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ÇAPAR U.*

FUSIONS OF STABLE MEASURES IN BANACH SPACES¹⁾

Показано, что в сепарабельном банаховом пространстве при растекании симметричной p -устойчивой ($0 < p < 1$) вероятностной меры может быть получена любая борелевская вероятностная мера.

Ключевые слова и фразы: растекании вероятности, барицентр, p -устойчивые меры в гильбертовом и банаховом пространствах, фюзия (fusion).

1. Introduction. The concept of fusions of a probability was initiated by J. Elton and T. P. Hill in 1992. The probabilistic motivation behind such a definition as well as the way it can be used in applied problems, e.g. optimal mixture problems, are described in their pioneering paper (see [2]).

Roughly speaking the fusions of a probability measure can be obtained by collapsing part of the masses of measurable sets to their respective barycenters as concentrated probabilities and continuing on with such a process. The concept is almost that of an «antibalayage», it has a strong interaction with Choquet theory and should possibly have interferences with ergodic theory. However in the present paper we concentrate only on the generalization of a single theoretical result stated in [2] and prove that in a separable Banach space, the class of all fusions of a p -stable probability measure ($0 < p < 1$) coincides with the set of all Borel probability measures.

The fusions are defined for compact, metrizable subsets of locally convex topological vector spaces or separable Banach spaces. In this paper we consider only the latter case. We start by recalling some basic terminology of Elton and Hill.

Let X be a separable Banach space and let $\mathcal{B}$ be the σ -algebra of Borel subsets of X . $\mathcal{P}$ denotes the set of Borel probability measures on $(X, \mathcal{B})$ and X^* is the continuous dual of X . Let $A \in \mathcal{B}$ and $P \in \mathcal{P}$. If $\int_A \|x\| dP(x) < \infty$ (i.e., A has finite first P -moment), then the Bochner integral $\int_A x dP(x)$ exists and the P -barycenter $b(A, P)$ of A is the uniquely determined element of $\overline{\text{co}(A)}$ satisfying

$$(f, b) = \frac{1}{P(A)} \int_A (f, x) dP(x) \quad \forall f \in X^*.$$

*The department of mathematics, Eastern Mediterranean University, Gazimagusa, Mersin 10, Turkey.

We say that P has a finite first moment if $b(X, P)$ exists.

$Q \in \mathcal{P}$ is an *elementary fusion* of P if it is of the form $Q = P|_{A^c} + tP(A)\delta(b(A, P)) + (1-t)P|_A$ for some $A \in \mathcal{B}$ with finite first moment and $P(A) > 0$ ($\delta(x)$ is the Dirac measure concentrated at x). Finite compositions of elementary fusions are called *simple fusions*.

$Q \in \mathcal{P}$ is called a *matrix simple fusion* (m.s.f.) of P if for some positive integers n and k , a Borel partition $\{A_i\}_{i=1}^n$ and a substochastic matrix $[t_{ij}]$ ($1 \leq i \leq n, 1 \leq j \leq k$) it assumes the form

$$Q = \sum_{i=1}^k \left(\sum_{i=1}^n t_{ij} P(A_i) \right) \delta(b_j) + \sum_{i=1}^n \left(1 - \sum_{j=1}^k t_{ij} \right) P|_{A_i}$$

($t_{ij} = 0$ if $b(A_i, P)$ does not exist), where b_j is the weighted barycenter given by

$$b_j = \frac{\sum_{i=1}^n t_{ij} b(A_i, P) P(A_i)}{\sum_{i=1}^n t_{ij} P(A_i)}.$$

If $k = 1$, $n = 3$ and $t_{31} = 0$, then Q is called a *binary* m.s.f. Every matrix simple fusion is the composition of a finite number of binary m.s.f.'s and every binary m.s.f. of P is a simple fusion of P . In fact Q is a simple fusion of P if and only if Q is a m.s.f. of P .

Finally the closure of the set of all simple fusions of P under the weak topology is the class of all *fusions of P* and is denoted by $\mathcal{F}(P)$. The set $\mathcal{F}(P)$ is convex and weakly closed.

Throughout the text by a «measure» we mean a probability measure.

2. Fusions of measures supported by the extreme points of simplices. In this section we generalize the result concerning the fusions of a two point mass distribution concentrated at α and β , i.e.,

$$\mathcal{F}(P) = \{Q \in \mathcal{P}: \text{supp } Q \subseteq [\alpha, \beta], b(X, Q) = \alpha p + \beta(1-p)\}$$

(see [2, Proposition 3.13]).

Lemma 2.1. *Let P be a probability distribution with a finite first moment. Then elementary, simple and matrix fusions do not change barycenter.*

P r o o f. This is a direct consequence of the related definitions.

Proposition 2.1. *Let $P = \{p_1, p_2, \dots, p_{n+1}\}$ be a purely atomic measure with $n+1$ atoms concentrated at the vertices $\alpha_1, \alpha_2, \dots, \alpha_{n+1}$ of an n -simplex. Let $\mathbf{b}$ be the barycenter of P . Then $\mathcal{F}(P)$ coincides with the class of measures supported by the closed simplex and having $\mathbf{b}$ as the barycenter.*

P r o o f. We consider the given simplex in an n -dimensional subspace of X . Let $\mathcal{F}_{\mathbf{b}}$ be the set of all probabilities supported in the closed simplex and having barycenter at $\mathbf{b}$. By Lemma 2.1 $\mathcal{F}(P) \subseteq \mathcal{F}_{\mathbf{b}}$. Now let $Q = \{q_1, q_2, \dots, q_k\} \in \mathcal{F}_{\mathbf{b}}$ be a measure, its k atoms being located at $\mathbf{a}_i \in \{\alpha_1, \alpha_2, \dots, \alpha_{n+1}\}$ ($i = 1, \dots, k$). We show that there is a uniquely

determined m.s.f. of P coinciding with Q . Take the Borel partition consisting of $A_i = \{\alpha_i\}$ ($i = 1, \dots, n+1$) and $A_{n+2} = (\cup_{i=1}^{n+1} \{\alpha_i\})^c$. As there is no mass outside the set of vertices, $t_{n+2,j} \equiv 0$. The given are

$$\sum_{i=1}^{n+1} p_i \alpha_i = \sum_{j=1}^k q_j \mathbf{a}_j = \mathbf{b}. \quad (1)$$

In order to create a mass q_j at $\mathbf{a}_j$, we fuse t_{ij} portion of p_i ($i = 1, \dots, n+1$) and collapse at $\mathbf{a}_j$. That would be possible if t_{ij} are suitably selected so that for fixed j , $\mathbf{a}_j$ is the weighted barycenter of the masses separated from p_i . Thus the equations governing t_{ij} are

$$\sum_{i=1}^{n+1} t_{ij} p_i \alpha_i = q_j \mathbf{a}_j, \quad j = 1, \dots, k, \quad (2)$$

$$\sum_{i=1}^{n+1} t_{ij} p_i = q_j, \quad j = 1, \dots, k, \quad (3)$$

$$t_{ij} \geq 0, \quad i = 1, \dots, n+1; \quad j = 1, \dots, k, \quad (4)$$

$$\sum_{j=1}^k t_{ij} \leq 1, \quad i = 1, \dots, n+1. \quad (5)$$

These are $(n+1)k$ coupled equations in $(n+1)k$ unknowns t_{ij} with the constraint that $[t_{ij}]$ be a substochastic matrix. (2) and (3) are equivalent to the system

$$\sum_{i=1}^{n+1} t_{ij} p_i (\alpha_i - \mathbf{a}_j) = \mathbf{0}, \quad j = 1, \dots, k, \quad (6)$$

$$\sum_{i=1}^{n+1} t_{ij} p_i = q_j, \quad j = 1, \dots, k. \quad (7)$$

Let τ_{ij} ($i = 1, \dots, n+1$) be the barycentric coordinates of $\mathbf{a}_j$. Then $t'_{ij} \geq 0$ obtained from the equations $t'_{ij} p_i = \tau_{ij}$ as well as any set $\{c_j t'_{ij}\}_{i=1}^{n+1}$ satisfy (6). The correct proportionality factor c_j is then obtained from (7). t_{ij} obtained in this way are obviously non-negative. Now we show that $S_i = \sum_j t_{ij} = 1$ ($i = 1, \dots, n+1$), so that t_{ij} constitute the unique solution of (2)–(5). (2), (3) and (1) imply

$$\sum_{i=1}^{n+1} S_i p_i \alpha_i = \sum_{i=1}^{n+1} p_i \alpha_i, \quad (8)$$

$$\sum_{i=1}^{n+1} S_i p_i = 1. \quad (9)$$

It can be directly verified from (8) and (9) that $S_i = 1$. However this is also evident by the observation that S_i are independent of the choice of the basis. Since the probabilities supported in the closed simplex and having barycenter $\mathbf{b}$ are dense in $\mathcal{F}_{\mathbf{b}}$ and $\mathcal{F}(P)$ is closed, the proof is complete.

Corollary 2.1. *Let α_i ($i = 1, \dots, n+k$; $k \geq 2$) be the vertices of a convex polyhedron in $\mathbf{R}^n$. Let $P = \{p_i\}_{i=1}^{n+k}$ be an atomic probability distribution concentrated at α_i and having barycenter $\mathbf{b}$. Then P can be fused to obtain any measure with support in the closure of the polyhedron and having $\mathbf{b}$ as barycenter.*

P r o o f. Proceeding as in the proof of Proposition 2.1 and letting $\mathcal{F}_{\mathbf{b}}$ be the set of all measures supported in the closed polyhedron and having barycenter $\mathbf{b}$, the equations governing t_{ij} now become

$$\sum_{i=1}^{n+k} t_{ij} p_i (\alpha_i - \mathbf{a}_j) = \mathbf{0}, \quad j = 1, \dots, k, \quad (10)$$

$$\sum_{i=1}^{n+k} t_{ij} p_i = q_j, \quad j = 1, \dots, k. \quad (11)$$

Now there are $(n+k)k$ unknowns and $(n+1)k$ equations. The infinite solution set contains members satisfying the constraints $t_{ij} \geq 0$ and $\sum t_{ij} = 1$. This can be proved by a simple continuity argument visualizing the given convex polyhedron as a limiting degenerate simplex of higher dimension and by the use of the proposition. For instance, for $k = 2$ consider the given masses $p_1, p_2, \dots, p_{n+2}$ as concentrated on the vertices of a $(n+1)$ -simplex constructed as follows: let α_i have components $\alpha_i = (\alpha_i^1, \dots, \alpha_i^n)$. Then the vertices of the $(n+1)$ -simplex are $\alpha'_i = (\alpha_i^1, \dots, \alpha_i^n, 0)$ ($i = 1, \dots, n+1$) and $\alpha'_{n+2} = (\alpha_{n+2}^1, \dots, \alpha_{n+2}^n, \delta)$ ($\delta > 0$). Let $\mathbf{b}(\delta)$ be the barycenter of the probability distribution P_δ consisting of atoms $p_1, \dots, p_{n+k}$ at the extreme points of the $(n+1)$ -simplex $[\alpha'_1, \dots, \alpha'_{n+2}]$ and $\mathcal{F}_{\mathbf{b}(\delta)}$ be the set of all measures with support in this simplex and having barycenter $\mathbf{b}(\delta)$. By Proposition 2.1 $\mathcal{F}(P_\delta) = \mathcal{F}_{\mathbf{b}(\delta)}$. As $\delta \rightarrow 0$, $\mathbf{b}(\delta) \rightarrow \mathbf{b}$ and $P_\delta \xrightarrow{w} P$ and $\mathcal{F}_{\mathbf{b}(\delta)}$ clearly coincides with $\mathcal{F}_{\mathbf{b}}$ in the limit. A similar argument applies to convex polyhedra with $k > 2$.

R e m a r k 2.1. Furthering the arguments in the last corollary and proposition 2.1, it is possible to show that the fusions of a measure on a smooth, closed and convex surface coincide with the class of measures supported by the convex closure of the surface having the same barycenter as the given one. For this purpose one utilizes the fact that atomic measures are weakly dense in the set of all measures on a surface. However we shall not pursue this direction since it is not needed in the sequel.

3. Fusions of the Cauchy distribution in $\mathbf{R}^n$. J. Elton and T. P. Hill prove that if $X = \mathbf{R}^1$ and P is the Cauchy distribution, then

$\mathcal{F}(P) = \mathcal{P}$ ([2, Proposition 3.14]). Recall that in $\mathbf{R}^n$ the symmetric Cauchy density is given by

$$f_n(x_1, \dots, x_n) = c_n(1 + x_1^2 + \dots + x_n^2)^{-(n+1)/2} \quad (12)$$

($c_1 = 1/\pi$, $c_2 = 1/(2\pi)$, $c_3 = 1/\pi^2$, $c_4 = 3/(4\pi^2)$ etc., see [3, p. 70]).

Barycenter of the Cauchy distribution does not exist, no positive cone possesses a barycenter either. The n -dimensional spherical coordinates mapping $T_n: \mathbf{R}^n \rightarrow \mathbf{R}^n$ is defined by

$$\begin{aligned} x_1 &= \rho \cos \alpha_1, & x_2 &= \rho \sin \alpha_1 \cos \alpha_2, \\ x_3 &= \rho \sin \alpha_1 \sin \alpha_2 \cos \alpha_3, & & \\ &\vdots & & \\ x_{n-1} &= \rho \sin \alpha_1 \cdots \sin \alpha_{n-2} \cos \theta, & x_n &= \rho \sin \alpha_1 \cdots \sin \alpha_{n-2} \sin \theta, \end{aligned}$$

where $0 < \rho < \infty$, $\alpha_i \in [0, \pi]$ ($i = 1, \dots, n-2$), $\theta \in [0, 2\pi]$ and the Jacobi an of T_n is of the form $\rho^{n-1} \Gamma(\alpha_1, \dots, \alpha_{n-2}, \theta)$. Therefore for a cone $\alpha_1 = g(\alpha_2, \dots, \alpha_{n-2}, \theta)$ the integrals that will yield the coordinates of the barycenter are of the form

$$b_j = c_n K_{g,j} \int_0^\infty \frac{\rho^n d\rho}{(1 + \rho^2)^{(n+1)/2}},$$

where $K_{g,j}$ is a constant depending on function g and comes out as a result of $(n-1)$ -integrals on $\alpha_1, \dots, \alpha_{n-2}, \theta$, e.g. for $n = 2$ and the cone $\alpha_1 = \alpha_0$ we have $b_1 = 0$, $b_2 = 2 \cos \alpha_0 c_2 \int_0^\infty \rho^2 d\rho / (1 + \rho^2)^{3/2}$. As the integrals are divergent, except for those j with $K_{g,j} = 0$, the barycenter of the cone does not exist. For positive integers $j(1) < j(2) < \dots < j(k)$ ($k \leq n$) ($j(k) \leq n$) let $V_{j(1), \dots, j(k)}$ denote the coordinate cone $V_{j(1), \dots, j(k)} = \{\mathbf{x} \in \mathbf{R}^n \mid x_i \geq 0 \text{ if } i = j(r), r = 1, \dots, k, x_i \leq 0 \text{ otherwise}\}$ (e.g., $V = \{\mathbf{x} \mid x_i \leq 0\}$ and $V_{1,2, \dots, n} = \{\mathbf{x} \mid x_i \geq 0\}$). There are 2^n such (positive) coordinate cones and due to the absolute continuity and the rotational invariance of the density (12), they all have the measure $1/2^n$. No coordinate cone has barycenter, but if we intersect any coordinate cone with a closed sphere S_R of radius R , $V_{j(1), \dots, j(k)} \cap S_R$ will have a barycenter $\mathbf{c}_{j(1), \dots, j(k)}(R)$ located on the bisector $l_{j(1), \dots, j(k)} = \{\mathbf{x} \in V_{j(1), \dots, j(k)} \mid |x_1| = \dots = |x_n|\}$. In the sequel we occasionally relabel the coordinate cones for the convenience of notation indicating them by V_i ($i = 1, \dots, 2^n$). Let $r_i = \|\mathbf{c}_i(R)\|$, then r_i is the same increasing function of R for all i . This function is obviously invertible, thus there exists a function $R = h(r_i)$ such that $R \rightarrow \infty$ as $r_i \rightarrow \infty$. Also notice that the correspondence $\mathbf{c}_i \leftrightarrow R$ is one-to-one for all i . On the other hand $P(V_i \setminus S_R) = \Delta(R)$ is also a known, monotonically decreasing function of R with $\Delta \rightarrow 0$ as $R \rightarrow \infty$.

Proposition 3.1. *Let $X = \mathbf{R}^n$ and let P be the n -dimensional symmetric Cauchy distribution. Then P can be fused to obtain any Borel measure in $\mathbf{R}^n$, i.e., $\mathcal{F}(P) = \mathcal{P}$.*

P r o o f. Suppose that a measure Q with barycenter $\mathbf{b} = (b_1, b_2, \dots, b_n)$ consisting of finitely many atoms has been given in $\mathbf{R}^n$. Thus there exists a sphere S_M containing all of the atoms. On the other hand given $\varepsilon > 0$, it is possible to determine $R' > 0$ such that $\Delta(R) = P(V_i \setminus S_R) < \varepsilon/2^n$ whenever $R \geq R'$ (R' is independent of i). Let $R_0 = \max\{2M, R'\}$. We can pick points $\mathbf{c}'_i \in (S_{R_0})^c$ ($i = 1, \dots, 2^n$) in the bisectors of coordinate cones and observe the following:

i) The closed, convex hull of $\mathbf{c}'_i$ contains the support of Q and $\mathbf{c}'_i$ become the extreme points of this hull.

ii) Let $R_i = h(\|\mathbf{c}'_i\|)$, then $\mathbf{c}'_i = \mathbf{c}_i(R_i) = b(V_i \cap S_{R_i}, P)$ and $P(\bigcup_{i=1}^{2^n} (V_i \cap S_R)) > 1 - \varepsilon$.

In fact these 2^n points $\mathbf{c}'_i$ or equivalently 2^n numbers R_i can be selected in such a way that $b(\bigcup_{i=1}^{2^n} (V_i \cap S_{R_i}), P) = \mathbf{b}$ is satisfied. Being able to make such a selection is equivalent to finding a solution $\{R_i\}$ ($i = 1, \dots, 2^n$) of the system

$$\sum_{i=1}^{2^n} \left(\frac{1}{2^n} - \Delta(R_i) \right) (\mathbf{c}_i(R_i))_j = \left(1 - \sum_{i=1}^{2^n} \Delta(R_i) \right) b_j, \quad j = 1, \dots, n. \quad (13)$$

If this can be done, the masses of $V_i \cap S_{R_i}$ can be collapsed at their respective barycenter $\mathbf{c}_i(R_i)$ after a sequence of 2^n elementary fusions (a simple fusion) and further fused in accordance with Corollary 2.1 to closely approximate Q . The proof is then completed by taking weak limits.

We can find a solution of (13) by the following algorithm:

i) For the 0th step select R_i so that the barycenter $\mathbf{c}_i(R_i)$ are on the sphere S_{R_0} .

ii) Suppose at the end of the m th iteration the radii selected are $\{R_i^{(m)}\}$: ($i = 1, \dots, 2^n$) with the cone barycenter $\{\mathbf{c}_i(R_i^{(m)})\}$ yielding an overall barycenter $\mathbf{b}^{(m)} = (b_1^{(m)}, \dots, b_n^{(m)})$ for the union of spheres intersected by the cones (see Fig. 1 for a two dimensional image).

If $\mathbf{b}^{(m)} = \mathbf{b}$ then the iterations terminate. Otherwise there is at least one index j , $b_j \neq b_j^{(m)}$. Suppose that

$$b_{j(r)} - b_{j(r)}^{(m)} \geq 0 \quad \text{if } r = 1, \dots, k, \quad (14)$$

$$b_l - b_l^{(m)} < 0 \quad \text{if } l \neq j(1), \dots, j(k). \quad (15)$$

First let us assume that all inequalities in (14) are strict. Then consider the cone $V_i \equiv V_{j(1), \dots, j(k)}$. If we give an increment to $R_{j(1), \dots, j(k)}^{(m)}$, then all components of $\mathbf{b}^{(m)}$ are improved in the correct direction. Suppose this increment is sustained until the first component of $\mathbf{b}$ is attained. (The barycenter are continuous functions of radii in the set up.) For simplicity assume that $j(1) = 1$ and b_1 is the first component attained. At this stage

$b_1 - b_1^{(m+1)} = 0$ and $b_j - b_j^{(m+1)}$ ($j \neq 1$) preserve the same sign as in the m th iteration. Corresponding radii $\{R_i^{(m+1)}\}$ and the overall barycenter $\mathbf{b}^{(m+1)}$ yield the $(m+1)$ th iteration. Obviously $\|\mathbf{b} - \mathbf{b}^{(m+1)}\| < \|\mathbf{b} - \mathbf{b}^{(m)}\|$.

Fig. 1

Now pick the cones $V_{1,j(2),\dots,j(k)}$ and $V_{j(2),\dots,j(k)}$ and give simultaneous and identical increments $\delta R > 0$ to both $R_{1,j(2),\dots,j(k)}$ and $V_{j(2),\dots,j(k)}$. This will further improve $b_j^{(m+1)}$ ($j \neq 1$) but at the same time alter $b_1^{(m+1)}$ so it will no more be equal to b_1 . However since the first coordinates of points in the two cones $V_{1,j(2),\dots,j(k)}$ and $V_{j(2),\dots,j(k)}$ have opposite signs, $b_1^{(m+1)}$ will be adversely effected by δR and its absolute change will be less than the others. In fact if $V_{1,j(2),\dots,j(k)}$ and $V_{j(2),\dots,j(k)}$ are denoted by V_s and V_t respectively, then using (13) the absolute change in $b_1^{(m+1)}$ and $b_j^{(m+1)}$ ($j \neq 1$) are easily estimated as follows (the superscripts $(m+1)$ are omitted):

$$|b_{1,\delta R} - b_1| = \frac{1}{\sqrt{n}2^n} \left| \left| \mathbf{c}_s(R_s + \delta R) \right| - \left| \mathbf{c}_s(R_s) \right| \right| - \left| \left| \mathbf{c}_t(R_t + \delta R) \right| - \left| \mathbf{c}_t(R_t) \right| \right| + O(\varepsilon),$$

$$|b_{j,\delta R} - b_j| = \frac{1}{\sqrt{n}2^n} \left| \left| \mathbf{c}_s(R_s + \delta R) \right| - \left| \mathbf{c}_s(R_s) \right| \right| \\ + \left| \left| \mathbf{c}_t(R_t + \delta R) \right| - \left| \mathbf{c}_t(R_t) \right| \right| + O(\varepsilon).$$

We increase δR until one of the components equals the corresponding component of $\mathbf{b}$. In view of the above estimates $\|\mathbf{b}^{(m+2)} - \mathbf{b}\| < \|\mathbf{b}^{(m+1)} - \mathbf{b}\|$ and this completes the $(m+2)$ th iteration etc.

R e m a r k 3.1. If at the end of the m th iteration more than one component, say

$$b_{j(r)}^{(m)}, \quad r = 1, \dots, p \ (p \leq k)$$

are found to be equal to the corresponding ones in $\mathbf{b}$, then the cones to be selected should be $V_{j(1), \dots, j(p), j(p+1), \dots, j(k)}$ and $V_{j(p+1), \dots, j(k)}$ and their radii should be given simultaneous and equal increments. Continuing this process we produce a bounded sequence $\{\mathbf{b}^{(m)}\}_{m=1}^{\infty}$; with $\|\mathbf{b} - \mathbf{b}^{(m)}\| \downarrow$. Assume $\inf_m \|\mathbf{b} - \mathbf{b}^{(m)}\| = K > 0$, then there exists a converging subsequence $\mathbf{b}^{(m(r))} \rightarrow \mathbf{d}$ with $\|\mathbf{b} - \mathbf{d}\| = K > 0$. Let γ be defined by $\gamma = \min_i \{|b_i - d_i| > 0\}$ and find an ε' , $0 < \varepsilon' < \gamma/2$, and fix r_0 sufficiently large to satisfy $\max_i |b_i^{(m(r_0))} - d_i| < \varepsilon'$. Now applying the algorithm, the $(m(r_0) + 1)$ th iteration yields $\mathbf{b}^{(m(r_0))}$ with $\|\mathbf{b} - \mathbf{b}^{(m(r_0)+1)}\| < K$ contradicting the assumption. Thus $\lim_{m \rightarrow \infty} \mathbf{b}^{(m)} = \mathbf{b}$.

If (14) holds with equalities on some indices then we act as in the above remark.

R e m a r k 3.2. It can be shown, following a more complicated and tedious procedure, that $\text{supp } Q$ can be placed into an n -simplex to the same effect as placing it into a convex polyhedron as in the proof of the proposition.

4. Fusions of stable measures in Banach spaces. In infinite dimensional spaces, we replace the Cauchy distribution by stable measures. In the construction of the previous section symmetry, rotational invariance and the non-existence of the cone barycenter were the sole properties of Cauchy distributions capitalized. A probability measure in an infinite dimensional space can not be rotation invariant in general. However symmetric Gauss measures in Hilbert spaces can be rotation invariant with respect to the rotations of another Hilbert space imbedded in the original one (see [4, p. 33]). On the other hand Bochner integrable functions are exactly those with integrable norms. It is well-known that (see [5, Sect. 6.7], [1]), if μ is a non-degenerated p -stable measure ($0 < p < 1$) in a Banach space, then $\int_B \|x\| d\mu(x) = \infty$, (see [5, Corollary 6.7.5]) so that the barycenter of the distribution can not exist.

With these considerations we undertake a detour via symmetric Gaussian measures in Hilbert spaces and then use the fact that each p -stable symmetric measure can be expressed as an integral average of symmetric

Gaussian measures. The last step is the passage to Banach spaces.

Theorem 4.1. *Let H be separable Hilbert space and let μ be a symmetric, p -stable measure in H . Further let U be the linear operator changing the signs of any number of fixed components of points in H . Then μ is rotation invariant under U , i.e., $\mu U^{-1} = \mu$.*

P r o o f. Let $\mathcal{G}$ denote the set of symmetric Gaussian measures in H . There exists a probability space $(\Omega, \mathbf{P})$ as well as measures $\rho_\omega \in \mathcal{G}$ such that:

- i) For each $B \in \mathcal{B}(H)$ the mapping $\omega \rightarrow \rho_\omega(B)$ is measurable,
- ii)

$$\mu(B) = \int_{\Omega} \rho_\omega(B) d\mathbf{P}(\omega), \quad B \in \mathcal{B}(H) \quad (16)$$

(see [5, Proposition 6.10.10]). Let S_ω be the covariance operator of ρ_ω . S_ω is an S -operator and $\sqrt{S_\omega}$ has the following representation with respect to a complete, orthonormal basis $\{e_n\}$;

$$\sqrt{S_\omega}x = \sum_{n=1}^{\infty} \sqrt{\lambda_n} x_n e_n, \quad x = (x_n)_{n=1}^{\infty} \in l_2, \quad (17)$$

where $\lambda_n \geq 0$ are the eigenvalues of S_ω .

In the image $\sqrt{S_\omega}(H)$ we define an inner product $\langle \cdot, \cdot \rangle_0$ by

$$\langle \sqrt{S_\omega}x, \sqrt{S_\omega}y \rangle_0 = \langle x, y \rangle; \quad x, y \in H, \quad \omega \in \Omega.$$

$\sqrt{S_\omega}$ is an isometry from H onto $\sqrt{S_\omega}(H)$, therefore it is unitary as a linear operator $H \rightarrow \sqrt{S_\omega}(H)$, where $\sqrt{S_\omega}(H)$ is a Hilbert space with inner product $\langle \cdot, \cdot \rangle_0$. U is unitary as an operator in $H_\omega = \sqrt{S_\omega}(H)$: for a finite number of changes of sign, let us assume without loss of generality that $x' = Ux = (-x_1, \dots, -x_s, x_{s+1}, \dots)$. Then by (17),

$$U\sqrt{S_\omega}x = -\sum_{i=1}^s \sqrt{\lambda_i} x_i e_i + \sum_{i=s+1}^{\infty} \sqrt{\lambda_i} x_i e_i = \sqrt{S_\omega}x',$$

thus $U(H_\omega) \subseteq H_\omega$. On the other hand if $y = (y_n)_{n=1}^{\infty}$: $\langle U\sqrt{S_\omega}x, U\sqrt{S_\omega}y \rangle_0 = \langle \sqrt{S_\omega}x', \sqrt{S_\omega}y' \rangle_0 = \langle x', y' \rangle = \langle x, y \rangle = \langle \sqrt{S_\omega}x, \sqrt{S_\omega}y \rangle_0$. A similar calculation applies to any number of changes of sign. Hence U is an isometry on H_ω , it is obviously onto. Then by [4, Theorem 2.4], $\rho_\omega U^{-1} = \rho_\omega$. Finally since $\mu(U^{-1}B)$ ($B \in \mathcal{B}(H)$) can be approximated in view of (17), by finite sums of the form $\sum_j \rho_{\omega_j}(U^{-1}B)P(A_j) = \sum_j \rho_{\omega_j}(B)P(A_j)$, ($\{A_j\}$ is a Borel partition of Ω), we have the conclusion of the theorem: $\mu U^{-1} = \mu$.

Corollary 4.1. *Let H be a separable Hilbert space and let μ be a p -stable measure with $0 < p < 1$. Then*

- i) *The whole space and the positive cones $V_{j(1), \dots, j(k)}^{(n)} = \{x \in H | x_i \geq 0 \text{ if } i = j(r), r = 1, \dots, k; x_i \leq 0 \text{ if } i \neq j(r), r = 1, \dots, k\}$ do not possess barycenter. (Here $j(k)$ is a permutation of $\{1, 2, \dots, n\}$ k at a time and $n = 1, 2, \dots$).*

ii) Let $c(R)$ be the barycenter of the set $V_{j(1), \dots, j(k)}^{(n)} \cap S_R$. Then

1. $(c(R))_j = 0$ for $j > n$ and $c(R)$ lies in the bisector set $\{x \mid x \in V_{j(1), \dots, j(k)}^{(n)} \cap S_R, |x_1| = \dots = |x_n|\}$.
2. $\|c(R)\|$ is an increasing function of R .
3. $\|c(R)\| \rightarrow \infty$ as $R \rightarrow \infty$.

P r o o f. i) H does not have a barycenter as pointed out at the beginning of this section. Now there are 2^n cones $V_{j(1), \dots, j(k)}^{(n)}$ and each of them can be mapped onto another by a unitary transformation U of the type in Theorem 4.1, hence μ is invariant under such a transformation. If any of the cones possesses a barycenter then so do all others. Considering that the coordinate cones intersect along the coordinate hyperplanes, that would imply $\int_H \|x\| d\mu(x) + \int_{\delta V} \|x\| d\mu(x) < \infty$, so that the barycenter of the whole space would also exist (δV denotes the union of the coordinate hyperplanes).

ii) 1. The first assertion follows from the unitary transformation $(\dots, x_j, \dots) \rightarrow (\dots, -x_j, \dots)$ and Theorem 4.1. It is sufficient to prove the second assertion for the set $V_{1, \dots, n}^{(n)} \cap S_R$. Consider the hyperplanes $M_{ij} = \{x \in H \mid x_i = x_j\}$ ($i, j = 1, \dots, n$). For fixed i and j let us select an orthonormal basis $\{e'_k\}$ ($k = 1, 2, \dots$) such that e'_1 is orthogonal to M_{ij} . This change of basis does not effect μ 's being symmetric and p -stable (which can be seen from the spectral representation $\hat{\mu}(a) = \exp\{-\int_{\delta U} |\langle x, a \rangle|^p d\sigma(x)\}$ of the characteristic function, δU is the unit sphere, σ is the spectral measure). Then considering the transformation $(x'_1, x'_2, \dots) \rightarrow (-x'_1, x'_2, \dots)$ and the Theorem, it is found that the barycenter should lie in the hyperplane M_{ij} . Repeating this for every M_{ij} ($i \neq j, 1 \leq i, j \leq n$) we conclude that the barycenter should be located in the bisector set.

2. Non-decreasing part is obvious; since μ is an integral average of Gaussian measures it is also strictly increasing.

3. This is an obvious consequence of i) and ii), part 2.

R e m a r k 4.1. As a result of this corollary, the counterparts of functions $h(r)$ and $\Delta(R)$ of Section 3 exist and are also well-defined for the Hilbert space case. Although their exact functional forms are not needed for the next proposition, one can find estimates, e.g., using Araujo and Giné's tail probability estimates (see, e.g., [5, p. 111]), such as $\lim_{R \rightarrow \infty} R^p 2^n \Delta(R) = \sigma(\delta U)/(pc_p)$ etc.

Proposition 4.1. Let H be a separable Hilbert space and let μ be a symmetric p -stable ($0 < p < 1$) measure. Then it can be fused to obtain any measure in H , i.e., $\mathcal{F}(\mu) = \mathcal{P}$.

P r o o f. Let Q be a distribution having finitely many atoms. The vectors where the atoms are located generate a finite dimensional subspace, say F , of H . An orthonormal basis can be chosen in such a way that the first n ($= \dim F$) vectors span F . This can be effected via a unitary transformation which again does not effect the symmetry and p -stability of the measure. Corollary 4.1 and Remark 4.1 imply that μ possesses all the prop-

erties of Cauchy's distribution utilized for the fusion scheme constructed in Proposition 3.1. Thus repeating this construction we can produce a polyhedron with 2^n extreme points supporting almost all of the probability mass and having the same barycenter as Q . Thus according to Corollary 2.1 it can be fused to closely approximate Q . We finally consider weak limits.

Corollary 4.2. *Let X be a separable Banach space and let μ be a symmetric p -stable ($0 < p < 1$) measure in X . Then it can be fused to obtain any Borel measure in X .*

P r o o f. Since X is separable we can pick up a countable dense set $\{a_n\}_{n=1}^\infty$ in X . Following [4], it is possible to find $f_n \in X^*$ such that $\|f_n\|_* = 1$ and $(f_n, a_n) = \|a_n\|$. Let $\{\lambda_n\}$ be a sequence of positive numbers such that $\sum_{n=1}^\infty \lambda_n = 1$. Define for $x, y \in X$

$$[x, y] = \sum_{n=1}^{\infty} \lambda_n (f_n, x) (f_n, y).$$

$[\cdot, \cdot]$ is an inner product. Let $|x|_0 = \sqrt{[x, x]}$, $x \in X$ ($|x|_0 \leq \|x\|$) and let H be the completion of X with respect to $|\cdot|_0$. It is shown (see [4, Ch. 1, Theorem 1.1]) that $\mathcal{B}(H) \cap X = \mathcal{B}(X)$. Define $\tilde{\mu}(B) = \mu(B \cap X)$, $B \in \mathcal{B}(H)$. $\tilde{\mu}$ is clearly symmetric and p -stable. Given a purely atomic measure Q in X , $\tilde{\mu}$ can be fused to yield Q . But since $\int_B x d\tilde{\mu} = \int_{B \cap X} x d\mu(x)$, $B \in \mathcal{B}(H)$, this implies that μ can serve the same purpose.

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