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THE RATE OF CONVERGENCE OF SPECTRA
OF SAMPLE COVARIANCE MATRICES¹⁾

Показано, что расстояние по Колмогорову между спектральной функцией распределения выборочной ковариационной матрицы $p^{-1}\mathbf{X}\mathbf{X}^T$, где $\mathbf{X}$ — $(n \times p)$ -матрица с независимыми элементами, и распределением Марченко–Пастура имеет порядок $O(n^{-1/2})$. Оценка справедлива для всех p , включая те, для которых p/n близко к единице или равно единице.

Ключевые слова и фразы: выборочная ковариационная матрица, распределение Марченко–Пастура, спектральная функция распределения.

1. Introduction. Let X_{ij} , $1 \leq i \leq p$, $1 \leq j \leq n$, be independent random variables with $\mathbf{E} X_{ij} = 0$ and $\mathbf{E} X_{ij}^2 = 1$, and let $\mathbf{X}_p = (X_{ij})_{\{1 \leq i \leq p, 1 \leq j \leq n\}}$. Denote by $\lambda_1 \leq \dots \leq \lambda_p$ the eigenvalues of the symmetric matrix $\mathbf{W} := \mathbf{W}_p := n^{-1}\mathbf{X}_p\mathbf{X}_p^T$ and define its empirical distribution by $F_p(x) = p^{-1} \sum_{k=1}^p I\{\lambda_k \leq x\}$, where $I\{B\}$ denotes the indicator of an event B . We investigate the rate of convergence of the expected spectral distribution $\mathbf{E} F_p(x)$ as well as of $F_p(x)$ to the Marchenko–Pastur distribution function $F_y(x)$ with density

$$f_y(x) = \frac{1}{2xy\pi} \sqrt{(b-x)(x-a)} I_{[a,b]}(x) + I_{[1,\infty)}(y)(1-y^{-1})\delta(x),$$

where $y \in (0, \infty)$ and $a = (1 - \sqrt{y})^2$, $b = (1 + \sqrt{y})^2$. Here we denote by $\delta(x)$ the Dirac delta-function and by $I_{[a,b]}(x)$ the indicator function of the interval $[a, b]$. As in [8] and [10] assume that X_{ij} , $i, j \geq 1$, are independent and identically distributed random variables such that $\mathbf{E} X_{ij} = 0$, $\mathbf{E} X_{ij}^2 = 1$, $\mathbf{E} X_{ij}^4 < \infty$, for all i, j . Then $\mathbf{E} F_p \rightarrow F_y$ and $F_p \rightarrow F_y$ in probability, where $y = \lim_{n \rightarrow \infty} y_p := \lim_{n \rightarrow \infty} p/n \in (0, \infty)$.

Let $y := y_p := p/n$. We introduce the Kolmogorov distance

$$\Delta_p := \sup_x |\mathbf{E} F_p(x) - F_y(x)|$$

between the expected distribution $\mathbf{E} F_p(x)$ and $F_y(x)$ as well as the expected Kolmogorov distance

$$\Delta_p^* := \mathbf{E} \sup_x |F_p(x) - F_y(x)|$$

between $F_p(x)$ and $F_y(x)$. We use the notation $\xi_n = O_{\mathbf{P}}(a_n)$ if, for any $\epsilon > 0$, there exists an $L > 0$ such that $\mathbf{P}\{|\xi_n| \geq La_n\} \leq \epsilon$. Note that, for any $L > 0$, $\mathbf{P}\{\sup_x |F_p(x) - F_y(x)| \geq$

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$L\} \leq \Delta_p^*/L$. Hence bounds for Δ_p^* provide bounds for the rate of convergence in probability of the quantity $\sup_x |F_p(x) - F_y(x)|$ to zero. Using our techniques it is straightforward though technical to prove that the rate of almost sure convergence is at least $O(n^{-1/2+\epsilon})$, for any $\epsilon > 0$. In view of the length of the proofs for the results stated above we refrain from including those details in this paper.

In [1] it was proved that $\Delta_p = O(n^{-1/2})$ provided that $\mathbf{E} X_{ij} = 0$, $\mathbf{E} X_{ij}^2 = 1$, $\sup_{i,j} \mathbf{E} X_{ij}^8 < \infty$ and $\sup_{i,j} n^2 \mathbf{E} X_{ij}^8 I\{|X_{ij}| > \delta\sqrt{n}\} \rightarrow 0$ as $n \rightarrow \infty$, when y stays away from 0 and 1. In [4] it was proved independently that $\Delta_p = O(n^{-1/2})$ under the condition that $\sup_{i,j} \mathbf{E} X_{ij}^8 < \infty$, when y stays away from 0 and 1. Under assumption $\mathbf{E} X_{jk}^{12} < \infty$, in [4] it was proved also that $\Delta_p^* = O(n^{-1/2})$. If y is close to 1, then the limit density and the Stieltjes transform of the limit density have a singularity. In this case the investigation of the rate of convergence is more difficult. In [1] it was shown for y_p equal to 1 or asymptotically near 1 that $\Delta_p = O(n^{-1/8})$ (see also [2]). It is clear that the case $y_p \approx 1$ requires different techniques. Results of the authors [3] show that for Gaussian random variables X_{ij} actually the rate $\Delta_p = O(n^{-1})$ is the correct rate of approximation for all y including the case $y = 1$.

By C (with an index or without it) we denote generic absolute constants, whereas $C(\cdot, \cdot)$ stands for positive constants depending on arguments. Our main results are the following.

Theorem 1.1. *Let $1 \geq y > \theta > 0$, for some positive constant θ . Assume that $\mathbf{E} X_{jk} = 0$, $\mathbf{E} X_{jk}^2 = 1$, and $M_4 := \sup_{1 \leq j,k \leq n} \mathbf{E} X_{jk}^4 < \infty$. Then there exists a positive constant $C(\theta) > 0$ depending on θ such that $\Delta_p \leq C(\theta) M_4^{1/2} n^{-1/2}$.*

Theorem 1.2. *Let $1 \geq y > \theta > 0$, for some positive constant θ . Assume that $\mathbf{E} X_{jk} = 0$, $\mathbf{E} X_{jk}^2 = 1$, and $M_{12} := \sup_{1 \leq j,k \leq n} \mathbf{E} X_{jk}^{12} < \infty$. Then there exists a positive constant $C(\theta) > 0$ depending on θ such that*

$$\Delta_p^* = \mathbf{E} \sup_x |F_p(x) - F_y(x)| \leq C(\theta) M_{12}^{1/6} n^{-1/2}.$$

We shall prove the same result for the following class of sparse matrices. Let ε_{jk} , $j = 1, \dots, n$, $k = 1, \dots, p$, be Bernoulli random variables which are independent in aggregate and independent of X_{jk} , and let $p_n := \mathbf{P}\{\varepsilon_{jk} = 1\}$. Consider the matrix

$$\mathbf{X}^{(\varepsilon)} = (np_n)^{-1/2} (\varepsilon_{jk} X_{jk}).$$

Let $\lambda_1, \dots, \lambda_n$ denote the eigenvalues of the matrix $\mathbf{X}^{(\varepsilon)} (\mathbf{X}^{(\varepsilon)})^T$, and denote by $F_p^{(\varepsilon)}(x)$ the empirical spectral distribution function of the matrix $\mathbf{X}^{(\varepsilon)} (\mathbf{X}^{(\varepsilon)})^T$, i.e.,

$$F_p^{(\varepsilon)}(x) := \frac{1}{p} \sum_{j=1}^p I\{\lambda_j \leq x\}.$$

Theorem 1.3. *Let $1 \geq y \geq \theta > 0$, and let $\mathbf{E} X_{jk} = 0$, $\mathbf{E} X_{jk}^2 = 1$, and $\mathbf{E} X_{jk}^4 \leq M_4 < \infty$. Assume that $np_n \rightarrow \infty$ as $n \rightarrow \infty$. Then*

$$\Delta_n^{(\varepsilon)} := \sup_x |\mathbf{E} F_p^{(\varepsilon)}(x) - F_y(x)| \leq C M_4^{1/2} (np_n)^{-1/2}.$$

We have developed a new approach to the investigation of convergence of spectra of sample covariance matrices based on the so-called Hadamar matrices. Note that this approach allows us to obtain a bound of the rate of convergence to the Marchenko–Pastur distribution uniformly in $1 \geq y \geq \theta$ (including $y = 1$). In this paper we give the proof of Theorem 1.1 only. To prove Theorems 1.2 and 1.3 it is enough to repeat the proofs of Theorem 1.2 and Corollary 1.3 in [4] with inessential changes.

2. Inequalities for the distance between distributions via Stieltjes transforms. We define the Stieltjes transform $s(z)$ of a random variables ξ with the distribution function $F(x)$ (the Stieltjes transform $s(z)$ of the distribution function $F(x)$)

$$s(z) := \mathbf{E} \frac{1}{\xi - z} = \int_{-\infty}^{\infty} \frac{1}{x - z} dF(x), \quad z = u + iv, \quad v > 0.$$

Lemma 2.1. *Let F and G be distribution functions such that*

$$\int_{-\infty}^{\infty} |F(x) - G(x)| dx < \infty.$$

Denote their Stieltjes transforms by $s(z)$ and $t(z)$, respectively. Assume that the distribution $G(x)$ has a support contained in the bounded interval $[a, b]$ and there exists the density $g(x) = G'(x)$ and $g(x) \leq c_g$ for some positive constant $c_g > 0$. Then there exists some constant $C > 0$ depending only on c_g such that, for any $0 < v < V$,

$$\begin{aligned} \Delta(F, G) \leq & C \int_{-\infty}^{\infty} |s(u + iV) - t(u + iV)| du + Cv \\ & + C \sup_{x \in [a, b]} \left| \operatorname{Re} \left\{ \int_v^V (s(x + iu) - t(x + iu)) du \right\} \right|, \end{aligned}$$

where $z = u + iv$.

The proof of Lemma 2.1 is given in [4].

3. The main lemma. Let $\xi \geq 0$ be a positive random variable with a distribution function $F(x)$. Let $\varkappa$ be a Rademacher random variable taking values ± 1 with probability $\frac{1}{2}$. Consider a random variable $\tilde{\xi} := \varkappa \xi$ and denote its distribution function by $\tilde{F}(x)$. For any x , we have $\tilde{F}(x) = \frac{1}{2}(1 + \operatorname{sign} x F(x^2))$. This equality implies that $\tilde{p}(x) := \frac{d}{dx} \tilde{F}(x) = |x|p(x)$, where $p(x) = \frac{d}{dx} F(x)$. For the Marchenko–Pastur distribution with parameter $y \in (0, 1]$, we have

$$\tilde{p}_y(x) = |x|p_y(x) = \frac{1}{2\pi y|x|} \sqrt{(x^2 - a)(b - x^2)}.$$

One can check directly that, for $y \in (0, 1]$,

$$\sup_x \tilde{p}_y(x) \leq \frac{1}{\pi \sqrt{y}(1 + \sqrt{y})}. \quad (3.1)$$

Note also that the distribution $\tilde{F}_y(x)$ has a support contained in the union of the intervals $[-(1 + \sqrt{y}), -(1 - \sqrt{y})] \cup [1 - \sqrt{y}, 1 + \sqrt{y}]$.

Introduce the matrix $\mathbf{H} := \begin{pmatrix} \mathbf{O} & \mathbf{X} \\ \mathbf{X}^T & \mathbf{O} \end{pmatrix}$, where $\mathbf{O}$ is the matrix with zero entries only.

Consider the resolvent matrix $\mathbf{R}(z) = (\mathbf{H} - z\mathbf{I}_{n+p})^{-1}$, where $\mathbf{I}_k$ denotes the identity matrix of order k .

Let $s_y(z)$ denote the Stieltjes transform of the Marchenko–Pastur distribution function with parameter y . Denote by $\tilde{s}_y(z)$ the Stieltjes transform of the distribution function $\tilde{F}_y(x)$. One can check directly that $\tilde{s}_y(z) = zs_y(z^2)$. For the Stieltjes transform of the expected spectral distribution function of the sample covariance matrix $s_p(z)$ and its «symmetrization» $\tilde{s}_p(z)$ we have $\tilde{s}_p(z) = zs_p(z^2)$. From the equation for $s_y(z)$,

$$s_y(z) = -\frac{1}{z + y - 1 + yzs_y(z)},$$

it follows that

$$\tilde{s}_y(z) = -\frac{1}{z + y\tilde{s}_y(z) + (y - 1)/z}.$$

By the inversion formula for block matrices (see [7, p. 18, Section 0.7.3]), we have

$$\mathbf{R}(z) = \begin{pmatrix} z(\mathbf{X}\mathbf{X}^T - z^2\mathbf{I}_n)^{-1} & (\mathbf{X}\mathbf{X}^T - z^2\mathbf{I}_p)^{-1}\mathbf{X} \\ (\mathbf{X}^T\mathbf{X} - z^2\mathbf{I}_p)^{-1}\mathbf{X}^T & z(\mathbf{X}^T\mathbf{X} - z^2\mathbf{I}_p)^{-1} \end{pmatrix}.$$

This equality implies that

$$\tilde{s}_p(z) = \frac{1}{n} \sum_{j=1}^n \mathbf{E} R_{jj}(z) = \frac{1}{n} \sum_{j=1}^p \mathbf{E} R_{j+n,j+n}(z) + \frac{y-1}{z}$$

and

$$\frac{1}{p} \sum_{j=1}^p R_{j+n,j+n}(z) = y \frac{1}{n} \sum_{j=1}^n R_{jj}(z) + \frac{1-y}{z}.$$

For the readers convenience we state here two lemmas, which follow from Shur's complement formula (see, for example, [4]). Let $\mathbf{A} = (a_{kj})$ denote a matrix of order n and let $\mathbf{A}_k$ denote the principal submatrix of order $n-1$, i.e., $\mathbf{A}_k$ is obtained from $\mathbf{A}$ by deleting the k th row and the k th column. Let $\mathbf{a}^{-1} = (a^{jk})$. Let $\mathbf{a}'_k$ denote the vector obtained from the k th row of $\mathbf{A}$ by deleting the k th entry and let $\mathbf{b}_k$ denote the vector obtained from the k th column of $\mathbf{A}$ by deleting the k th entry.

Lemma 3.1. *Assume that $\mathbf{A}$ and $\mathbf{A}_k$ are nonsingular. Then*

$$a^{kk} = \frac{1}{a_{kk} - \mathbf{A}'_k \mathbf{A}_k^{-1} \mathbf{b}_k}.$$

Lemma 3.2. *Let $z = u + iv$, and let $\mathbf{A}$ be an $n \times n$ symmetric matrix. Then*

$$\begin{aligned} \operatorname{Tr}(\mathbf{A} - z\mathbf{I}_n)^{-1} - \operatorname{Tr}(\mathbf{A}_k - z\mathbf{I}_{n-1})^{-1} &= \frac{1 + \mathbf{a}'_k(\mathbf{A}_k - z\mathbf{I}_{n-1})^{-2}\mathbf{a}_k}{a_{kk} - z - \mathbf{a}'_k(\mathbf{A}_k - z\mathbf{I}_{n-1})^{-1}\mathbf{a}_k} \\ &= (1 + \mathbf{a}'_k(\mathbf{A}_k - z\mathbf{I}_{n-1})^{-2}\mathbf{a}_k) a^{kk}. \end{aligned}$$

and

$$\left| \operatorname{Tr}(\mathbf{A} - z\mathbf{I}_n)^{-1} - \operatorname{Tr}(\mathbf{A}_k - z\mathbf{I}_{n-1})^{-1} \right| \leq v^{-1}.$$

Let $\mathbf{A} = \mathbf{H}$, $\mathbf{A}_j = \mathbf{H}^{(k)}$, and $\mathbf{R}^{(j)} := (\mathbf{H}^{(j)} - z\mathbf{I}_{n+p-1})^{-1}$. Applying Lemma 3.1, we may write, for $j = 1, \dots, p$

$$\begin{aligned} R_{jj} &= -\frac{1}{z + y\tilde{s}_p(z) + (y-1)/z - \varepsilon_j} = -\frac{1}{z + y\tilde{s}_p(z) + (y-1)/z} \\ &\quad + \frac{\varepsilon_j}{(z + y\tilde{s}_p(z) + (y-1)/z)(z + y\tilde{s}_p(z) + (y-1)/z - \varepsilon_j)} \\ &= -\frac{1}{z + y\tilde{s}_p(z) + (y-1)/z} (1 - \varepsilon_j R_{jj}), \end{aligned} \tag{3.2}$$

where

$$\varepsilon_j = \varepsilon_j^{(1)} + \varepsilon_j^{(2)} + \varepsilon_j^{(3)} + \varepsilon_j^{(4)} \tag{3.3}$$

with

$$\begin{aligned} \varepsilon_j^{(1)} &= \frac{1}{p} \sum_{1 \leq k \neq l \leq p} X_{jk} X_{jl}^T R_{k+n,l+n}^{(j)}, \quad \varepsilon_j^{(2)} = \frac{1}{p} \sum_{k=1}^p (X_{jk}^2 - 1) R_{k+n,k+n}^{(j)}, \\ \varepsilon_j^{(3)} &= \frac{1}{p} \sum_{k=1}^p R_{k+n,k+n}^{(j)} - \frac{1}{p} \sum_{k=1}^p R_{k+n,k+n}, \\ \varepsilon_j^{(4)} &= \frac{1}{p} \sum_{k=1}^p R_{k+n,k+n} - \frac{1}{p} \mathbf{E} \left(\sum_{k=1}^p R_{k+n,k+n} \right). \end{aligned}$$

This implies that

$$\tilde{s}_p(z) = -\frac{1}{z + y\tilde{s}_p(z) + (y-1)/z} + \delta_p(z), \quad (3.4)$$

where

$$\delta_p(z) = \frac{1}{n(z + y\tilde{s}_p(z) + (y-1)/z)} \sum_{j=1}^n \varepsilon_j R_{jj}.$$

Throughout this paper we consider $z = u + iv$ with $a \leq |u| \leq b$ and $0 < v < C$. The main result of this section is the following lemma.

Lemma 3.3. *Let $\text{Im} \{y\delta_p(z) + z + (y-1)/z\} \geq 0$. Then*

$$\left| z + \frac{y-1}{z} + y s_p(z) \right| \geq 1.$$

P r o o f. From representation (3.4) it follows that

$$\text{Im} \left\{ y s_p(z) + z + \frac{y-1}{z} \right\} = \frac{\text{Im} \{y s_p(z) + z + (y-1)/z\}}{|y s_p(z) + z + (y-1)/z|^2} + \text{Im} \left\{ \delta_p(z) + z + \frac{y-1}{z} \right\}.$$

This equality concludes the proof.

4. Bounds for $\delta_p(z)$. We start with a simple bound for $\delta_p(z)$.

Lemma 4.1. *Under the conditions of Theorem 1.1 the following bound holds for $1 \geq v \geq CM_4^{1/2} n^{-1/2}$:*

$$|\delta_p(z)| \leq \frac{1}{|z + y\tilde{s}_p(z) + (y-1)/z|^2} \frac{C}{nv^4}.$$

P r o o f. Note that

$$|\delta_p(z)| \leq \frac{1}{|z + y\tilde{s}_p(z) + (y-1)/z|^2} \left(\frac{1}{n} \sum_{j=1}^n |\mathbf{E} \varepsilon_j| + \frac{1}{n} \sum_{j=1}^n \mathbf{E} |\varepsilon_j|^2 |R_{jj}| \right).$$

Using inequalities (4.2), (4.3), (4.6), and (4.7) below and inequality $|R_{jj}| \leq 1/v$, we get

$$\begin{aligned} |\delta_p(z)| &\leq \frac{1}{|z + y\tilde{s}_p(z) + (y-1)/z|^2} \left(\frac{1}{nv} + \frac{1}{nv} \sum_{j=1}^n \mathbf{E} |\varepsilon_j|^2 \right) \\ &\leq \frac{1}{|z + y\tilde{s}_p(z) + (y-1)/z|^2} \left(\frac{1}{nv} + \frac{C}{nv^3} \right). \end{aligned}$$

Thus Lemma 4.1 is proved.

Furthermore, we give improved bounds for the remainder term $\delta_p(z)$ in equation (3.4). We first start with bounds assuming that there exist positive constants a_1, a_2 such that

$$a_1 \leq \left| z + \frac{y-1}{z} + y s_p(z) \right| \leq a_2. \quad (4.1)$$

Lemma 4.2. *There exists a positive absolute constant C such that*

$$\mathbf{E} |\varepsilon_j^{(1)}|^2 \leq \frac{C(1 + |\tilde{s}_p(z)|)}{nv}, \quad (4.2)$$

$$\mathbf{E} |\varepsilon_j^{(2)}|^2 \leq \frac{C(1 + |\tilde{s}_p(z)|)}{nv}, \quad (4.3)$$

and

$$\mathbf{E} |\varepsilon_j^{(1)}|^4 \leq \frac{CM_4^2(1 + |\tilde{s}_p(z)|)^2}{n^2v^2}. \quad (4.4)$$

P r o o f. Consider inequality (4.2). We have

$$\mathbf{E} |\varepsilon_j^{(1)}|^2 \leq \frac{2}{p^2} \sum_{k,l=1}^p \mathbf{E} |R_{kl}^{(j)}|^2 \leq \frac{1}{p^2} \mathbf{E} \operatorname{Tr} \mathbf{R}^{(j)} (\mathbf{R}^{(j)})^* \leq \frac{2}{p^2 v} \mathbf{E} \operatorname{Im} \operatorname{Tr} \mathbf{R}^{(j)}. \quad (4.5)$$

Applying Lemma 3.2, we get $|\operatorname{Tr} \mathbf{R} - \operatorname{Tr} \mathbf{R}^{(j)}| \leq 1/v$. Note that

$$\frac{1}{2n} \mathbf{E} \operatorname{Im} \operatorname{Tr} \mathbf{R}(z) \leq (1+y) |\tilde{s}_p(z)| + \left| \operatorname{Im} \frac{1-y}{z} \right|.$$

One can check directly that $|\operatorname{Im} \{(1-y)/z\}| \leq 1$. The last inequalities together conclude the proof of (4.2). The proof of (4.3) is similar. Furthermore,

$$\mathbf{E} |\varepsilon_j^{(1)}|^4 \leq \frac{CM_4^2}{p^4} \mathbf{E} \left(\sum_{k,l=1}^p |R_{kl}^{(j)}|^2 \right)^2 \leq \frac{CM_4^2}{p^2 v^2} \mathbf{E} \left(\frac{1}{p} \operatorname{Im} \operatorname{Tr} \mathbf{R}^{(j)} \right)^2.$$

Similarly to inequality (4.2) we get $\mathbf{E} |\varepsilon_j^{(1)}|^4 \leq CM_4^2 (1 + |\tilde{s}_p(z)|)^2 / (p^2 v^2)$. Lemma 4.2 is proved.

Lemma 4.3. *For any $j = 1, \dots, n$ the following inequality holds:*

$$|\varepsilon_j^{(3)}| \leq \frac{1}{nv}. \quad (4.6)$$

P r o o f. The result is an immediate consequence of Lemma 3.2 with $\mathbf{A} = \mathbf{H}$.

Lemma 4.4. *The following bound holds for all $v > 0$:*

$$\mathbf{E} |\varepsilon_j^{(4)}|^2 \leq \frac{4}{nv^2}. \quad (4.7)$$

There exist positive constants c and C depending on a_1 and a_2 such that for any $v \geq cn^{-1/2}$

$$\mathbf{E} |\varepsilon_j^{(4)}|^2 \leq \frac{CM_4(1 + |\tilde{s}_p(z)|)}{n^2 v^3}, \quad (4.8)$$

$$\mathbf{E} |\varepsilon_j^{(4)}|^3 \leq \frac{CM_4(1 + |\tilde{s}_p(z)|)}{n^{5/2} v^4}, \quad (4.9)$$

and

$$\mathbf{E} |\varepsilon_j^{(4)}|^4 \leq \frac{CM_4(1 + |\tilde{s}_p(z)|)}{n^3 v^5}. \quad (4.10)$$

P r o o f. Note that

$$\varepsilon_j^{(4)} = \frac{1}{p} \left(\sum_{j=1}^p R_{j+n,j+n} - \mathbf{E} \sum_{j=1}^p R_{j+n,j+n} \right) = \frac{1}{p} (\operatorname{Tr} \mathbf{R}(z) - \mathbf{E} \operatorname{Tr} \mathbf{R}(z)).$$

Let $\mathbf{E}_k$ denote the conditional expectation given X_{lm} , $1 \leq l \leq k$, $1 \leq m \leq p$. We may write

$$\mathbf{E} |\varepsilon_j^{(4)}|^2 = \frac{1}{p^2} \sum_{k=1}^n \mathbf{E} |\gamma_k|^2, \quad (4.11)$$

where $\gamma_k = \mathbf{E}_k \operatorname{Tr} \mathbf{R} - \mathbf{E}_{k-1} \operatorname{Tr} \mathbf{R}$. Since $\mathbf{E}_k \operatorname{Tr} \mathbf{R}^{(k)} = \mathbf{E}_{k-1} \operatorname{Tr} \mathbf{R}^{(k)}$, we have $\gamma_k = \mathbf{E}_k \sigma_k - \mathbf{E}_{k-1} \sigma_k$, where $\sigma_k = \operatorname{Tr} \mathbf{R} - \operatorname{Tr} \mathbf{R}^{(k)}$. According to Lemma 3.2, we may represent σ_k as follows:

$$\sigma_k = \sigma_k^{(1)} + \sigma_k^{(2)} + \sigma_k^{(3)} + \sigma_k^{(4)},$$

where

$$\begin{aligned}\sigma_k^{(1)} &= \frac{1 + p^{-1} \sum_{r=1}^n \sum_{s=1}^p X_{kr} X_{ks} (\mathbf{R}^{(k)})_{rs}^2}{z + y \tilde{s}_p(z) + (y-1)/z}, \\ \sigma_k^{(2)} &= \frac{\varepsilon_k \sigma_k}{z + y \tilde{s}_p(z) + (y-1)/z}, \\ \sigma_k^{(3)} &= \frac{p^{-1} \left(\sum_{r=1}^n \sum_{s=1}^p X_{kr} X_{ks} (\mathbf{R}^{(k)})_{rs}^2 - \text{Tr}(\mathbf{R}^{(k)})^2 \right)}{z + y \tilde{s}_p(z) + (y-1)/z}.\end{aligned}$$

Since $\mathbf{E}_k \sigma_k^{(1)} = \mathbf{E}_{k-1} \sigma_k^{(1)}$, we get

$$\mathbf{E} |\gamma_k|^2 \leq 2 \left(\mathbf{E} |\sigma_k^{(2)}|^2 + \mathbf{E} |\sigma_k^{(3)}|^2 \right) \leq C \left(\frac{1}{v^2} \mathbf{E} |\varepsilon_k|^2 + \mathbf{E} |\sigma_k^{(3)}|^2 \right). \quad (4.12)$$

By definition of ε_k , we have

$$\mathbf{E} |\varepsilon_k|^2 \leq 4 \mathbf{E} |\varepsilon_k^{(1)}|^2 + 4 \mathbf{E} |\varepsilon_k^{(2)}|^2 + 4 \mathbf{E} |\varepsilon_k^{(3)}|^2 + 4 \mathbf{E} |\varepsilon_k^{(4)}|^2.$$

According to Lemmas 4.2 and 4.3, we have

$$\mathbf{E} |\varepsilon_k|^2 \leq \frac{C(1 + |\tilde{s}_p(z)|)}{nv} + 4 \mathbf{E} |\varepsilon_k^{(4)}|^2. \quad (4.13)$$

Furthermore,

$$\mathbf{E} |\sigma_k^{(3)}|^2 \leq \frac{C}{n^2 v^3} \text{Im Tr } \mathbf{R}^{(k)} \leq \frac{C(1 + |\tilde{s}_p(z)|)}{nv^3}. \quad (4.14)$$

Inequalities (4.12)–(4.14) together imply that

$$\mathbf{E} |\gamma_k|^2 \leq \frac{C(1 + |\tilde{s}_p(z)|)}{nv^3} + \frac{C}{v^2} \mathbf{E} |\varepsilon_k^{(4)}|^2. \quad (4.15)$$

From (4.11) and (4.15) it follows that

$$\mathbf{E} |\varepsilon_k^{(4)}|^2 \leq \frac{C(1 + |\tilde{s}_p(z)|)}{n^2 v^3} + \frac{C}{nv^2} \mathbf{E} |\varepsilon_k^{(4)}|^2.$$

For $v \geq cn^{-1/2}$ with some sufficiently small positive absolute constant c , we get

$$\mathbf{E} |\varepsilon_k^{(4)}|^2 \leq \frac{C(1 + |\tilde{s}_p(z)|)}{n^2 v^3}.$$

Thus inequality (4.8) is proved. To prove (4.10) we use the Burkholder inequality for martingales (see [6, p. 24]). We get

$$\mathbf{E} |\varepsilon_k^{(4)}|^4 \leq \frac{n}{p^4} \sum_{l=1}^n \mathbf{E} |\gamma_l|^4. \quad (4.16)$$

Using that $|\gamma_l| \leq 2/v$, we get

$$\mathbf{E} |\gamma_l|^4 \leq \frac{2}{v^2} \mathbf{E} |\gamma_l|^2 \leq \frac{CM_4(1 + |\tilde{s}_p(z)|)}{nv^5}. \quad (4.17)$$

Inequalities (4.16) and (4.17) together imply that

$$\mathbf{E} |\varepsilon_k^{(4)}|^4 \leq \frac{CM_4(1 + |\tilde{s}_p(z)|^4)}{n^3 v^5}.$$

The proof of (4.9) is similar. Lemma 4.4 is proved.

Lemma 4.5. *There exist some positive constants c and C such that, for any $1 \geq v \geq cn^{-1/2}$, the following inequality holds:*

$$\frac{1}{n} \sum_{j=1}^n \mathbf{E} |R_{jj}|^2 \leq C.$$

P r o o f. To prove this lemma we repeat the proof of Lemma 5.4 in [4]. Let $U^2 = n^{-1} \sum_{j=1}^{n+p} \mathbf{E} |R_{jj}|^2$. By (3.2), we have

$$U^2 \leq C \left(1 + \frac{1}{n} \sum_{j=1}^n \mathbf{E} |\varepsilon_j|^2 |R_{jj}|^2 \right). \quad (4.18)$$

Applying Lemmas 4.2–4.4, we obtain

$$\frac{1}{n} \sum_{j=1}^n \mathbf{E} |\varepsilon_j^{(1)}|^2 |R_{jj}|^2 \leq \frac{CM_4}{nv^2} \left(\frac{1}{n} \sum_{j=1}^n \mathbf{E} |R_{jj}|^2 \right)^{1/2}. \quad (4.19)$$

Furthermore,

$$\frac{1}{n} \sum_{j=1}^n \mathbf{E} |\varepsilon_j^{(3)}|^2 |R_{jj}|^2 \leq \frac{C}{n^2 v^4}. \quad (4.20)$$

To bound $n^{-1} \sum_{j=1}^n \mathbf{E} |\varepsilon_j^{(4)}|^2 |R_{jj}|^2$ we use that $\varepsilon_j^{(4)}$ does not depend on j . We write

$$\begin{aligned} \frac{1}{n} \sum_{j=1}^n \mathbf{E} |\varepsilon_j^{(4)}|^2 |R_{jj}|^2 &= \mathbf{E} |\varepsilon_1^{(4)}|^2 \left(\frac{1}{n} \sum_{j=1}^n |R_{jj}|^2 \right) \leq \frac{C}{v} \mathbf{E} |\varepsilon_1^{(4)}|^2 \frac{1}{n} \operatorname{Im} \operatorname{Tr} \mathbf{R}(z) \\ &\leq \frac{C |\tilde{s}_p(z)|}{v} \mathbf{E} |\varepsilon_1^{(4)}|^2 + \frac{C}{v} \mathbf{E} |\varepsilon_1^{(4)}|^2 \left| \frac{1}{n} (\operatorname{Tr} \mathbf{R}(z) - \mathbf{E} \operatorname{Tr} \mathbf{R}(z)) \right| \\ &\leq \frac{C(1 + |\tilde{s}_p(z)|)}{v} \mathbf{E} |\varepsilon_1^{(4)}|^2 + \frac{C}{v} \mathbf{E} |\varepsilon_1^{(4)}|^3. \end{aligned} \quad (4.21)$$

Inequalities (4.8), (4.10), and (4.21) together imply

$$\frac{1}{n} \sum_{j=1}^n \mathbf{E} |\varepsilon_j^{(4)}|^2 |R_{jj}|^2 \leq \frac{CM_4(1 + |\tilde{s}_p(z)|)}{n^2 v^4} + \frac{CM_4(1 + |\tilde{s}_p(z)|)}{\sqrt{n^5 v^{10}}}. \quad (4.22)$$

Let $T := n^{-1} \sum_{j=1}^{n+p} \mathbf{E} |\varepsilon_j^{(2)}|^2 |R_{jj}|^2$. From inequalities (4.18)–(4.20) and (4.22) it follows that $U^2 \leq C + \delta U + T$ for $v \geq cn^{-1/2}$. Solving this inequality with respect to U , we get

$$U^2 \leq C + T. \quad (4.23)$$

To bound T we start with the obvious inequality

$$T \leq \frac{1}{v^2} \frac{1}{n} \sum_{j=1}^{n+p} \mathbf{E} |\varepsilon_j^{(2)}|^2 \leq \frac{C}{nv^2} \frac{1}{n} \sum_{j=1}^{n+p} \left(\frac{1}{n} \sum_{k=1}^{(j)} \mathbf{E} |R_{kk}^{(j)}|^2 \right), \quad (4.24)$$

where $\sum^{(j)}$ denotes the sum over all $k = 1, \dots, n+p$ except $k = j$. Introduce some integer $m = m(n)$ depending on n such that $mv^{-1} \leq a_1/4$. Without loss of generality we may assume that $m \leq n/2$. Since $|\tilde{s}_{p-l}(z) - \tilde{s}_{p-l-1}(z)| \leq 1/(n-l)$, we get

$$\frac{a_1}{2} \leq \min_{1 \leq l \leq m} \left| \tilde{s}_{p-l}(z) + z + \frac{y-1}{z} \right| \leq \max_{1 \leq l \leq m} \left| y \tilde{s}_{p-l}(z) + z + \frac{y-1}{z} \right| \leq \frac{3}{2} a_2.$$

Let $\mathbf{j}^{(r)} = (j_1, \dots, j_r)$ with $1 \leq j_1 \neq j_2 \neq \dots \neq j_r \leq n$, $r = 1, \dots, m$. Denote by $\mathbf{H}^{(\mathbf{j}^{(r)})}$ the matrix which is obtained from $\mathbf{H}$ by deleting the j_1 th, $\dots$, j_r th rows and columns, and let

$$\mathbf{R}^{(\mathbf{j}^{(r)})} = \left(\frac{1}{\sqrt{n-r}} \mathbf{H}^{(\mathbf{j}^{(r)})} - z \mathbf{I}_{n+p-r} \right)^{-1}.$$

Arguing similarly to inequality (4.24) we get that uniformly for $r = 1, \dots, m-1$, and for $v \geq C_1(a_1, a_2)n^{-\frac{1}{2}}M^{\frac{1}{2}}$

$$\begin{aligned} & \frac{1}{n} \sum_{k=1, k \notin \mathbf{j}^{(r)}}^n \mathbf{E} |R_{kk}^{(\mathbf{j}^{(r)})}|^2 \\ & \leq \frac{C_0(a_1, a_2)M_4}{nv^2} \left(\frac{1}{n} \sum_{k=1, k \notin \mathbf{j}^{(r)}}^n \left(\frac{1}{n} \sum_{j=1, j \notin \mathbf{j}^{(r+1)}}^n \mathbf{E} \left| R_{jj}^{(\mathbf{j}^{(r+1)})} \right|^2 \right) \right) + C_0(a_1, a_2), \end{aligned} \quad (4.25)$$

here $\mathbf{j}^{(r+1)} = (j_1, \dots, j_r, j_{r+1})$ with $j_{r+1} = k$. Note that the constants $C_0(a_1, a_2)$ and $C_1(a_1, a_2)$ do not depend on $l = 1, \dots, m$. Applying inequality (4.25) recursively we get for $1 \geq v \geq C_1(a_1, a_2)n^{-1/2}M_4^{1/2}$

$$\begin{aligned} & \frac{1}{n} \sum_{k=1}^n \mathbf{E} |R_{kk}|^2 \leq C_0(a_1, a_2) \sum_{r=0}^{m-1} \left(\frac{C_0(a_1, a_2)M_4}{nv^2} \right)^r \\ & + \left(\frac{C_0(a_1, a_2)M_4}{nv^2} \right)^m \left(\frac{1}{n} \sum_{k=1, k \notin \mathbf{j}^{(m-1)}}^n \left(\frac{1}{n} \sum_{j=1, j \notin \mathbf{j}^{(m)}}^n \mathbf{E} \left| R_{jj}^{(\mathbf{j}^{(m)})} \right|^2 \right) \right), \end{aligned} \quad (4.26)$$

here $\mathbf{j}^{(m)} = (j_1, \dots, j_r, j_m)$ with $j_m = k$. Without loss of generality we may assume that $C_0(a_1, a_2)M_4/(nv^2) \leq \frac{1}{2}$. Similarly to (4.5) we get that

$$\frac{1}{n} \sum_{j=1, j \notin \mathbf{j}^{(m)}}^n \mathbf{E} \left| R_{jj}^{(\mathbf{j}^{(m)})} \right|^2 \leq \mathbf{E} \operatorname{Tr} \mathbf{R}^{(\mathbf{j}^{(m)})} \left(\mathbf{R}^{(\mathbf{j}^{(m)})} \right)^* \leq \frac{C_0(a_1, a_2)}{v}. \quad (4.27)$$

Inequalities (4.26) and (4.27) together imply that

$$\frac{1}{n} \sum_{k=1}^n \mathbf{E} |R_{kk}|^2 \leq 2C_0(a_1, a_2) + \frac{1}{2^m} \frac{C}{v}. \quad (4.28)$$

Choosing $m = \lceil C \ln n \rceil$ such that $2^{-m} \leq Cv$ concludes the proof.

Lemma 4.6. *Assume that condition (4.1) holds. Then there exist positive constants $C_3(a_1, a_2)$ and $C_4(a_1, a_2)$ such that for $v \geq C_3(a_1, a_2)n^{-1/2}M_4^{1/2}$ the following inequality holds:*

$$|\delta_p(z)| \leq \frac{C_4(a_1, a_2)M_4}{nv}.$$

P r o o f. Equalities (3.2) and (3.3) imply that

$$|\delta_p(z)| \leq \frac{C}{|z + y\tilde{s}_p(z) + (y-1)/z|^2} \left(\frac{1}{p} \sum_{k=1}^{n+p} |\mathbf{E} \varepsilon_k| + \frac{1}{p} \sum_{k=1}^{n+p} \mathbf{E} |\varepsilon_k|^2 |R_{kk}| \right). \quad (4.29)$$

According to Lemma 4.3 and inequalities (4.1) we get

$$\frac{C}{|z + y\tilde{s}_n(z) + (y-1)/z|^2} \left(\frac{1}{n} \sum_{k=1}^n |\mathbf{E} \varepsilon_k| \right) \leq \frac{C}{nv a_1^2} \leq \frac{C(a_1, a_2)}{nv}. \quad (4.30)$$

Using representation (3.3), we obtain

$$\frac{C}{|z + y s_n(z) + (y-1)/z|^2} \left(\frac{1}{n} \sum_{k=1}^n \mathbf{E} |\varepsilon_k|^2 |R_{jj}| \right) \leq C(a_1, a_2) \sum_{\nu=1}^4 \left(\frac{1}{n} \sum_{k=1}^n \mathbf{E} |\varepsilon_k^{(\nu)}|^2 |R_{jj}| \right). \quad (4.31)$$

Similarly to inequality (4.26) and by Lemma 3.3 we arrive at

$$\frac{1}{n} \sum_{k=1}^n \mathbf{E} |\varepsilon_k^{(1)}|^2 |R_{kk}| \leq \left(\frac{1}{n} \sum_{k=1}^n \mathbf{E} |\varepsilon_k^{(1)}|^4 \right)^{1/2} \left(\frac{1}{n} \sum_{k=1}^n \mathbf{E} |R_{kk}|^2 \right)^{1/2} \leq \frac{C(a_1, a_2) M_4^{1/2}}{nv}. \quad (4.32)$$

Lemma 4.3 and the inequality $|\varepsilon_k^{(3)}| \leq (nv)^{-1}$ imply

$$\frac{1}{n} \sum_{k=1}^n \mathbf{E} |\varepsilon_k^{(3)}|^2 |R_{kk}| \leq \frac{1}{n^2 v^3} \leq \frac{C(a_1, a_2)}{nv}. \quad (4.33)$$

Finally, note that

$$\frac{1}{n} \sum_{k=1}^n \mathbf{E} |\varepsilon_k^{(2)}|^2 |R_{kk}| \leq \frac{1}{nv} \sum_{k=1}^n \mathbf{E} |\varepsilon_k^{(2)}|^2 \leq \frac{C(a_1, a_2) M_4}{nv} \left(\frac{1}{n} \sum_{j=1, j \neq k}^n \mathbf{E} |R_{jj}^{(k)}|^2 \right).$$

Applying Lemma 4.5 to the matrix $\mathbf{H}^{(k)}$ we get

$$\frac{1}{n} \sum_{k=1}^n \mathbf{E} |\varepsilon_k^{(2)}|^2 |R_{kk}| \leq \frac{C(a_1, a_2) M_4}{nv}. \quad (4.34)$$

Inequalities (4.28)–(4.34) together imply that inequality $|\delta_n(z)| \leq C(a_1, a_2) M_4 / (nv)$ holds for $1 \geq v \geq C_1(a_1, a_2) n^{-1/2} M_4^{1/2}$, which proves Lemma 4.6.

Lemma 4.7. *Assuming the conditions of Theorem 1.1, there exists an absolute positive constant C such that for any $1 \geq v \geq C M_4^{1/2} n^{-1/2}$ and $u \in [a, b]$, the following inequality holds:*

$$\operatorname{Im} \left\{ z + y \tilde{s}_p(z) + \frac{y-1}{z} \right\} > 0, \quad z = u + iv. \quad (4.35)$$

P r o o f. Assume that for $r_n(z) := z + y \delta_p(z) + (y-1)/z$ the following equality holds:

$$\operatorname{Im} \{r_n(z)\} = 0. \quad (4.36)$$

Let $\alpha(z) := y \tilde{s}_p(z) + (y-1)/z + z$. Equalities $\alpha(z) = -y/\alpha(z) + r_n(z)$ and (4.36) implies immediately that $\operatorname{Im} \alpha(z) = -\operatorname{Im} \{y/\alpha(z)\}$. Since $\operatorname{Im} \alpha(z) \geq \operatorname{Im} z = v > 0$ this implies that $|\alpha(z)| = \sqrt{y}$. Hence condition (4.1) holds with $a_1 = a_2 = \sqrt{y}$ and we have $|\delta_p(z)| \leq C M_4 / (nv)$. Then, inequality $|\delta_n(z)| \leq \frac{1}{4} v < v$ holds for any $v \geq 2n^{-1/2} \sqrt{C M_4}$. But condition (4.36) implies that $|\delta_p(z)| \geq v$, which is a contradiction. Hence we conclude that $\operatorname{Im} \{z + y \delta_p(z) + (y-1)/z\} \neq 0$ in the region $v \geq 2n^{-1/2} \sqrt{C M_4}$. From Lemma 4.1 it follows, for example, that, for $v = 1$, $\operatorname{Im} r_n(z) > 0$. Since the function $\operatorname{Im} r_n(z)$ is continuous in the region $v \geq C_1 n^{-1/2} \sqrt{M_4}$ we get that $\operatorname{Im} r_n(z) > 0$ for $v \geq C_1 n^{-1/2} \sqrt{M_4}$. This proves Lemma 4.7.

P r o o f o f T h e o r e m 1.1. Recall that $1 \geq y \geq \theta > 0$. Let $v_0 = \max\{\gamma_0 \Delta_p, 2n^{-1/2} C_1 M_4^{1/2}\}$ with a γ_0 such that $1 > \gamma_0 > 0$ to be chosen later. By Lemma 4.7 for any $1 \geq v \geq v_0$ we have $\operatorname{Im} \{z + y \tilde{s}_p(z) + (y-1)/z\} > 0$. Note that the constant C_1 does not depend on γ_0 . In addition we find that

$$|\tilde{s}_p(z) - \tilde{s}_y(z)| = \left| \int_{-\infty}^{\infty} \frac{1}{x-z} d(\mathbf{E} \tilde{F}_p(x) - \tilde{F}_y(x)) \right| = \left| \int_{-\infty}^{\infty} \frac{\mathbf{E} \tilde{F}_p(x) - \tilde{F}_y(x)}{(x-z)^2} dx \right| \leq \frac{\Delta_p}{v} \leq \frac{1}{\gamma_0}.$$

This implies that for $z = u + iv$ such that $|u| \in [a, b]$, $1 \geq v \geq v_0$, we have

$$\left| y\tilde{s}_p(z) + z + \frac{y-1}{z} \right| \leq \frac{1}{\gamma_0} + 5. \quad (4.37)$$

From equality (3.4) it follows that

$$s_p(z) = -\frac{1}{2y} \left(z + \frac{y-1}{z} - y\delta_p(z) - \sqrt{\left(z + \frac{y-1}{z} + y\delta_p(z) \right)^2 - 4y} \right). \quad (4.38)$$

Introduce the function

$$q(z) := -\frac{1}{2y} (z - \sqrt{z^2 - 4y}). \quad (4.39)$$

Equalities (4.38) and (4.39) together imply that for $v \geq v_0$

$$z + y\tilde{s}_p(z) + \frac{y-1}{z} = q(\omega + y\delta_p(z)), \quad (4.40)$$

where $\omega := z + (y-1)/z$. Let $s(z)$ denote the Stieltjes transform of the semicircular law. Then $q(z) = s(z/\sqrt{y})/\sqrt{y}$. This implies in particular that $|q(z)| \leq 1/\sqrt{y}$. Since $\text{Im}\{y\delta_p(z) + \omega\} > 0$, the equality (4.40) implies immediately that

$$\left| z + y\tilde{s}_p(z) + \frac{y-1}{z} \right| \geq \frac{1}{\sqrt{y}}, \quad \text{for } v \geq v_0 \quad (4.41)$$

From (4.40) and (4.41) it follows that condition (4.1) holds with $a_1 = 1$ and $a_2 = 1/\gamma_0 + 5$. Relation (4.40) implies that

$$|\tilde{s}_p(z) - \tilde{s}_y(z)| \leq \frac{1}{\sqrt{y}} |q(\omega) - q(\omega + y\delta_p(z))|. \quad (4.42)$$

After a simple calculation we get

$$|\tilde{s}_p(z) - \tilde{s}_y(z)| \leq \frac{y|\delta_n(z)|}{\sqrt{|(\omega + y\delta_p(z))^2 - 4y|} + \sqrt{|\omega^2 - 4y|}}. \quad (4.43)$$

By Lemma 4.6 we obtain, for $1 \geq v \geq v_0$,

$$|\delta_n(z)| \leq \frac{1}{4}v, \quad (4.44)$$

and for $z = u + iv$ such that $u \in [a, b]$ we find that

$$\min \left\{ \sqrt{|\omega^2 - 4y|}, \sqrt{|(\omega + y\delta_n(z))^2 - 4y|} \right\} \geq C\sqrt{v}. \quad (4.45)$$

Inequalities (4.43)–(4.45) imply that for $z = u + iv$ such that $u \in [a, b]$ and $1 \geq v \geq v_0$

$$|\tilde{s}_p(z) - \tilde{s}_y(z)| \leq \frac{C|\delta_p(z)|}{\sqrt{v}}. \quad (4.46)$$

By Lemma 4.6, we have

$$|\delta_p(z)| \leq \frac{C(\gamma_0)M_4}{nv}. \quad (4.47)$$

From (4.46) and (4.47) it follows that $|\tilde{s}_p(z) - \tilde{s}(z)| \leq C(\gamma_0)M_4/(nv^{3/2})$. Choosing in Lemma 2.1 $V = 1$ and using (4.29) we get after integrating in u and v ,

$$\Delta_n \leq C_1M_4n^{-1} + C_2v_0 + C_3(\gamma_0)M_4n^{-1}v_0^{-1}.$$

Since $v_0 \geq 2n^{-1/2}\sqrt{C_1(\gamma_0)M_4}$, we infer that $\Delta_n \leq C(\gamma_0)M_4^{1/2}n^{-1/2} + C_3v_0$. Recall that C_2 does not depend on γ_0 . If $v_0 = 2n^{-1/2}C_1(\gamma_0)M_4^{1/2}$, then $\Delta_n \leq C(\gamma_0)M_4^{1/2}n^{-1/2}$. We choose $\gamma_0 = 1/(2C_3)$. If $v_0 = \gamma_0\Delta_n$, then

$$\Delta_n \leq C(\gamma_0)M_4^{1/2}(1 - C_3\gamma_0)^{-1}n^{-1/2} \leq 2C(\gamma_0)M_4^{1/2}n^{-1/2}.$$

This completes the proof of Theorem 1.1.

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