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CONDITIONAL ZERO-ONE LAWS

Мы говорим, что для некоторого класса событий выполняется условный закон нуля или единицы, если этот класс является подмножеством пополнения «обуславливающей» σ -алгебры. В этом случае условная вероятность события из этого класса есть индикаторная функция. Поэтому условная вероятность почти наверное принимает только значения нуля и единицы; в безусловном случае индикаторные функции суть почти наверное константы.

Рассматриваются два частных случая закона нуля или единицы: если последовательность случайных величин условно независима, то для ее хвостовой σ -алгебры выполняется закон нуля или единицы — это является обобщением закона нуля или единицы Колмогорова; если же последовательность дополнительно условно одинаково распределена, тогда для ее перестановочной σ -алгебры, содержащей хвостовую σ -алгебру, верен закон нуля или единицы — это обобщает закон нуля или единицы Хьюитта и Сэвиджа.

Ключевые слова и фразы: условная вероятность, условная независимость, законы нуля или единицы.

1. Introduction. In many situations we are interested in the properties of the events of special σ -algebras. E.g., for a sequence of random variables the limit of the sequence of its means is measurable with respect to the tail σ -algebra and the limit of the sequence of its partial sums is measurable with respect to the permutable σ -algebra.

In classical probability theory a class of events fulfills a Zero-One Law if the class is degenerate in the sense that every event of the class has either the probability zero or one. The Zero-One Laws by Kolmogorov and by Hewitt and Savage say that the tail σ -algebra and the permutable σ -algebra are degenerate if the sequence of random variables is independent or independent and identically distributed, respectively.

In many models we do not have independence but we have conditional independence, given a σ -algebra which may be supposed to be generated by a structure variable. Such an assumption is reasonable, e.g., in two-stage statistical models: The first stage is to realize the structure variable and the second stage is to realize a sequence of random variables with a given conditional distribution with respect to the structure variable.

For the case of conditional independence Hunt [9] proved a generalization of Kolmogorov's Zero-One Law. Combining this result with the consideration of conditional independence by Chow and Teicher [4] and Döhler [6] we are able to give a uniform theory of conditional Zero-One Laws.

It is well known that in general there is no implication between conditional independence and independence; however conditionally identically distributed random variables are identically distributed. Therefore we show a generalization of Hewitt–Savage Zero-One

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Law for special identically distributed sequences of random variables which need not be (unconditionally) independent; by contrast, Horn and Schach [8], Blum and Pathak [3], Sendler [13], and Aldous and Pitman [1] considered independent but not necessarily identically distributed sequences of random variables having a degenerate permutable σ -algebra.

The paper is organized as follows. First we give a review of conditional independence including the Kolmogorov's conditional Zero-One Law and a conditional Law of Large Numbers. In Section 3 we introduce the concept of completion and in Section 4 we consider universal σ -algebras which lead us to the tail and permutable σ -algebra. Finally we prove in the main Section 5 some assertions on conditional Zero-One Laws including the conditional version of the Zero-One Law by Hewitt and Savage. We finish the section with an overview over the relations between the different σ -algebras considered in the paper. In Section 6 we give some remarks.

Throughout this paper, let $(\Omega, \mathcal{F}, \mathbf{P})$ be a fixed probability space and let $\mathcal{G}$ be a sub- σ -algebra of $\mathcal{F}$. For random variables X and Y we will write $X = Y$, if $\mathbf{P}\{X = Y\} = 1$. We denote by $\mathbf{E}(X|\mathcal{G})$ the $\mathcal{G}$ -conditional expectation of the random variable X . The $\mathcal{G}$ -conditional expectation is well defined, whenever X is nonnegative. In the general case we say that the $\mathcal{G}$ -conditional expectation exists, if $\min\{\mathbf{E}(X^+|\mathcal{G}), \mathbf{E}(X^-|\mathcal{G})\} < \infty$ holds. In this case the $\mathcal{G}$ -conditional expectation is defined by $\mathbf{E}(X|\mathcal{G}) := \mathbf{E}(X^+|\mathcal{G}) - \mathbf{E}(X^-|\mathcal{G})$. If X is integrable this definition, which is due to Hunt [9], agrees with the usual one. The $\mathcal{G}$ -conditional probability of an event $A \in \mathcal{F}$ is defined by $\mathbf{P}(A|\mathcal{G}) := \mathbf{E}(\chi_A|\mathcal{G})$. A nonempty collection $\mathcal{C} \subseteq \mathcal{F}$ is said to be a class. A class $\mathcal{C}$ is a π -class, if the intersection of two events of $\mathcal{C}$ is again an event of $\mathcal{C}$ (for details see [4]).

2. Conditional independence. Throughout this section let I be an index set and let $F(I)$ denote the set of all finite subsets of I .

A family of classes $\{\mathcal{C}_i\}_{i \in I}$ is $\mathcal{G}$ -conditionally independent if

$$\mathbf{P}\left(\bigcap_{i \in J} A_i \mid \mathcal{G}\right) = \prod_{i \in J} \mathbf{P}(A_i \mid \mathcal{G})$$

holds for each $J \in F(I)$ and every family $\{A_i\}_{i \in J}$ with $A_i \in \mathcal{C}_i$ for all $i \in J$.

Simple examples show that in general neither $\mathcal{G}$ -conditional independence implies independence nor independence implies $\mathcal{G}$ -conditional independence.

Proposition 2.1. *For a family of π -classes $\{\mathcal{C}_i\}_{i \in I}$ the following are equivalent:*

- (i) $\{\mathcal{C}_i\}_{i \in I}$ is $\mathcal{G}$ -conditionally independent;
- (ii) $\{\sigma(\mathcal{C}_i)\}_{i \in I}$ is $\mathcal{G}$ -conditionally independent.

For the proof see [4, p. 230 f].

The tail σ -algebra $\mathcal{C}$ of a family of classes $\{\mathcal{C}_i\}_{i \in I}$ is defined by

$$\mathcal{C} := \bigcap_{J \in F(I)} \sigma\left(\bigcup_{i \in I \setminus J} \mathcal{C}_i\right).$$

Proposition 2.2 (Kolmogorov's $\mathcal{G}$ -Zero-One Law). *Let $\{\mathcal{C}_i\}_{i \in I}$ be a $\mathcal{G}$ -conditionally independent family of π -classes. Then for every event C of the tail σ -algebra of this family there exists some $G \in \mathcal{G}$ with $\chi_C = \chi_G$ and hence $\mathbf{P}(C|\mathcal{G}) = \chi_G$.*

For the proof see [9, p. 59].

The identity $\mathbf{P}(C|\mathcal{G}) = \chi_G$ explains the name Zero-One Law; in the classical case we have $\mathcal{G} = \{\emptyset, \Omega\}$ and the indicator functions are constant.

A sequence of random variables $\{X_n\}_{n \in \mathbf{N}}$ is called

— $\mathcal{G}$ -conditionally independent, if the sequence $\{\sigma(X_n)\}_{n \in \mathbf{N}}$ is $\mathcal{G}$ -conditionally independent,

— $\mathcal{G}$ -conditionally identically distributed, if $\mathbf{P}(X_n \in B|\mathcal{G}) = \mathbf{P}(X_1 \in B|\mathcal{G})$ holds for all $n \in \mathbf{N}$ and $B \in \mathcal{B}(\mathbf{R})$, and

— $\mathcal{G}$ -conditionally i.i.d., if the sequence is both $\mathcal{G}$ -conditionally independent and $\mathcal{G}$ -conditionally identically distributed.

Obviously, every $\mathcal{G}$ -conditionally identically distributed sequence of random variables is identically distributed. This follows by integrating the conditional probabilities in the definition.

Using Proposition 2.1 we get an equivalent characterization of $\mathcal{G}$ -conditionally independent sequences of random variables.

Lemma 2.1. *A sequence of random variables $\{X_n\}_{n \in \mathbf{N}}$ is $\mathcal{G}$ -conditionally independent if and only if*

$$\mathbf{P} \left(\bigcap_{k=1}^n \{X_k \leq x_k\} \mid \mathcal{G} \right) = \prod_{k=1}^n \mathbf{P}(X_k \leq x_k \mid \mathcal{G})$$

holds for all $n \in \mathbf{N}$ and $x_1, \dots, x_n \in \mathbf{R}$.

If $\{X_n\}_{n \in \mathbf{N}}$ is $\mathcal{G}$ -conditionally identically distributed we denote by X a random variable which has the same $\mathcal{G}$ -conditional distribution as each X_n . Obviously the $\mathcal{G}$ -conditional expectation of X and each X_n are almost surely identical.

Let us finish this section with a Law of Large Numbers for $\mathcal{G}$ -conditionally i.i.d. sequences of random variables.

Proposition 2.3 (Law of Large Numbers). *Let $\{X_n\}_{n \in \mathbf{N}}$ be a $\mathcal{G}$ -conditionally i.i.d. sequence of random variables for which the $\mathcal{G}$ -conditional expectation exists. Then*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n X_k = \mathbf{E}(X \mid \mathcal{G})$$

almost surely.

For the proof see [9, p. 62].

3. Completion. For a class $\mathcal{E}$ the completion of $\mathcal{E}$ in $\mathcal{F}$ is defined to be

$$\mathcal{E}^* := \left\{ A \in \mathcal{F} \mid \exists B \in \mathcal{E}: \mathbf{P}(A \triangle B) = 0 \right\}.$$

If $\mathcal{E}$ is a σ -algebra, then $\mathcal{E}^*$ is a σ -algebra as well; if $\mathcal{E} = \{\Omega, \emptyset\}$, then $\mathcal{E}^*$ is a degenerate σ -algebra, which consists of all events with probabilities zero or one.

Lemma 3.1. *Let $\mathcal{C}$ and $\mathcal{E}$ be classes. Then the following are equivalent:*

- (i) $\mathcal{C} \subseteq \mathcal{E}^*$;
- (ii) $\mathcal{C}^* \subseteq \mathcal{E}^*$;
- (iii) for every $A \in \mathcal{C}$ there exists some $B \in \mathcal{E}$ with $\mathbf{P}(A \triangle B) = 0$;
- (iv) for every $A \in \mathcal{C}$ there exists some $B \in \mathcal{E}$ with $\chi_A = \chi_B$.

The proof of this lemma is evident.

Lemma 3.2. *Let X be a random variable for which the $\mathcal{G}$ -conditional expectation exists. Then the $\mathcal{G}^*$ -conditional expectation of X exists and we have*

$$\mathbf{E}(X \mid \mathcal{G}) = \mathbf{E}(X \mid \mathcal{G}^*).$$

P r o o f. The assertion follows from the definition of the conditional expectation and Lemma 3.1 (iv).

From Lemma 3.2 we get that the $\mathcal{G}$ -conditional probability and the $\mathcal{G}^*$ -conditional probability are almost surely equal. Furthermore we see that a family of classes is $\mathcal{G}$ -conditionally independent or $\mathcal{G}$ -conditionally i.i.d. if and only if it is $\mathcal{G}^*$ -conditionally independent or $\mathcal{G}^*$ -conditionally i.i.d., respectively.

4. Universal σ -algebras. Let

$$\mathcal{T}_n := \mathbf{R}^n \times \mathcal{B}(\mathbf{R}^\infty) = \{\mathbf{R}^n \times B \mid B \in \mathcal{B}(\mathbf{R}^\infty)\} \subseteq \mathcal{B}(\mathbf{R}^\infty).$$

Then $\{\mathcal{T}_n\}_{n \in \mathbf{N}}$ is a decreasing sequence of σ -algebras. The tail σ -algebra of the sequence $\{\mathcal{T}_n\}_{n \in \mathbf{N}}$ is called the (universal) tail σ -algebra and is denoted by $\mathcal{T}$. We have

$$\mathcal{T} = \bigcap_{n=1}^{\infty} \mathcal{T}_n.$$

A one-to-one map $\pi: \mathbf{N} \rightarrow \mathbf{N}$ with $\pi(m) = m$ for all $m > n$ is called n -permutation. We denote the set of all n -permutations by Π_n . For $\pi \in \bigcup_{n \in \mathbf{N}} \Pi_n$ we define the map $\tilde{\pi}: \mathbf{R}^\infty \rightarrow \mathbf{R}^\infty$ by $\tilde{\pi}(x_1, \dots, x_k, \dots) = (x_{\pi(1)}, \dots, x_{\pi(k)}, \dots)$.

Let

$$\mathcal{S}_n := \{B \in \mathcal{B}(\mathbf{R}^\infty) \mid \forall \pi \in \Pi_n: B = \tilde{\pi}(B)\}.$$

Then $\{\mathcal{S}_n\}_{n \in \mathbf{N}}$ is a decreasing sequence of σ -algebras. The tail σ -algebra of the sequence $\{\mathcal{S}_n\}_{n \in \mathbf{N}}$ is called the (*universal*) *permutable σ -algebra* and is denoted by $\mathcal{S}$. We have

$$\mathcal{S} = \bigcap_{n=1}^{\infty} \mathcal{S}_n.$$

The definitions of both universal σ -algebras yield $\mathcal{T}_n \subseteq \mathcal{S}_n$ for all $n \in \mathbf{N}$ and therefore

$$\mathcal{T} \subseteq \mathcal{S}.$$

For a sequence of random variables $X = \{X_n\}_{n \in \mathbf{N}}$ we define

$$\mathcal{T}_X := X^{-1}(\mathcal{T}), \quad \mathcal{S}_X := X^{-1}(\mathcal{S}).$$

It is not difficult to see that $\mathcal{T}_X$ agrees with the tail σ -algebra of $\{\sigma(X_n)\}_{n \in \mathbf{N}}$, which is also called the tail σ -algebra of $\{X_n\}_{n \in \mathbf{N}}$. By analogy, $\mathcal{S}_X$ is called the *permutable σ -algebra* of $\{X_n\}_{n \in \mathbf{N}}$. We have

$$\mathcal{T}_X \subseteq \mathcal{S}_X.$$

5. Conditional Zero-One Laws. In this section we generalize the concept of Zero-One Laws. Recall that Kolmogorov's $\mathcal{G}$ -Zero-One Law asserts that for every event C of the tail σ -algebra, there exists some $G \in \mathcal{G}$ such that $\chi_C = \chi_G$. Lemma 3.1 shows that this is equivalent with the condition that the tail σ -algebra is a subset of $\mathcal{G}^*$. This leads us to the definition of conditional Zero-One Laws: We say that a class $\mathcal{C}$ fulfills a $\mathcal{G}$ -Zero-One Law if $\mathcal{C} \subseteq \mathcal{G}^*$ holds.

Let us consider some properties of $\mathcal{G}$ -Zero-One Laws.

Lemma 5.1. *Let $\mathcal{C}$ be a class. Then the following are equivalent:*

- (i) $\mathcal{C} \subseteq \mathcal{G}^*$;
- (ii) the pair $\{\mathcal{C}, \mathcal{C}\}$ is $\mathcal{G}$ -conditionally independent;
- (iii) the pair $\{\mathcal{C}, \mathcal{F}\}$ is $\mathcal{G}$ -conditionally independent.

P r o o f. Assume that (i) holds. Let $A \in \mathcal{C} \subseteq \mathcal{G}^*$ and $B \in \mathcal{F}$. Then we have

$$\mathbf{P}(A \cap B \mid \mathcal{G}^*) = \mathbf{E}(\chi_A \chi_B \mid \mathcal{G}^*) = \chi_A \mathbf{E}(\chi_B \mid \mathcal{G}^*) = \mathbf{P}(A \mid \mathcal{G}^*) \mathbf{P}(B \mid \mathcal{G}^*)$$

and Lemma 3.2 implies (iii).

It is clear that (iii) implies (ii).

Assume now that (ii) holds and consider $A \in \mathcal{C}$. Then we have $\mathbf{P}(A \cap B \mid \mathcal{G}) = \mathbf{P}(A \mid \mathcal{G}) \mathbf{P}(B \mid \mathcal{G})$ for all $B \in \mathcal{C}$, hence $\mathbf{P}(A \mid \mathcal{G}) = (\mathbf{P}(A \mid \mathcal{G}))^2$ and thus

$$\mathbf{P}\{\mathbf{P}(A \mid \mathcal{G}) \in \{0, 1\}\} = 1.$$

Since $\mathbf{P}(A \mid \mathcal{G})$ is $\mathcal{G}$ -measurable, there exists some $G \in \mathcal{G}$ with

$$\mathbf{P}(A \mid \mathcal{G}) = \chi_G.$$

This yields $\mathbf{P}(A) = \mathbf{P}(G)$, and from $\chi_G = \chi_G^2 = \chi_G \mathbf{P}(A \mid \mathcal{G}) = \mathbf{P}(G \cap A \mid \mathcal{G})$ we get $\mathbf{P}(G) = \mathbf{P}(G \cap A)$. Therefore, we have $\mathbf{P}(A) = \mathbf{P}(G) = \mathbf{P}(G \cap A)$, hence $\mathbf{P}(A \triangle G) = 0$ and thus $A \in \mathcal{G}^*$.

Theorem 5.1. *Let $\mathcal{C}$ be a class with $\mathcal{C} \subseteq \mathcal{G}^*$. Then*

- (a) for all $A \in \mathcal{C}$ there exists some $G \in \mathcal{G}$ such that $\chi_A = \chi_G$,
- (b) $\mathbf{P}(A \cap B \mid \mathcal{G}) = \mathbf{P}(A \mid \mathcal{G}) \mathbf{P}(B \mid \mathcal{G})$ for every $A, B \in \mathcal{C}$, and
- (c) $\mathbf{P}(A \mid \mathcal{G}) = \chi_A$ for every $A \in \mathcal{C}$.

P r o o f. Because of $\mathcal{C} \subseteq \mathcal{G}^*$ and Lemma 3.1, respectively, Lemma 5.1, we obtain (a) and (b).

Let $A \in \mathcal{C}$. Then by (a) there exists some $G \in \mathcal{G}$ with $\chi_A = \chi_G$, and we get (c) from $\mathbf{P}(A \mid \mathcal{G}) = \chi_G = \chi_A$.

We now state Kolmogorov's $\mathcal{G}$ -Zero-One Law for sequences of random variables in the notation introduced in the previous section.

Theorem 5.2 (Kolmogorov's $\mathcal{G}$ -Zero-One Law). *Let $X = \{X_n\}_{n \in \mathbf{N}}$ be a $\mathcal{G}$ -conditionally independent sequence of random variables. Then*

$$\mathcal{T}_X \subseteq \mathcal{G}^*.$$

Theorem 5.3. *Let $X = \{X_n\}_{n \in \mathbf{N}}$ be a $\mathcal{G}$ -conditionally i.i.d. sequence of random variables. Then*

(a) $\{X_n\}_{n \in \mathbf{N}}$ is $\mathcal{T}_X$ -conditionally i.i.d.;

(b) $\{X_n\}_{n \in \mathbf{N}}$ is $\mathcal{S}_X$ -conditionally i.i.d.

Furthermore

$$\mathbf{P}(X_1 \in B \mid \mathcal{G}) = \mathbf{P}(X_1 \in B \mid \mathcal{T}_X) = \mathbf{P}(X_1 \in B \mid \mathcal{S}_X)$$

holds for all $B \in \mathcal{B}(\mathbf{R})$.

P r o o f. It is easy to see that $\mathcal{G}$ -conditionally i.i.d. sequences of random variables are exchangeable, in the sense that

$$\mathbf{P}(X_1, \dots, X_n) = \mathbf{P}(X_{\pi(1)}, \dots, X_{\pi(n)})$$

holds for all $n \in \mathbf{N}$ and $\pi \in \Pi_n$. Therefore, the proof of (a) and (b) is similar to that of Theorem 7.3.2 by [4, p. 232 ff], but with an other concept of convergence: We can show that for every $x \in \mathbf{R}$ the sequence $\{\chi_{\{X_i < x\}}\}_{i \in \mathbf{N}}$ is again $\mathcal{G}$ -conditionally i.i.d. Because of the conditional Law of Large Numbers we get almost sure convergence of $\{(1/n) \sum_{i=1}^n \chi_{\{X_i < x\}}\}_{n \in \mathbf{N}}$ to $\mathbf{P}(X < x \mid \mathcal{G})$. On the other hand we see that the sequence converges almost surely to a $\mathcal{T}_X$ -measurable random variable $Z(x)$. By Kolmogorov's $\mathcal{G}$ -Zero-One Law we have $Z(x) = \mathbf{P}(X_n < x \mid \mathcal{T}_X)$ for all $n \in \mathbf{N}$. Obviously, $Z(x)$ is $\mathcal{S}_X$ -measurable as well. Now we can follow the proof by Chow and Teicher, with almost sure convergence in the place of convergence in probability. Lemma 3.2 finishes the proof.

From the last theorem we know that every $\mathcal{G}$ -conditionally i.i.d. sequences X is $\mathcal{T}_X$ -conditionally i.i.d.; furthermore, it is also $\mathcal{G}^*$ -conditionally i.i.d. On the other hand we have $\mathcal{T}_X \subseteq \mathcal{G}^*$, by Kolmogorov's $\mathcal{G}$ -Zero-One Law. The next theorem shows us that this sequence is conditionally i.i.d. with respect to every σ -algebra between $\mathcal{T}_X$ and $\mathcal{G}^*$.

Theorem 5.4. *Let $X = \{X_n\}_{n \in \mathbf{N}}$ be a $\mathcal{G}$ -conditionally i.i.d. sequence of random variables and let $\mathcal{H}$ be a sub- σ -algebra of $\mathcal{F}$ with $\mathcal{T}_X \subseteq \mathcal{H} \subseteq \mathcal{G}^*$. Then*

(a) $\{X_n\}_{n \in \mathbf{N}}$ is $\mathcal{H}$ -conditionally i.i.d.;

(b) the identity $\mathbf{P}(X_n \in B \mid \mathcal{T}_X) = \mathbf{P}(X_n \in B \mid \mathcal{H}) = \mathbf{P}(X_n \in B \mid \mathcal{G})$ holds for all $n \in \mathbf{N}$ and $B \in \mathcal{B}(\mathbf{R})$.

P r o o f. For all $n \in \mathbf{N}$ we get from Lemma 3.2 and Theorem 5.3 for all $B \in \mathcal{B}(\mathbf{R})$

$$\begin{aligned} \mathbf{P}(X_n \in B \mid \mathcal{H}) &= \mathbf{E}(\mathbf{P}(X_n \in B \mid \mathcal{G}) \mid \mathcal{H}) = \mathbf{E}(\mathbf{P}(X_n \in B \mid \mathcal{T}_X) \mid \mathcal{H}) \\ &= \mathbf{P}(X_n \in B \mid \mathcal{T}_X) = \mathbf{P}(X_1 \in B \mid \mathcal{T}_X) \end{aligned}$$

which proves (b) and part of (a). Furthermore, for all $x_1, \dots, x_n \in \mathbf{R}$, we have

$$\begin{aligned} \mathbf{P}\left(\bigcap_{k=1}^n \{X_k \leq x_k\} \mid \mathcal{H}\right) &= \mathbf{E}\left(\mathbf{P}\left(\bigcap_{k=1}^n \{X_k \leq x_k\} \mid \mathcal{G}\right) \mid \mathcal{H}\right) \\ &= \mathbf{E}\left(\prod_{k=1}^n \mathbf{P}(X_k \leq x_k \mid \mathcal{G}) \mid \mathcal{H}\right) \\ &= \mathbf{E}\left(\prod_{k=1}^n \mathbf{P}(X_k \leq x_k \mid \mathcal{H}) \mid \mathcal{H}\right) = \prod_{k=1}^n \mathbf{P}(X_k \leq x_k \mid \mathcal{H}) \end{aligned}$$

which because of Lemma 2.1 completes the proof of (a).

A σ -algebra which fulfills the condition of Theorem 5.4 is the intersection of the completion of the conditioning σ -algebra and the σ -algebra generated by the sequence of random variables. For this σ -algebra we get the following theorem.

Theorem 5.5. *Let $X = \{X_n\}_{n \in \mathbf{N}}$ be a $\mathcal{G}$ -conditionally i.i.d. sequence of random variables and $\mathcal{H} := \mathcal{G}^* \cap \sigma(\{X_n\}_{n \in \mathbf{N}})$. Then*

$$\mathcal{H}^* = \mathcal{T}_X^*.$$

P r o o f. Kolmogorov's $\mathcal{G}$ -Zero-One Law implies $\mathcal{I}_X \subseteq \mathcal{G}^*$.

(1) Because of Theorem 5.3 we get for all $G \in \mathcal{G}^*$, $m \in \mathbf{N}$, and $x_1, \dots, x_m \in \mathbf{R}$

$$\begin{aligned} \mathbf{P} \left(G \cap \bigcap_{n=1}^m \{X_n \leq x_n\} \mid \mathcal{I}_X \right) &= \mathbf{E} \left(\mathbf{P} \left(G \cap \bigcap_{n=1}^m \{X_n \leq x_n\} \mid \mathcal{G}^* \right) \mid \mathcal{I}_X \right) \\ &= \mathbf{E} \left(\chi_G \mathbf{P} \left(\bigcap_{n=1}^m \{X_n \leq x_n\} \mid \mathcal{G}^* \right) \mid \mathcal{I}_X \right) = \mathbf{E} \left(\chi_G \prod_{n=1}^m \mathbf{P} \{X_n \leq x_n \mid \mathcal{G}^*\} \mid \mathcal{I}_X \right) \\ &= \mathbf{E} \left(\chi_G \prod_{n=1}^m \mathbf{P} \{X_n \leq x_n \mid \mathcal{I}_X\} \mid \mathcal{I}_X \right) = \mathbf{P} (G \mid \mathcal{I}_X) \prod_{n=1}^m \mathbf{P} (X_n \leq x_n \mid \mathcal{I}_X) \\ &= \mathbf{P} (G \mid \mathcal{I}_X) \mathbf{P} \left(\bigcap_{n=1}^m \{X_n \leq x_n\} \mid \mathcal{I}_X \right). \end{aligned}$$

Hence Lemma 2.1 yields $\{\mathcal{G}^*, \sigma(X_1, \dots, X_m)\}$ is $\mathcal{I}_X$ -conditionally independent for all $m \in \mathbf{N}$. Thus $\{\mathcal{G}^*, \sigma(\{X_n\}_{n \in \mathbf{N}})\}$ is $\mathcal{I}_X$ -conditionally independent, and therefore $\{\mathcal{H}, \mathcal{H}^*\}$ is $\mathcal{I}_X$ -conditionally independent, as well. By Lemma 5.1 we get $\mathcal{H} \subseteq \mathcal{I}_X^*$.

(2) By definition we have $\mathcal{I}_X \subseteq \sigma(\{X_n\}_{n \in \mathbf{N}})$ and therefore $\mathcal{I}_X \subseteq \mathcal{H}^*$.

(1) and (2) imply $\mathcal{H}^* \subseteq \mathcal{I}_X^*$ and $\mathcal{I}_X^* \subseteq \mathcal{H}^*$, and we get the assertion.

Now we are able to prove the Hewitt-Savage $\mathcal{G}$ -Zero-One Law.

Theorem 5.6 (Hewitt-Savage $\mathcal{G}$ -Zero-One Law). *Let $X = \{X_n\}_{n \in \mathbf{N}}$ be a $\mathcal{G}$ -conditionally i.i.d. sequence of random variables. Then*

$$\mathcal{I}_X \subseteq \mathcal{G}^*.$$

P r o o f. By Theorem 5.3 $\{X_n\}_{n \in \mathbf{N}}$ is $\mathcal{I}_X$ -conditionally i.i.d. By definition we have $\mathcal{S}_X \subseteq \sigma(\{X_n\}_{n \in \mathbf{N}})$. Therefore Theorem 5.5 implies $\mathcal{S}_X^* = \mathcal{I}_X^*$. By Kolmogorov's $\mathcal{G}$ -Zero-One Law we get $\mathcal{I}_X^* \subseteq \mathcal{G}^*$, and the assertion follows.

Let us summarize the relations between the different σ -algebras: Without any conditions on X we have $\mathcal{I}_X \subseteq \mathcal{S}_X$. If the sequence is $\mathcal{G}$ -conditionally independent, then we have by Kolmogorov's $\mathcal{G}$ -Zero-One Law $\mathcal{I}_X \subseteq \mathcal{G}^*$. If the sequence is even $\mathcal{G}$ -conditionally identically distributed, then Hewitt-Savage $\mathcal{G}$ -Zero-One Law gives $\mathcal{S}_X \subseteq \mathcal{G}^*$. Let us now illustrate these relations:

6. Remarks. Usually in the literature the (unconditional) notion of exchangeability is considered. De Finetti's theorem shows, that exchangeable sequences of random variables are conditionally i.i.d. given some σ -algebra and that this σ -algebra can be chosen either as the terminal or as the permutable σ -algebra (see [5] and [11]); this, however, is not true for finite sequences. Many proofs about exchangeable sequences of random variables use, via de Finetti's theorem, conditionally i.i.d. sequences. Therefore it appears to be preferable to consider first the conditional properties and to proceed later to the unconditional property of exchangeability. Moreover, many models, e.g., in insurance mathematics, contain a sub- σ -algebra, with respect to which a sequence of random variables is conditionally independent or even conditionally i.i.d. Our notion of conditional Zero-One Laws yields a homogeneous approach which is completely analogous to the classical case. The unconditional properties then follow easily; for special cases, see [4].

Some properties of conditional independence were proved by Döhler [6]. He also used the concept of completion, as well as Chow and Teicher [4]. The definitions of the universal σ -algebras follow Port [12]. Kolmogorov's conditional Zero-One Law is due to Hunt [9]. The classical Hewitt–Savage Zero-One Law was proved by Hewitt and Savage [7]. Another way to prove its conditional version is to use the results by Letta [10], who proved a Hewitt–Savage Zero-One Law for exchangeable families of random variables with functional analytic tools. His result asserts that for exchangeable families of random variables the tail σ -algebra and the permutable σ -algebra are identical in the sense of completion.

Starting with the conditional Zero-One Laws it is possible to obtain further results similar to the unconditional case. E.g., it is possible to prove a theorem on random walks of conditionally i.i.d. sequences of random variables following the classic proof by Bauer [2] which leads to the Chung–Fuchs theorem.

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