambda_0 \in (0, 1]$ and $\lambda = 2/\lambda_0$.

For $\lambda_0 \in (0, 1]$, x^{λ_0} is a Bernstein function on $[0, \infty)$. Composing it with the completely monotone function $(1 + \alpha x)^{-1/\lambda_0 - 1}$, $x \geq 0$, we obtain another completely monotone function $(1 + \alpha x^{\lambda_0})^{-1/\lambda_0 - 1}$ on $[0, \infty)$, whose integral

$$\int_0^x (1 + \alpha u^{\lambda_0})^{-1/\lambda_0 - 1} du = \frac{x}{(1 + \alpha x^{\lambda_0})^{1/\lambda_0}} = \left(\frac{x^{\lambda_0}}{1 + \alpha x^{\lambda_0}} \right)^{1/\lambda_0}, \quad x \geq 0,$$

is thus a Bernstein function on $[0, \infty)$. By Corollary 3.2, $\{\|\mathbf{x}\|^{\lambda_0}/(1 + \alpha \|\mathbf{x}\|^{\lambda_0})\}^{2/\lambda_0}$, $\mathbf{x} \in \mathbf{R}^d$, is a variogram, and furthermore, by Lemma 2.2, (8) is a variogram in $\mathbf{R}^d$ for every $\lambda \in (0, 2/\lambda_0]$. In this case, we obtain a power-law decaying covariance function

$$C(\mathbf{x}) = \begin{cases} c_0 & \text{if } \mathbf{x} = \mathbf{0}, \\ c_1 \{1 - (1 + \alpha \|\mathbf{x}\|^{-\lambda_0})^{-\lambda}\} & \text{if } \mathbf{x} \neq \mathbf{0}, \quad \mathbf{x} \in \mathbf{R}^d, \end{cases}$$

where α, c_0 , and c_1 are positive constants with $c_0 \geq c_1$.

It would be of interest to see if $\lambda \in (0, 2/\lambda_0]$ is also a sufficient condition for (8) being a variogram in $\mathbf{R}^d$ when $\lambda_0 \in (1, 2) \cup (2, \infty)$.

Example 3.4. A necessary condition for $\ln^\lambda(1 + \alpha \|\mathbf{x}\|^{\lambda_0})$ to be a variogram in $\mathbf{R}^d$ is $0 < \lambda_0 \lambda \leq 2$, by applying Theorem 3.2, (i) to the Bernstein function $\ln(1 + x)$, $x \geq 0$, where α and λ_0 are positive constants.

In particular, $\ln(1 + \alpha \|\mathbf{x}\|^{\lambda_0})$, $\mathbf{x} \in \mathbf{R}^d$, is a variogram if and only if $0 < \lambda_0 \leq 2$, while for $\lambda_0 = 1$ or 2 , $\ln^\lambda(1 + \alpha \|\mathbf{x}\|^{\lambda_0})$ is a variogram on $\mathbf{R}^d$ if and only if $\lambda \in (0, 2/\lambda_0]$. It remains to find out the largest λ for the permissibility of $\ln^\lambda(1 + \alpha \|\mathbf{x}\|^{\lambda_0})$, $\mathbf{x} \in \mathbf{R}^d$.

4. Two extreme cases for the power of an intrinsically stationary variogram in space and time. For an intrinsically stationary variogram $\gamma(\mathbf{s}; t)$ on a spatio-temporal index domain $\mathcal{S} \times \mathcal{T}$, this section considers two extreme cases, where its power $\gamma^\lambda(\mathbf{s}; t)$ is a variogram only if $\lambda \in (0, 1]$.

Theorem 4.1. Assume that $B(x)$ is a Bernstein function on $[0, \infty)$ with a positive right-hand derivative at the origin, $g(x)$ is continuous on $[0, \infty)$ and twice continuously differentiable on $(0, \infty)$ with $g''(0_+) > 0$, $g''(|x|)$ is a covariance function on $\mathbf{R}$, and $\gamma_S(\mathbf{s})$ is a purely spatial variogram on $\mathcal{S}$. Then the function

$$\gamma(\mathbf{s}; t) = B^\lambda(\gamma_S(\mathbf{s}) + g(|t|) - g(0) - g'(0)|t|), \quad \mathbf{s} \in \mathcal{S}, \quad t \in \mathbf{R}, \quad (9)$$

is an intrinsically stationary variogram on $\mathcal{S} \times \mathbf{R}$ if and only if $\lambda \in (0, 1]$.

Clearly, Theorem 4.1 holds but the spatio-temporal variogram (9) is intrinsically stationary in time, if one replaces the intrinsically stationary variogram $\gamma_S(\mathbf{s})$ in (9) by a purely spatial variogram $\gamma_S(\mathbf{s}_1, \mathbf{s}_2)$ on $\mathcal{S}$.

The family formed by (9) is pretty large, through appropriate choice of $B(x)$, $g(x)$, and $\gamma_S(\mathbf{s})$. The following corollary is obtained by setting $g(x) = \alpha x^2$, $x \geq 0$.

Corollary 4.1. Let α be a positive constant. The function $B^\lambda(\gamma_S(\mathbf{s}) + \alpha t^2)$ is a variogram on $\mathcal{S} \times \mathbf{R}$ if and only if $\lambda \in (0, 1]$.

In particular, $\lambda \in (0, 1]$ is a necessary and sufficient condition for each of the following functions to be a variogram on $\mathcal{S} \times \mathbf{R}$:

- (i) $\{\gamma_S(\mathbf{s}) + \alpha t^2\}^\lambda$, where $B(x) = x$, $x \geq 0$,
- (ii) $\ln^\lambda\{1 + \gamma_S(\mathbf{s}) + \alpha t^2\}$, where $B(x) = \ln(1 + x)$, $x \geq 0$, and
- (iii) $\{\gamma_S(\mathbf{s}) + \alpha t^2\}^\lambda \{1 + \gamma_S(\mathbf{s}) + \alpha t^2\}^{-\lambda}$, where $B(x) = x(1 + x)^{-1}$, $x \geq 0$.

It follows from the last variogram that $1 - \{\gamma_S(\mathbf{s}) + \alpha t^2\}^\lambda \{1 + \gamma_S(\mathbf{s}) + \alpha t^2\}^{-\lambda}$ is a stationary covariance function on $\mathcal{S} \times \mathbf{R}$ with long-range dependence.

Starting from a continuous function $g(x)$, $x \geq 0$, which is twice continuously differentiable on $(0, \infty)$ such that $g''(|x|)$ is a covariance function on $\mathbf{R}$, we are able to obtain a variogram on the real line

$$\gamma(t) = \int_0^{|t|} (|t| - u) g''(u) du = g(|t|) - g(0) - g'(0)|t|, \quad t \in \mathbf{R},$$

along the procedure of Lemma 2 in [17]. An extension is given in the following corollary.

Corollary 4.2. The function $B^\lambda(g(|t|) - g(0) - g'(0)|t|)$, $t \in \mathbf{R}$, is a variogram on $\mathbf{R}$ if and only if $\lambda \in (0, 1]$.

Working on another space-time setting, we derive the following theorem by replacing the temporal distance $|t|$ in Theorem 4.1 by the spatial distance $\|\mathbf{s}\|$, under an additional assumption that $g''(x)$ is completely monotone on $[0, \infty)$.

Theorem 4.2. Assume that $B(x)$ is a Bernstein function on $[0, \infty)$ with a positive right-hand derivative at the origin, $g(x)$ is continuous on $[0, \infty)$, $g''(x)$ is completely monotone on $[0, \infty)$ with $g''(0_+) > 0$, and $\gamma_T(t)$ is a purely temporal variogram on $\mathcal{T}$. Then the function

$$\gamma(\mathbf{s}; t) = B^\lambda(g(\|\mathbf{s}\|) - g(0) - g'(0)\|\mathbf{s}\| + \gamma_T(t)), \quad \mathbf{s} \in \mathbf{R}^d, \quad t \in \mathcal{T}, \quad (10)$$

is an intrinsically stationary variogram on $\mathbf{R}^d \times \mathcal{T}$ if and only if $\lambda \in (0, 1]$.

Once $\gamma_T(t)$ in (10) is replaced by a purely temporal variogram $\gamma_T(t_1, t_2)$ on $\mathcal{T}$, Theorem 4.2 still holds except that (10) is intrinsically stationary in space.

In particular, $\lambda \in (0, 1]$ is a necessary and sufficient condition for each of the following functions to be a variogram on $\mathbf{R}^d \times \mathcal{T}$:

- (i) $\{\alpha \|\mathbf{s}\|^2 + \gamma_T(t)\}^\lambda$, where $B(x) = x$, $x \geq 0$,
- (ii) $\{\alpha \|\mathbf{s}\|^2 + \gamma_T(t)\}^\lambda \{1 + \alpha \|\mathbf{s}\|^2 + \gamma_T(t)\}^{-\lambda}$, where $B(x) = x(1 + x)^{-1}$, $x \geq 0$, and
- (iii) $\ln^\lambda\{1 + \alpha \|\mathbf{s}\|^2 + \gamma_T(t)\}$, where $B(x) = \ln(1 + x)$, $x \geq 0$.

Here the semiparametric model (ii) is a bounded variogram, in which results a long-range dependent covariance model $1 - \{\alpha \|\mathbf{s}\|^2 + \gamma_T(t)\}^\lambda \{1 + \alpha \|\mathbf{s}\|^2 + \gamma_T(t)\}^{-\lambda}$, $\mathbf{s} \in \mathbf{R}^d$, $t \in \mathcal{T}$.

The following corollary follows from (10) by choosing $B(x) = x$, $x \geq 0$.

Corollary 4.3. (i) The function $\gamma(\mathbf{s}; t) = \{g(\|\mathbf{s}\|) - g(0) - g'(0)\|\mathbf{s}\| + \gamma_T(t)\}^\lambda$, $\mathbf{s} \in \mathbf{R}^d$, $t \in \mathcal{T}$, is a variogram on $\mathbf{R}^d \times \mathcal{T}$ if and only if $\lambda \in (0, 1]$.

(ii) The function $\gamma(\mathbf{s}) = \{g(\|\mathbf{s}\|) - g(0) - g'(0)\|\mathbf{s}\|\}^\lambda$, $\mathbf{s} \in \mathbf{R}^d$, is a variogram in $\mathbf{R}^d$ if and only if $\lambda \in (0, 1]$.

E x a m p l e 4.1. Consider the function $g(x) = (1+x)\ln(1+x) - x$, $x \geq 0$. Since $g''(x) = (1+x)^{-1}$ is completely monotone on $[0, \infty)$, it follows from Corollary 4.3 that

$$\gamma(\mathbf{s}; t) = \{(1 + \|\mathbf{s}\|)\ln(1 + \|\mathbf{s}\|) - \|\mathbf{s}\| + \gamma_T(t)\}^\lambda, \quad \mathbf{s} \in \mathbf{R}^d, \quad t \in \mathcal{T},$$

is a variogram on $\mathbf{R}^d \times \mathcal{T}$ provided that λ is a positive constant between 0 and 1.

E x a m p l e 4.2. Let $\lambda_0 \in (-\infty, 0) \cup (1, 2]$ be a constant. For the function $g(x) = (1+x)^{\lambda_0}$, $x \geq 0$, its second-order derivative $g''(x) = \lambda_0(\lambda_0 - 1)(1+x)^{\lambda_0-2}$ is completely monotone on $[0, \infty)$. It follows from Corollary 4.3, (ii) that

$$\gamma(\mathbf{s}) = -\lambda_0\|\mathbf{s}\| + (1 + \|\mathbf{s}\|)^{\lambda_0} - 1, \quad \mathbf{s} \in \mathbf{R}^d, \quad (11)$$

is a variogram in $\mathbf{R}^d$. When $1 < \lambda_0 \leq 2$, (11) is the difference of two variograms $(1 + \|\mathbf{s}\|)^{\lambda_0} - 1$ and $\lambda_0\|\mathbf{s}\|$, and when $\lambda_0 < 0$, (11) is the difference of two other variograms $-\lambda_0\|\mathbf{s}\|$ and $1 - (1 + \|\mathbf{s}\|)^{\lambda_0}$.

For $\lambda_0 \in [0, 1]$, selecting $g(x) = -(1+x)^{\lambda_0}$ in Corollary 4.3 yields that the function

$$\gamma(\mathbf{s}) = \lambda_0\|\mathbf{s}\| + 1 - (1 + \|\mathbf{s}\|)^{\lambda_0}, \quad \mathbf{s} \in \mathbf{R}^d,$$

is a variogram in $\mathbf{R}^d$, as the difference of two variograms $\lambda_0\|\mathbf{s}\|$ and $(1 + \|\mathbf{s}\|)^{\lambda_0} - 1$.

5. Proofs. **P r o o f o f L e m m a 2.3.** (i) For an intrinsically stationary variogram $\gamma^\lambda(\mathbf{x})$, its square root $\gamma^{\lambda/2}(\mathbf{x})$ is subadditive (see [1, Proposition 3.3, p. 99]) in the sense that

$$\gamma^{\lambda/2}(\mathbf{x}_1 + \mathbf{x}_2) \leq \gamma^{\lambda/2}(\mathbf{x}_1) + \gamma^{\lambda/2}(\mathbf{x}_2), \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{D}.$$

It implies that for every positive integer k ,

$$\gamma^{\lambda/2}(k\mathbf{x}) \leq k\gamma^{\lambda/2}(\mathbf{x}) \quad \text{or} \quad \gamma(k\mathbf{x}) \leq k^{2/\lambda}\gamma(\mathbf{x}), \quad \mathbf{x} \in \mathcal{D}.$$

Letting $\lambda \rightarrow \infty$ yields $\gamma(k\mathbf{x}) \leq \gamma(\mathbf{x})$.

(ii) When $\gamma(\mathbf{x})$ is defined on $\mathbf{R}^d$, it follows from (1) that $\gamma(\mathbf{x}) \leq \gamma(\frac{\mathbf{x}}{k})$. Letting k tend to infinity, we obtain

$$\gamma(\mathbf{x}) \leq \lim_{k \rightarrow \infty} \gamma\left(\frac{\mathbf{x}}{k}\right) = \gamma(\mathbf{0}) = 0.$$

In other words, $\gamma(\mathbf{x})$ is identical to zero.

P r o o f o f L e m m a 2.4. As an illustration, let us consider the case, where $\gamma(x_1, 0, \dots, 0)$ has the mentioned derivatives at the origin. The same method applies to other cases.

(i) Let $\gamma(x, 0, \dots, 0)$ possess a positive right-hand derivative at the origin, that is,

$$\lim_{\delta \rightarrow 0_+} \frac{\gamma(\delta, 0, \dots, 0)}{\delta} = \gamma_{x_1}(0_+, 0, \dots, 0) > 0.$$

If $\gamma^\lambda(\mathbf{x})$ is a variogram on $\mathbf{R}^d$, then so are its margin $\gamma^\lambda(x, 0, \dots, 0)$ and $\delta^{-\lambda}\gamma^\lambda(\delta x, 0, \dots, 0)$ on the real line, for every positive constant δ . Taking the limit as $\delta \rightarrow 0_+$ and applying L'Hôpital's rule, we obtain that

$$\begin{aligned} \lim_{\delta \rightarrow 0_+} \frac{\gamma^\lambda(\delta x, 0, \dots, 0)}{\delta^\lambda} &= \left\{ \lim_{\delta \rightarrow 0_+} \frac{\gamma(\delta x, 0, \dots, 0)}{\delta} \right\}^\lambda = \left\{ \lim_{\delta \rightarrow 0_+} |\gamma_{x_1}(\delta x, 0, \dots, 0)| \right\}^\lambda \\ &= \{\gamma_{x_1}(0_+, 0, \dots, 0)\}^\lambda |x|^\lambda, \quad x \in \mathbf{R}, \end{aligned}$$

is a variogram on $\mathbf{R}$, for which $\lambda \in (0, 2]$ is necessary.

(ii) Suppose that $\gamma(x, 0, \dots, 0)$ possesses a zero right-hand derivative and a positive second-order right-hand derivative at the origin, that is,

$$\lim_{\delta \rightarrow 0_+} \frac{\gamma(\delta, 0, \dots, 0)}{\delta} = 0, \quad \text{and} \quad \lim_{\delta \rightarrow 0_+} \frac{\gamma(\delta, 0, \dots, 0)}{\delta^2} = \frac{1}{2} \gamma_{x_1 x_1}(0_+, 0, \dots, 0) > 0.$$

If $\gamma^\lambda(\mathbf{x})$ is a variogram in $\mathbf{R}^d$, so are its margin $\gamma^\lambda(x, 0, \dots, 0)$ and $(\frac{1}{2}\delta^2)^{-\lambda}\gamma^\lambda(\delta x, 0, \dots, 0)$ on $\mathbf{R}$, for every positive constant δ . Letting $\delta \rightarrow 0_+$ and applying L'Hôpital's rule twice yields that

$$\begin{aligned} \lim_{\delta \rightarrow 0_+} \frac{\gamma^\lambda(\delta x, 0, \dots, 0)}{(\delta^2/2)^\lambda} &= \left\{ \lim_{\delta \rightarrow 0_+} \frac{\gamma(\delta x, 0, \dots, 0)}{\delta^2/2} \right\}^\lambda = \left\{ \lim_{\delta \rightarrow 0_+} \frac{|x|\gamma_{x_1}(\delta x, 0, \dots, 0)}{\delta} \right\}^\lambda \\ &= \left\{ \lim_{\delta \rightarrow 0_+} |x|^2 \gamma_{x_1 x_1}(\delta x, 0, \dots, 0) \right\}^\lambda = \{\gamma_{x_1 x_1}(0_+, 0, \dots, 0)\}^\lambda |x|^{2\lambda}, \quad x \in \mathbf{R}, \end{aligned}$$

is a variogram on $\mathbf{R}$, for which $\lambda \in (0, 1]$ is necessary. Moreover, $\lambda \in (0, 1]$ is necessary and sufficient.

P r o o f o f T h e o r e m 3.1. (i) Being a Bernstein function on $[0, \infty)$, $B_k(x)$ can be expressed as

$$B_k(x) = \int_0^\infty \{1 - \exp(-xu)\} u^{-1} dF_k(u), \quad x \geq 0,$$

where $F_k(u)$, $u > 0$, is nondecreasing, and $\int_0^1 u^{-1} dF_k(u)$ exists, $k = 1, 2$. Consequently, $\gamma(\mathbf{x})$ is represented as

$$\gamma(\mathbf{x}) = \int_0^\infty \int_0^\infty \{1 - \exp(-\|\mathbf{x}\|u_1)\} \{1 - \exp(-\|\mathbf{x}\|u_2)\} (u_1 u_2)^{-1} dF_1(u_1) dF_2(u_2),$$

a nonnegative mixture of

$$\{1 - \exp(-\|\mathbf{x}\|u_1)\} \{1 - \exp(-\|\mathbf{x}\|u_2)\}, \quad \mathbf{x} \in \mathbf{R}^d,$$

which is a variogram on $\mathbf{R}^d$ by [20, Theorem 3]. By Lemma 2.1, $\gamma(\mathbf{x})$ is a variogram on $\mathbf{R}^d$.

(ii) When (4) is a variogram,

$$\frac{\gamma(\delta \mathbf{x})}{\delta^2} = \frac{B_1(g(\delta \mathbf{x})) B_2(g(\delta \mathbf{x}))}{\delta^2} = \frac{B_1(\delta g(\mathbf{x})) B_2(\delta g(\mathbf{x}))}{\delta^2}, \quad \mathbf{x} \in \mathbf{R}^d,$$

is a variogram on $\mathbf{R}^d$, where δ is a positive constant. Letting $\delta \rightarrow 0_+$, we obtain that

$$\lim_{\delta \rightarrow 0_+} \frac{\gamma(\delta \mathbf{x})}{\delta^2} = g^2(\mathbf{x}) \lim_{\delta \rightarrow 0_+} \frac{B_1(\delta)}{\delta} \lim_{\delta \rightarrow 0_+} \frac{B_2(\delta)}{\delta}, \quad \mathbf{x} \in \mathbf{R}^d,$$

is a variogram. It then follows from [20, Lemma 1] that $g(\mathbf{x})$ must be of the form (5).

P r o o f o f T h e o r e m 3.2. (i) Suppose that (7) is a variogram in $\mathbf{R}^d$. So are $\delta^{-\lambda} B^\lambda(\delta \|\mathbf{x}\|^{\lambda_0})$ for every positive constant δ , and

$$\lim_{\delta \rightarrow 0_+} \frac{B^\lambda(\delta \|\mathbf{x}\|^{\lambda_0})}{\delta^\lambda} = \left(\lim_{\delta \rightarrow 0_+} \frac{B(\delta \|\mathbf{x}\|^{\lambda_0})}{\delta} \right)^\lambda = \{B'(0_+)\}^\lambda \|\mathbf{x}\|^{\lambda_0 \lambda}, \quad \mathbf{x} \in \mathbf{R}^d.$$

For the latter it requires $0 < \lambda_0 \lambda \leq 2$.

(ii) When $\lambda_0 \in (0, 1]$, Corollary 3.2 claims that $B^2(\|\mathbf{x}\|^{\lambda_0})$ is a variogram in $\mathbf{R}^d$, and furthermore, by Lemma 2.2, $B^\lambda(\|\mathbf{x}\|^{\lambda_0})$ is a variogram in $\mathbf{R}^d$ for every $\lambda \in (0, 2]$.

When $\lambda_0 \in (0, 2]$, $\|\mathbf{x}\|^{\lambda_0}$ is known to be a variogram in $\mathbf{R}^d$. For each fixed $u \geq 0$, it follows from Schoenberg's theorem (see, e.g., [1, p. 74]) that $\exp(-\|\mathbf{x}\|^{\lambda_0} u)$, $\mathbf{x} \in \mathbf{R}^d$, is a covariance function, and thus, $1 - \exp(-\|\mathbf{x}\|^{\lambda_0} u)$ is a variogram on $\mathbf{R}^d$. In terms of the representation (5), $B(\|\mathbf{x}\|^{\lambda_0})$ is a nonnegative mixture of $1 - \exp(-u \|\mathbf{x}\|^{\lambda_0})$ and so, by Lemma 2.1, is a variogram in $\mathbf{R}^d$. Therefore, by Lemma 2.2, $B^\lambda(\|\mathbf{x}\|^{\lambda_0})$ is a variogram on $\mathbf{R}^d$ provided that $\lambda \in (0, 1]$.

P r o o f o f T h e o r e m 4.1. If $B^\lambda(\gamma_S(\mathbf{s}) + g(|t|) - g(0) - g'(0)|t|)$ is a variogram on $\mathcal{S} \times \mathbf{R}$, then so is its temporal margin $B^\lambda(g(|t|) - g(0) - g'(0)|t|)$, $t \in \mathbf{R}$. We now apply Lemma 2.4, (ii) to show that λ has to be limited in the interval $(0, 1]$, by checking the derivatives of the underlying variogram $B(g(|t|) - g(0) - g'(0)|t|)$, $t \in \mathbf{R}$. Actually, by L'Hôpital's rule we have

$$\lim_{x \rightarrow 0_+} \frac{B(g(x) - g(0) - g'(0)x)}{x} = \lim_{x \rightarrow 0_+} B'(g(x) - g(0) - g'(0)x) \{g'(x) - g'(0)\} = 0,$$

and

$$\begin{aligned}
& \lim_{x \rightarrow 0+} \frac{B'(g(x) - g(0) - g'(0)x)\{g'(x) - g'(0)\} - 0}{x} \\
&= \lim_{x \rightarrow 0+} [B''(g(x) - g(0) - g'(0)x)\{g'(x) - g'(0)\}^2 \\
&\quad + B'(g(x) - g(0) - g'(0)x)g''(x)] = B'(0+)g''(0+) > 0.
\end{aligned}$$

On the other hand, let $\lambda \in (0, 1]$. In order to show that $B^\lambda(\gamma_S(s) + g(|t|) - g(0) - g'(0)|t|)$ is a variogram on $\mathcal{S} \times \mathbf{R}$, it suffices to prove that $B(\gamma_S(s) + g(|t|) - g(0) - g'(0)|t|)$ is a variogram on $\mathcal{S} \times \mathbf{R}$, according to Lemma 2.2. By [17, Lemma 2], $g(|t|) - g(0) - g'(0)|t|$ is a variogram on the real line, and then, $\gamma_S(s) + g(|t|) - g(0) - g'(0)|t|$ is a separable space-time variogram. It follows from Schoenberg's theorem that $\exp\{-(\gamma_S(s) + g(|t|) - g(0) - g'(0)|t|)u\}$ is a stationary covariance function, and thus, $1 - \exp\{-(\gamma_S(s) + g(|t|) - g(0) - g'(0)|t|)u\}$ is a variogram on $\mathcal{S} \times \mathbf{R}$ for every nonnegative constant u . Based on the representation (5) for the Bernstein function $B(x)$, $B(\gamma_S(s) + g(|t|) - g(0) - g'(0)|t|)$ possesses a nonnegative mixture representation

$$\begin{aligned}
& B(\gamma_S(s) + g(|t|) - g(0) - g'(0)|t|) \\
&= \int_0^\infty [1 - \exp\{-(\gamma_S(s) + g(|t|) - g(0) - g'(0)|t|)u\}] u^{-1} dF(u), \quad s \in \mathcal{S}, \quad t \in \mathbf{R},
\end{aligned}$$

and is a variogram on $\mathcal{S} \times \mathbf{R}$ by Lemma 2.1.

P r o o f o f T h e o r e m 4.2. The *only if* part follows by looking at $B^\lambda(g(|s|) - g(0) - g'(0)|s| + \gamma_T(t))$, $s \in \mathbf{R}$, and using the approach in the proof of Theorem 4.1.

The proof for the *if* part is also similar to that in the proof of Theorem 4.1, except for verifying that $g(\|s\|) - g(0) - g'(0)\|s\|$ is a variogram on $\mathbf{R}^d$.

Since $g''(x)$ is completely monotone on $[0, \infty)$, there exists a bounded and nondecreasing function $F(x)$ on $[0, \infty)$ such that

$$g''(x) = \int_0^\infty \exp(-ux) dF(u), \quad x \geq 0.$$

Consequently,

$$\begin{aligned}
g'(x) &= \int_0^x g''(w) dw + g'(0) = \int_0^\infty \int_0^x \exp(-uw) dw dF(u) + g'(0) \\
&= \int_0^\infty (1 - \exp(-ux)) u^{-1} dF(u) + g'(0),
\end{aligned}$$

and

$$\begin{aligned}
g(x) &= \int_0^x g'(w) dw + g(0) = \int_0^\infty \int_0^x (1 - \exp(-uw)) u^{-1} dw dF(u) + g'(0)x + g(0) \\
&= \int_0^\infty \{ux + \exp(-ux) - 1\} u^{-2} dF(u) + g'(0)x + g(0).
\end{aligned}$$

Therefore, $g(\|s\|) - g(0) - g'(0)\|s\|$ is alternatively expressed as

$$g(\|s\|) - g(0) - g'(0)\|s\| = \int_0^\infty \{u\|s\| + \exp(-u\|s\|) - 1\} u^{-2} dF(u), \quad s \in \mathbf{R}^d.$$

It is a variogram in $\mathbf{R}^d$ by Lemma 2.1, since for every nonnegative u , $u\|s\| + \exp(-u\|s\|) - 1$ is a variogram in $\mathbf{R}^d$ (see [15, p. 216–219]).

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A CLASS OF t POSTERIOR DISTRIBUTIONS WITH APPLICATION

Устанавливается класс апостериорных распределений параметра положения для некоторых генеральных совокупностей. В качестве априорного распределения для параметра масштаба берется распределение Стьюдента. В вычислениях задействованы некоторые специальные функции. Дается применение к описанию распределений коэффициентов дискретного косинус-преобразования.

Ключевые слова и фразы: дискретное косинус-преобразование, параметр положения, апостериорное распределение, распределение Стьюдента.

1. Introduction. Suppose we have a population and that we wish study the distribution of its mean. The most commonly encountered situation is the study of the mean of a normal population. The fashionable approach these days is to use Bayesian statistics, i.e., try to derive the posterior distribution of the population mean based on some available information.

Let $g(\cdot; \lambda, \mu)$ denote the probability density function (pdf) of the observed values of the desired population. Here, μ and λ denote some location and scale parameters, respectively, representing the population mean and population variance, respectively. For example, if g is a normal pdf, then μ is the population mean and λ is the population standard deviation.

Usually, the observer will have some prior knowledge about the scale parameter λ . For the past 40 to 50 years, the Student's t distribution has been the most popular prior distribution because elicitation of prior information in various physical, engineering, and financial phenomena is closely associated with that distribution, see [2]. So, if we assume that

$$p(\lambda) \propto \left(1 + \frac{\lambda^2}{\nu}\right)^{-(1+\nu)/2},$$

then the joint posterior will be

$$p(\mu, \lambda | x) \propto g(x; \lambda, \mu) \left(1 + \frac{\lambda^2}{\nu}\right)^{-(1+\nu)/2}.$$

Hence, the marginal posterior of μ will be

$$p(\mu | x) \propto \int_0^\infty g(x; \lambda, \mu) \left(1 + \frac{\lambda^2}{\nu}\right)^{-(1+\nu)/2} d\lambda. \quad (1)$$

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