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MAGDZIARZ M.*

A NOTE ON MARUYAMA'S MIXING THEOREM

Г. Маруяма в фундаментальной статье [3] (1970, теорема 6) привел необходимые и достаточные условия для того, чтобы стационарный безгранично делимый процесс был процессом с перемешиванием. Мы показываем, что последнее условие в теореме Маруямы следует из предпоследнего. Этот результат позволяет получить существенно более простую версию теоремы Маруямы.

Ключевые слова и фразы: безгранично делимые процессы, стационарные процессы с перемешиванием.

Let $(X_t)_{t \in \mathbf{R}}$ be an infinitely divisible (i.d.) stochastic process, i.e., a process whose all finite dimensional distributions are i.d. Recall that a probability distribution μ on $\mathbf{R}^d$ is i.d. if its characteristic function has the form (Lévy–Khinchine formula)

$$\mathcal{F}_\mu(z) = \exp \left\{ -i\langle a, z \rangle - \frac{1}{2} G(z) - \int_{\mathbf{R}^d} (1 - e^{i\langle x, z \rangle} + i\langle x, z \rangle \mathbf{1}_{\{|x| < 1\}}) Q(dx) \right\}, \quad z \in \mathbf{R}^d.$$

Here $a \in \mathbf{R}^d$, G is a positive semi-definite quadratic form on $\mathbf{R}^d$ called the Gaussian coefficient, and Q is the so-called Lévy measure, i.e., a measure on $\mathbf{R}^d \setminus \{0\}$ satisfying $\int (1 \wedge |x|^2) Q(dx) < \infty$.

A stationary i.d. process $(X_t)_{t \in \mathbf{R}}$ defined on the canonical space $(\mathbf{R}^{\mathbf{R}}, \mathcal{F}, \mathbf{P})$ is said to be mixing if

$$\mathbf{P}(A \cap S^t B) \longrightarrow \mathbf{P}(A) \mathbf{P}(B) \quad (1)$$

as $t \rightarrow \infty$ for every $A, B \in \mathcal{F}$, where (S^t) is a group of shift transformations on $\mathbf{R}^{\mathbf{R}}$.

The description of the mixing property for stationary i.d. processes in terms of their Lévy characteristics dates back to the fundamental paper by Maruyama [3] (Theorema 6). He proved that an i.d. stationary process $(X_t)_{t \in \mathbf{R}}$ is mixing if and only if

(C1) the covariance function $r(t)$ of its Gaussian part converges to 0 as $t \rightarrow \infty$,

(C2) $\lim_{t \rightarrow \infty} Q_{0t}(|xy| > \delta) = 0$ for every $\delta > 0$, and

(C3) $\lim_{t \rightarrow \infty} \int_{0 < x^2 + y^2 \leq 1} xy Q_{0t}(dx, dy) = 0$, where Q_{0t} is the Lévy measure of (X_0, X_t) .

The above result was crucial for further scientific research on the subject of ergodic properties of stochastic processes, and has been extensively exploited by many authors (see, e.g., [1], [2], [4]). In what follows, we show that condition (C2) implies (C3), and therefore the necessary and sufficient conditions for an i.d. process to be mixing can be reduced only to (C1) and (C2). As a consequence, the proofs and methods used in the above-cited references, which based on Maruyama's theorem, can be essentially simplified.

Lemma 1. Assume that $\lim_{t \rightarrow \infty} Q_{0t}(|xy| > \delta) = 0$ for every $\delta > 0$. Then, we get

$$\lim_{t \rightarrow \infty} \int_{0 < x^2 + y^2 \leq 1} xy Q_{0t}(dx, dy) = 0.$$

P r o o f. First, let us notice that the assumption in the lemma implies that

$$\lim_{t \rightarrow \infty} Q_{0t}(|x| \wedge |y| > l) = 0 \quad \text{for every } l > 0, \quad (2)$$

where $a \wedge b = \min\{a, b\}$. Indeed, putting $\delta = l^2$, we get

$$Q_{0t}(|x| \wedge |y| > l) \leq Q_{0t}(|xy| > \delta) \longrightarrow 0 \quad \text{as } t \rightarrow \infty.$$

* Institute of Mathematics and Computer Science, Wrocław University of Technology, 50-370 Wrocław, Poland; e-mail: marcin.magdziarz@pwr.wroc.pl

Now, fix $\varepsilon > 0$, put $B_\delta = \{x^2 + y^2 \leq \delta^2\}$ and $R_\delta = \{\delta^2 < x^2 + y^2 \leq 1\}$. Then, we obtain

$$\int_{0 < x^2 + y^2 \leq 1} |xy| Q_{0t}(dx, dy) = \int_{B_\delta} |xy| Q_{0t}(dx, dy) + \int_{R_\delta} |xy| Q_{0t}(dx, dy) =: I_1 + I_2.$$

We will estimate the terms I_1 and I_2 separately.

Taking advantage of stationarity of Q_{0t} , we get for the first term

$$\begin{aligned} I_1 &\leq \frac{1}{2} \int_{B_\delta} x^2 Q_{0t}(dx, dy) + \frac{1}{2} \int_{B_\delta} y^2 Q_{0t}(dx, dy) \\ &\leq \frac{1}{2} \int_{\{x^2 \leq \delta^2\}} x^2 Q_{0t}(dx, dy) + \frac{1}{2} \int_{\{y^2 \leq \delta^2\}} y^2 Q_{0t}(dx, dy) = \int_{|x| \leq \delta} x^2 Q_0(dx). \end{aligned}$$

Here Q_0 is the Lévy measure of X_0 . Thus, for some appropriately small δ_0 we have

$$I_1 = \int_{B_{\delta_0}} |xy| Q_{0t}(dx, dy) \leq \frac{\varepsilon}{2}. \quad (3)$$

For the second term, put $l_0 = \min\{\delta_0/2, \varepsilon/(8q)\}$, with $q = Q_0(|x| > \delta_0/2) < \infty$. Then, for $C = R_{\delta_0} \cap \{|x| \wedge |y| > l_0\}$ we obtain

$$\begin{aligned} I_2 &= \int_C |xy| Q_{0t}(dx, dy) + \int_{R_{\delta_0} \setminus C} |xy| Q_{0t}(dx, dy) \leq Q_{0t}(C) + \int_{R_{\delta_0} \setminus C} \frac{\varepsilon}{8q} Q_{0t}(dx, dy) \\ &\leq Q_{0t}(|x| \wedge |y| > l_0) + \frac{\varepsilon}{8q} Q_{0t}(R_{\delta_0} \setminus C) \\ &\leq Q_{0t}(|x| \wedge |y| > l_0) + \frac{\varepsilon}{8q} Q_{0t}\left(\left\{|x| > \frac{\delta_0}{2}\right\} \cup \left\{|y| > \frac{\delta_0}{2}\right\}\right) \\ &\leq Q_{0t}(|x| \wedge |y| > l_0) + \frac{\varepsilon}{8q} Q_{0t}\left(|x| > \frac{\delta_0}{2}\right) + \frac{\varepsilon}{8q} Q_{0t}\left(|y| > \frac{\delta_0}{2}\right) \\ &= Q_{0t}(|x| \wedge |y| > l_0) + \frac{\varepsilon}{4q} Q_0\left(|x| > \frac{\delta_0}{2}\right) = Q_{0t}(|x| \wedge |y| > l_0) + \frac{\varepsilon}{4}. \end{aligned}$$

Using (2), for t large enough we have $Q_{0t}(|x| \wedge |y| > l_0) < \varepsilon/4$, and therefore

$$I_2 = \int_{R_{\delta_0}} |xy| Q_{0t}(dx, dy) < \frac{\varepsilon}{2}. \quad (4)$$

Finally, combining (3) and (4), and letting $\varepsilon \searrow 0$, we obtain the desired result.

The above result allows us to formulate the following simplified version of Maruyama's mixing theorem.

Theorem 1. *An i.d. stationary process $(X_t)_{t \in \mathbf{R}}$ is mixing if and only if the following two conditions hold:*

(C1) *the covariance function $r(t)$ of its Gaussian part converges to 0 as $t \rightarrow \infty$, and*

(C2) *$\lim_{t \rightarrow \infty} Q_{0t}(|xy| > \delta) = 0$ for every $\delta > 0$, where Q_{0t} is the Lévy measure of (X_0, X_t) .*

P r o o f. Necessity follows directly from the Maruyama theorem. For sufficiency, let us notice that from Lemma 1 we see that condition (C2) implies (C3). Thus, the process must be mixing.

R e m a r k. Condition (C2) says that the Lévy measure Q_{0t} is asymptotically concentrated on the axes, which for an i.d. distribution is equivalent to the asymptotic independence of the Poissonian parts of X_0 and X_t . Therefore, conditions (C1) and (C2) yield the asymptotic independence of X_0 and X_t , which, in view of definition (1), is the natural interpretation of mixing property.

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