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В замечании 2 для примера проверим пункт 2. Пусть положительная функция $\eta = \eta(\varepsilon)$ достаточно медленно убывает к нулю при $\varepsilon \searrow 0$. Имеем

$$\begin{aligned} \psi_\alpha(\varepsilon) &= \int_1^{1/\varepsilon} g(u) \frac{du}{u} + \int_1^{\eta/\varepsilon} g(u)(R_\alpha(\varepsilon u) - 1) \frac{du}{u} \\ &\quad + \int_{\eta/\varepsilon}^{1/\varepsilon} g(u)(R_\alpha(\varepsilon u) - 1) \frac{du}{u} + \int_1^\infty g\left(\frac{u}{\varepsilon}\right) R_\alpha(u) \frac{du}{u} = I_1 + \dots + I_4. \end{aligned}$$

Далее, используя свойства медленно меняющихся функций (например, [3]), получим при $\varepsilon \searrow 0$

$$\begin{aligned} I_1 &\geq \int_{\eta/\varepsilon}^{1/\varepsilon} g(u) \frac{du}{u} \sim g\left(\frac{1}{\varepsilon}\right) \ln \frac{1}{\eta}, \quad I_2 = o(1) \int_1^{\eta/\varepsilon} g(u) \frac{du}{u} = o(I_1), \\ I_4 + I_3 &= g\left(\frac{1}{\varepsilon}\right) \left(\int_1^\infty R_\alpha(u) \frac{du}{u} - \int_0^1 (1 - R_\alpha(u)) \frac{du}{u} + o(1) \right). \end{aligned}$$

Отсюда следует соотношение (1.23).

СПИСОК ЛИТЕРАТУРЫ

1. Gut A., Spataru A. Precise asymptotics in the Baum-Katz and Davis laws of large numbers. — J. Math. Anal. Appl., 2000, v. 248, № 1, p. 233–246.
2. Spataru A. Precise asymptotics in Spitzer's law of large numbers. — J. Theor. Probab., 1999, v. 12, № 3, p. 811–819.
3. Феллер В. Введение в теорию вероятностей и ее приложения. Т. 2. М.: Мир, 1984, 752 с.
4. Петров В. В. Предельные теоремы для сумм независимых случайных величин. М.: Наука, 1987.
5. Розовский Л. В. Одна оценка для вероятностей больших отклонений. — Матем. заметки, 1987, v. 42, p. 145–156.

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ADJUSTED EULER-MACLAURIN PREDICTOR FOR INTEGRATING SMOOTH SPATIAL PROCESSES

Рассматривается задача нахождения предсказующих интегралов пространственного стационарного процесса Z над единичным квадратом. При помощи аппроксимации среднеквадратичных производных процесса в двумерной формуле Эйлера-Маклорена по конечным разностям вплоть до некоторого порядка построены предикторы, основанные на систематической выборке размера m^2 . Показано, что если спектральная плотность удовлетворяет соотношению $f_Z(\omega) = o(|\omega|^{-p})$ для любого фиксированного положительного целого p , то соответствующая среднеквадратичная ошибка имеет порядок m^{-p} .

Ключевые слова и фразы: пространственный стационарный процесс, предиктор, формула Эйлера-Маклорена.

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1. Introduction

The problem of finding optimal designs when a process is defined on a real line was extensively studied by many authors. When the number of sampling points is fixed, optimal designs can be derived for some specific processes. For a more general class of processes, asymptotic optimal solution to the design problem was given by Sacks and Ylvisaker [12], [13], [14], Eubank, Smith, and Smith [5], Benhenni and Cambanis [2], [3], Benhenni [1], Stein [16], Ritter [11], Istas and Laredo [6] for the equivalent problems of predicting integrals of stochastic processes, estimation of regression coefficients and signal detection. In problems related to geology, environment, hydrology, it is important to consider the best possible spatial design in order to choose the observation sites. There are different type of designs one could use but optimal spatial designs are hard to derive. Many practitioners of geostatistical environmetrics used some regular designs such as triangular, square, or hexagonal grid, and compared them through some criteria, see for instance [7], [9], and [18]. According to [4], regular (systematic) designs are usually more efficient than simple and stratified random sampling in order to predict the integral of a spatial process.

Center regular sampling design was considered for predicting integrals of a spatial process over a square $[0, 1]^2$. The square was divided into $m \times m$ grid of smaller squares and observations are placed at the center of these m^2 squares. Then the predictor of the integral was constructed by taking the average of these observations, see [15], [10]. Matern [8] used also the average predictor based on a systematic design when the observations are not centered. The rates of convergence for the mean squared error of approximation were derived when the observations get dense in the fixed region of integration (in fill asymptotics). Stein [15] considered a weakly stationary process with a spectral density that satisfies $\lim_{\omega \rightarrow +\infty} |\omega|^p f(\omega) = C$. Then he showed that the rate of convergence of the mean squared error is of order m^{-4} for $p = 4$. For smoother process such that $p = 8$, he used a corrected simple average predictor by adjusting the weights near the edges, and obtained a faster rate of convergence m^{-8} . However the problem remains open for the integral prediction of a general smooth stationary process in terms of the underlying spectral density that satisfies for any fixed p , $\lim_{\omega \rightarrow +\infty} |\omega|^p f(\omega) = C$. Tubilla [17] used the extended Euler–MacLaurin formula with a systematic sampling, and derived the rate of convergence with an asymptotic constant that depends on the covariance function.

In this paper, we derive the rates of convergence and asymptotic constant in terms of the spectral density for smooth stationary process. Under similar conditions on the spectral density, $\lim_{\omega \rightarrow +\infty} |\omega|^p f(\omega) = C$, $p \geq 4$, we construct predictors by approximating the existing mean squared derivatives of the process in the extended Euler–MacLaurin formula by finite differences up to some appropriate order. The final predictor will only depend on the observations of the process and thus does not use its existing mean squared derivatives that cannot be observed in practice. First we give the corresponding rates of convergence and the asymptotic constant in terms of the spectral density when the two dimensional Euler–MacLaurin predictor is used. Then we show that the asymptotic results are preserved when we replace the mean squared derivatives by the corresponding finite differences in each direction using the forward and backward Newton's interpolation formula.

2. Presentation and preliminaries

This work is devoted to the prediction of the integral in the unit square

$$I = \iint_{[0,1]^2} Z(x) dx \quad (1)$$

of a stationary centered process $Z = (Z(x))_{x \in \mathbb{R}^2}$ expressed in the following Cramer's representation: for $x = (x_1, x_2)$

$$Z(x) = \iint_{\mathbb{R}^2} e^{i(\omega_1 x_1 + \omega_2 x_2)} dW_Z(\omega), \quad (2)$$

where W_Z is an orthogonal stochastic measure associated with the spectral measure F_Z with Radon–Nikodym derivative f_Z with respect to the Lebesgue measure. Consequently,

the mixed partial derivatives of the process satisfy

$$Z^{(l,l')}(x) = \iint_{\mathbf{R}^2} e^{i(\omega_1 x_1 + \omega_2 x_2)} (i\omega_1)^l (i\omega_2)^{l'} dW_Z(\omega) \quad (3)$$

for any nonnegative integers l, l' such that

$$\iint_{\mathbf{R}^2} |\omega_1|^{2l} |\omega_2|^{2l'} dF_Z(\omega) < +\infty. \quad (4)$$

2.1. Euler–MacLaurin predictor. The Euler–MacLaurin predictor for estimating a simple integral of a process $(Z(x))_{x \in [0,1]}$ having k mean squared derivatives (for any positive integer k) is constructed from the observations of the process and its existing mean squared derivatives at $n+1$ sample points taken from the interval $[0, 1]$; $0 = t_0 < t_1 < \dots < t_n = 1$:

$$\begin{aligned} I_n &= \frac{1}{2n} \sum_{j=0}^{n-1} [Z(t_j) + Z(t_{j+1})] - \sum_{l=1}^k \frac{1}{n^{l+1}} \frac{B_{l+1}}{(l+1)!} [Z^{(l)}(t_n) - Z^{(l)}(t_0)] \\ &= \sum_{j=0}^{n-1} \left\{ \frac{1}{2n} [Z(t_j) + Z(t_{j+1})] - \sum_{l=1}^k \frac{1}{n^{l+1}} \frac{B_{l+1}}{(l+1)!} [Z^{(l)}(t_{j+1}) - Z^{(l)}(t_j)] \right\}. \end{aligned}$$

Now, we assume that the spatial process Z has the necessary mean squared partial derivatives. A two-dimensional Euler–MacLaurin predictor of the integral I given in (1), is obtained by applying the previous Euler–MacLaurin formula in each direction at $m+1$ sample points to $0 = t_0 < t_1 < \dots < t_m = 1$ twice:

$$\begin{aligned} I_m &= \frac{1}{4m^2} \sum_{j,j'=0}^{m-1} [Z(t_j, t_{j'}) + Z(t_{j+1}, t_{j'}) + Z(t_j, t_{j'+1}) + Z(t_{j+1}, t_{j'+1})] \\ &\quad + \frac{1}{2} \sum_{l=1}^k \frac{1}{m^{l+2}} \frac{B_{l+1}}{(l+1)!} \sum_{j'=0}^{m-1} [Z^{(l,0)}(t_0, t_{j'}) - Z^{(l,0)}(t_m, t_{j'}) \\ &\quad \quad \quad + Z^{(l,0)}(t_0, t_{j'+1}) - Z^{(l,0)}(t_m, t_{j'+1})] \\ &\quad + \frac{1}{2} \sum_{l'=1}^k \frac{1}{m^{l'+2}} \frac{B_{l'+1}}{(l'+1)!} \sum_{j=0}^{m-1} [Z^{(0,l')}(t_j, t_0) - Z^{(0,l')}(t_j, t_m) \\ &\quad \quad \quad + Z^{(0,l')}(t_{j+1}, t_0) - Z^{(0,l')}(t_{j+1}, t_m)] \\ &\quad + \sum_{l,l'=1}^k \frac{1}{m^{l+l'+2}} \frac{B_{l+1}}{(l+1)!} \frac{B_{l'+1}}{(l'+1)!} [Z^{(l,l')}(t_0, t_0) - Z^{(l,l')}(t_0, t_m) \\ &\quad \quad \quad - Z^{(l,l')}(t_m, t_0) + Z^{(l,l')}(t_m, t_m)]; \end{aligned}$$

which can also be written as

$$I_m = \sum_{j,j'=0}^{m-1} M_{jj'} \quad (5)$$

with

$$\begin{aligned} M_{jj'} &= \frac{1}{4m^2} [Z(t_j, t_{j'}) + Z(t_{j+1}, t_{j'}) + Z(t_j, t_{j'+1}) + Z(t_{j+1}, t_{j'+1})] \\ &\quad + \frac{1}{2} \sum_{l=1}^k \frac{1}{m^{l+2}} \frac{B_{l+1}}{(l+1)!} [Z^{(l,0)}(t_j, t_{j'}) - Z^{(l,0)}(t_{j+1}, t_{j'}) \\ &\quad \quad \quad + Z^{(l,0)}(t_j, t_{j'+1}) - Z^{(l,0)}(t_{j+1}, t_{j'+1})] \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \sum_{l'=1}^k \frac{1}{m^{l'+2}} \frac{B_{l+1}}{(l'+1)!} \left[Z^{(0,l')}(t_j, t_{j'}) - Z^{(0,l')}(t_j, t_{j'+1}) \right. \\
& \quad \left. + Z^{(0,l')}(t_{j+1}, t_{j'}) - Z^{(0,l')}(t_{j+1}, t_{j'+1}) \right] \\
& + \sum_{l,l'=1}^k \frac{1}{m^{l+l'+2}} \frac{B_{l+1}}{(l+1)!} \frac{B_{l'+1}}{(l'+1)!} \left[Z^{(l,l')}(t_j, t_{j'}) - Z^{(l,l')}(t_j, t_{j'+1}) \right. \\
& \quad \left. - Z^{(l,l')}(t_{j+1}, t_{j'}) + Z^{(l,l')}(t_{j+1}, t_{j'+1}) \right], \quad (6)
\end{aligned}$$

where B_l is the l th Bernoulli number.

In fact, we will study another predictor of I slightly different from the previous one since the last sum will be for l, l' such that $l, l' \geq 1$ and $l + l' \leq k$ instead of $1 \leq l, l' \leq k$. We then replace $M_{jj'}$ in (5) with

$$\begin{aligned}
M_{jj'} &= \frac{1}{4m^2} \left[Z(t_j, t_{j'}) + Z(t_{j+1}, t_{j'}) + Z(t_j, t_{j'+1}) + Z(t_{j+1}, t_{j'+1}) \right] \\
&+ \frac{1}{2} \sum_{l=1}^k \frac{1}{m^{l+2}} \frac{B_{l+1}}{(l+1)!} \left[Z^{(l,0)}(t_j, t_{j'}) - Z^{(l,0)}(t_{j+1}, t_{j'}) \right. \\
& \quad \left. + Z^{(l,0)}(t_j, t_{j'+1}) - Z^{(l,0)}(t_{j+1}, t_{j'+1}) \right] \\
&+ \frac{1}{2} \sum_{l'=1}^k \frac{1}{m^{l'+2}} \frac{B_{l+1}}{(l'+1)!} \left[Z^{(0,l')}(t_j, t_{j'}) - Z^{(0,l')}(t_j, t_{j'+1}) \right. \\
& \quad \left. + Z^{(0,l')}(t_{j+1}, t_{j'}) - Z^{(0,l')}(t_{j+1}, t_{j'+1}) \right] \\
&+ \sum_{\substack{l,l' \geq 1, \\ l+l' \leq k}} \frac{1}{m^{l+l'+2}} \frac{B_{l+1}}{(l+1)!} \frac{B_{l'+1}}{(l'+1)!} \left[Z^{(l,l')}(t_j, t_{j'}) - Z^{(l,l')}(t_j, t_{j'+1}) \right. \\
& \quad \left. - Z^{(l,l')}(t_{j+1}, t_{j'}) + Z^{(l,l')}(t_{j+1}, t_{j'+1}) \right]. \quad (7)
\end{aligned}$$

2.2. Final predictor. The existing partial mean squared derivatives of the process cannot be used in practice. Therefore, we should introduce another predictor that depends only on the observations of the process Z and is of the form

$$\tilde{I}_m = \sum_{j,j'=0}^{m-1} a_{m,j,j'} Z(t_j, t_{j'}).$$

The basic idea is to replace the partial derivatives of Z in the expression of I_m with their finite difference approximations. For any positive integer n , let Δ^n be the increment of n th order operator

$$\Delta^n g(t_j) = \sum_{r=0}^n C_n^r (-1)^{n-r} g(t_{j+r}) \quad \text{with } 0 \leq j+n \leq m.$$

We introduce

$$\begin{aligned}
\Delta^{-n} g(t_j) &= (-1)^n \Delta^n g(t_{j-n}) = \sum_{r=0}^n C_n^{n-r} (-1)^{2n-r} g(t_{j-n+r}) \\
&= \sum_{r'=-n}^0 C_n^{-r'} (-1)^{n-r'} g(t_{j+r'}) \quad (r' = r - n)
\end{aligned}$$

and then, for $\sigma = \pm 1$, $\Delta^\sigma g(t_j) = \sum_{r=0}^{\sigma n} C_n^{[r]} (-1)^{n-r} g(t_{j+r})$. Denoting w_n the function defined by $w_n(u) = \prod_{r=0}^{n-1} (u - r)$, we introduce, for $\sigma = \pm 1$, the approximation $\tilde{g}_\sigma^{(l)}$

(backward $\sigma = -1$ and forward $\sigma = 1$) of the derivative $g^{(l)}$, $l = 1, \dots, k$, based on the Newton interpolation formula up to the order k :

$$\tilde{g}_{\sigma}^{(l)}(t_j) = \sum_{n=l}^k \frac{\sigma^n}{m^{n-l}} \frac{w_n^{(l)}(0)}{n!} \Delta^{\sigma n} g(t_j) \quad \text{with } 0 \leq j, j + \sigma l \leq m.$$

For any function $f(\cdot, \cdot)$ and positive integers n, n' , we define

$$\begin{aligned} (\Delta^{\sigma n, 0} f)(t_j, t_{j'}) &= (\Delta^{\sigma n} f(\cdot, t_{j'}))(t_j), \\ (\Delta^{0, \sigma' n'} f)(t_j, t_{j'}) &= (\Delta^{\sigma' n'} f(t_j, \cdot))(t_{j'}), \\ (\Delta^{\sigma n, \sigma' n'} f)(t_j, t_{j'}) &= (\Delta^{\sigma n, 0} \circ \Delta^{0, \sigma' n'}) f(t_j, t_{j'}) = (\Delta^{0, \sigma' n'} \circ \Delta^{\sigma n, 0}) f(t_j, t_{j'}). \end{aligned}$$

The difference $\Delta^{\sigma n, \sigma' n'} f(t_j, t_{j'})$ with respect to both arguments $(t_j, t_{j'})$ can be derived by applying first the n th order difference $\Delta^{\sigma n, 0} f(t_j, t_{j'})$ with respect to the first argument t_j (with a fixed second argument $t_{j'}$) and then the n' th order difference $\Delta^{0, \sigma' n'} f(t_j, t_{j'})$ with respect to the second argument $t_{j'}$.

For $\sigma, \sigma' = \pm 1$ and any positive integers l, l' , the approximations of the partial derivatives $f^{(l, l')}(t_j, t_{j'})$ are given by

$$\begin{aligned} \tilde{f}_{\sigma}^{(l, 0)}(t_j, t_{j'}) &= \sum_{n=l}^k \frac{\sigma^n}{m^{n-l}} \frac{w_n^{(l)}(0)}{n!} \Delta^{\sigma n, 0} f(t_j, t_{j'}), \\ \tilde{f}_{\sigma'}^{(0, l')}(t_j, t_{j'}) &= \sum_{n'=l'}^k \frac{\sigma'^{n'}}{m^{n'-l'}} \frac{w_{n'}^{(l')}(0)}{n'!} \Delta^{0, \sigma' n'} f(t_j, t_{j'}), \\ \tilde{f}_{\sigma, \sigma'}^{(l, l')}(t_j, t_{j'}) &= \sum_{\substack{n \geq l, n' \geq l', \\ n+n' \leq k}} \frac{\sigma^n \sigma'^{n'}}{m^{n+n'-(l+l')}} \frac{w_n^{(l)}(0)}{n!} \frac{w_{n'}^{(l')}(0)}{n'!} \Delta^{\sigma n, \sigma' n'} f(t_j, t_{j'}). \end{aligned}$$

We then define the approximation $\tilde{f}^{(l, l')}(t_j, t_{j'})$ of the partial derivatives $f^{(l, l')}(t_j, t_{j'})$, $0 \leq l, l' \leq k$ and $0 \leq j, j' \leq m$, by

$$\tilde{f}^{(l, l')}(t_j, t_{j'}) = \begin{cases} \tilde{f}_{+1}^{(l, l')}(t_j, t_{j'}) & \text{if } (j = 0 \text{ and } l' = 0) \text{ or } (j' = 0 \text{ and } l = 0), \\ \tilde{f}_{-1}^{(l, l')}(t_j, t_{j'}) & \text{if } (j = m \text{ and } l' = 0) \text{ or } (j' = m \text{ and } l = 0), \\ \tilde{f}_{+1, +1}^{(l, l')}(t_j, t_{j'}) & \text{if } (j, j' = 0 \text{ and } l, l' > 0), \\ \tilde{f}_{-1, -1}^{(l, l')}(t_j, t_{j'}) & \text{if } (j, j' = m \text{ and } l, l' > 0), \\ \tilde{f}_{+1, -1}^{(l, l')}(t_j, t_{j'}) & \text{if } (j = 0 \text{ and } j' = m \text{ and } l, l' > 0), \\ \tilde{f}_{-1, +1}^{(l, l')}(t_j, t_{j'}) & \text{if } (j = m \text{ and } j' = 0 \text{ and } l, l' > 0). \end{cases}$$

Hence, we introduce a new predictor by replacing the mean squared derivatives with their corresponding approximations in the Euler-MacLaurin predictor:

$$\begin{aligned} \tilde{I}_m &= \frac{1}{4m^2} \sum_{j, j'=0}^{m-1} \left[Z(t_j, t_{j'}) + Z(t_{j+1}, t_{j'}) + Z(t_j, t_{j'+1}) + Z(t_{j+1}, t_{j'+1}) \right] \\ &+ \frac{1}{2} \sum_{l=1}^k \frac{1}{m^{l+2}} \frac{B_{l+1}}{(l+1)!} \sum_{j'=0}^{m-1} \left[\tilde{Z}^{(l, 0)}(t_0, t_{j'}) - \tilde{Z}^{(l, 0)}(t_m, t_{j'}) \right. \\ &\quad \left. + \tilde{Z}^{(l, 0)}(t_0, t_{j'+1}) - \tilde{Z}^{(l, 0)}(t_m, t_{j'+1}) \right] \\ &+ \frac{1}{2} \sum_{l'=1}^k \frac{1}{m^{l'+2}} \frac{B_{l'+1}}{(l'+1)!} \sum_{j=0}^{m-1} \left[\tilde{Z}^{(0, l')}(t_j, t_0) - \tilde{Z}^{(0, l')}(t_j, t_m) \right. \\ &\quad \left. + \tilde{Z}^{(0, l')}(t_{j+1}, t_0) - \tilde{Z}^{(0, l')}(t_{j+1}, t_m) \right] \end{aligned}$$

$$+ \sum_{\substack{l, l' \geq 1, \\ l+l' \leq k}} \frac{1}{m^{l+l'+2}} \frac{B_{l+1}}{(l+1)!} \frac{B_{l'+1}}{(l'+1)!} \left[\tilde{Z}^{(l, l')}(t_0, t_0) - \tilde{Z}^{(l, l')}(t_0, t_m) \right. \\ \left. - \tilde{Z}^{(l, l')}(t_m, t_0) + \tilde{Z}^{(l, l')}(t_m, t_m) \right].$$

Setting

$$D_{k, f} = \frac{1}{2} \sum_{l=1}^k \frac{1}{m^{l+2}} \frac{B_{l+1}}{(l+1)!} \sum_{j'=0}^{m-1} \left[D^{(l, 0)} f(t_0, t_{j'}) - D^{(l, 0)} f(t_m, t_{j'}) \right. \\ \left. + D^{(l, 0)} f(t_0, t_{j'+1}) - D^{(l, 0)} f(t_m, t_{j'+1}) \right] \\ + \frac{1}{2} \sum_{l'=1}^k \frac{1}{m^{l'+2}} \frac{B_{l'+1}}{(l'+1)!} \sum_{j=0}^{m-1} \left[D^{(0, l')} f(t_j, t_0) - D^{(0, l')} f(t_j, t_m) \right. \\ \left. + D^{(0, l')} f(t_{j+1}, t_0) - D^{(0, l')} f(t_{j+1}, t_m) \right] \\ + \sum_{\substack{l, l' \geq 1, \\ l+l' \leq k}} \frac{1}{m^{l+l'+2}} \frac{B_{l+1}}{(l+1)!} \frac{B_{l'+1}}{(l'+1)!} \left[D^{(l, l')} f(t_0, t_0) - D^{(l, l')} f(t_0, t_m) \right. \\ \left. - D^{(l, l')} f(t_m, t_0) + D^{(l, l')} f(t_m, t_m) \right],$$

where $D^{(l, l')} f(t_j, t_{j'}) = f^{(l, l')}(t_j, t_{j'}) - \tilde{f}^{(l, l')}(t_j, t_{j'})$, we obtain the residual error of approximation from using the predictor $\tilde{I}_m$ instead of I_m : $I_m - \tilde{I}_m = D_{k, Z}$.

3. Main results

In what follows we use a systematic sampling design where the observations of the spatial process are taken at the sample points $(t_j, t_{j'}) = (j/m, j'/m)$ for $0 \leq j, j' \leq m$. First, we need to express the mean squared error of the predictor. Since

$$I - I_m = \sum_{j, j'=1}^m Y_{jj'},$$

where $Y_{jj'} = \iint_{[(j-1)/m, j/m] \times [(j'-1)/m, j'/m]} (Z(x) - M_{jj'}) dx$, we obtain

$$\mathbf{E}((I - I_m)^2) = \text{Var}(I - I_m) = \sum_{j_1, j'_1, j_2, j'_2=1}^m \text{Cov}(Y_{j_1 j'_1}, Y_{j_2 j'_2}).$$

By a straightforward calculation using relation (3), we obtain

$$Y_{jj'} = \iint_{\mathbf{R}^2} e^{i(\omega_1 j/m + \omega_2 j'/m)} e^{-i\omega_1/(2m)} e^{-i\omega_2/(2m)} H(\omega) dW(\omega),$$

where $H(\omega) = \sum_{i=1}^5 H_i(\omega)$ with

$$H_1(\omega) = 4 \frac{\sin(\omega_1/(2m)) \sin(\omega_2/(2m))}{\omega \omega'},$$

$$H_2(\omega) = -\frac{1}{m^2} \cos\left(\frac{\omega_1}{2m}\right) \cos\left(\frac{\omega_2}{2m}\right),$$

$$H_3(\omega) = 2i \sum_{l=1}^k \frac{1}{m^{l+2}} \frac{B_{l+1}}{(l+1)!} \left((i\omega_1)^l \sin\left(\frac{\omega_1}{2m}\right) \cos\left(\frac{\omega_2}{2m}\right) \right),$$

$$H_4(\omega) = 2i \sum_{l'=1}^k \frac{1}{m^{l'+2}} \frac{B_{l'+1}}{(l'+1)!} \left((i\omega_2)^{l'} \cos\left(\frac{\omega_1}{2m}\right) \sin\left(\frac{\omega_2}{2m}\right) \right) = H_3(\omega_2, \omega_1),$$

$$H_5(\omega) = 4 \sum_{\substack{l, l' \geq 1, \\ l+l' \leq k}} \frac{1}{m^{l+l'+2}} \frac{B_{l+1}}{(l+1)!} \frac{B_{l'+1}}{(l'+1)!} (i\omega_1)^l (i\omega_2)^{l'} \sin\left(\frac{\omega_1}{2m}\right) \sin\left(\frac{\omega_2}{2m}\right).$$

This leads to

$$\text{Cov}(Y_{j_1 j'_1}, Y_{j_2 j'_2}) = \int_{\mathbf{R}^2} \left(\frac{j_1 - j_2}{m} \omega_1 \right) \cos \left(\frac{j'_1 - j'_2}{m} \omega_2 \right) |H(\omega)|^2 dF_Z(\omega),$$

so that the mean squared error of approximation can be written as

$$\mathbf{E}((I - I_m)^2) = \int_{\mathbf{R}^2} \frac{(1 - \cos(\omega_1))(1 - \cos(\omega_2))}{4 \sin^2(\omega_1/(2m)) \sin^2(\omega_2/(2m))} |H(\omega)|^2 dF_Z(\omega)$$

by applying the result

$$\sum_{j_1, j_2=1}^m \cos((j_1 - j_2)t) = \frac{1 - \cos(mt)}{2 \sin^2(t/2)}.$$

The asymptotic behavior of the mean squared error of the predictor I_m follows directly from that of $H(\omega)$. Let us denote h the nonnegative integer such that

$$k = \begin{cases} 2h & \text{if } k \text{ even,} \\ 2h - 1 & \text{if } k \text{ odd.} \end{cases}$$

Theorem 1. *If a systematic sampling design of size m^2 is used, and if the spectral density f_Z of the process Z satisfies*

- (i) *assumption (4) for any nonnegative l, l' such that $l + l' \leq 2h + 2$, and*
 - (ii) *$f_Z(\omega) = o(|\omega|^{-4h-4})$ as $|\omega| \rightarrow +\infty$,*
- then*

$$\lim_{m \rightarrow +\infty} m^{4h+4} \mathbf{E}((I - I_m)^2) = \int_{\mathbf{R}^2} \frac{(1 - \cos(\omega_1))(1 - \cos(\omega_2))}{\omega_1^2 \omega_2^2} C(\omega)^2 f_Z(\omega) d\omega,$$

where for any ω in $\mathbf{R}^2$

$$C(\omega) = \frac{1}{(2h+2)!} \sum_{q=0}^{h+1} C_{2h+2}^{2q} (\omega_1^{2q} |B_{2q}|) (\omega_2^{2h+2-2q} |B_{2h+2-2q}|).$$

Assumption (ii) ensures that the mean squared partial derivatives of the process Z in the predictor I_m exist.

R e m a r k 1. In the case when $H_5(\omega)$ is replaced with

$$H'_5(\omega) = 4 \sum_{l, l'=1}^{2h-1} \frac{1}{m^{l+l'+2}} \frac{B_{l+1}}{(l+1)!} \frac{B_{l'+1}}{(l'+1)!} (i\omega_1)^l (i\omega_2)^{l'} \sin\left(\frac{\omega_1}{2m}\right) \sin\left(\frac{\omega_2}{2m}\right)$$

this will not affect the asymptotic behavior of $H(\omega)$ in terms of the rate of convergence, but will have a different asymptotic constant $C'(\omega)$:

$$C'(\omega) = \frac{|B_{2h+2}|}{(2h+2)!} (\omega_1^{2h+2} + \omega_2^{2h+2}).$$

The mean squared error of the final predictor $\tilde{I}_m$ can be decomposed as the mean squared error corresponding to the original predictor I_m , a residual mean squared error from using $\tilde{I}_m$ instead of I_m and a cross-covariance term:

$$\mathbf{E}((I - \tilde{I}_m)^2) = \mathbf{E}((I - I_m)^2) + \mathbf{E}((I_m - \tilde{I}_m)^2) + 2\mathbf{E}((I - I_m)(I_m - \tilde{I}_m)).$$

We show that the rates of convergence of the residual mean squared error and the cross-covariance terms are of higher orders than the rate of convergence of the mean squared error corresponding to the Euler-MacLaurin predictor. Therefore the result of Theorem 1 remains true when the final predictor is used.

Theorem 2. *Under the same assumptions as in Theorem 1, we have*

$$\lim_{m \rightarrow +\infty} m^{4h+4} \mathbf{E}((I - \tilde{I}_m)^2) = \int_{\mathbf{R}^2} \frac{(1 - \cos(\omega_1))(1 - \cos(\omega_2))}{\omega_1^2 \omega_2^2} C(\omega)^2 f_Z(\omega) d\omega,$$

where the constant $C(\omega)$ is defined in Theorem 1.

4. Proofs

P r o o f o f T h e o r e m 1. Using the fact that the Bernoulli coefficients B_l vanish for l odd with $l \geq 2$, the functions H_3 and H_4 can also be written as

$$H_3(\omega) = 2i \sum_{l=1}^{2h-1} \frac{1}{m^{l+2}} \frac{B_{l+1}}{(l+1)!} \left((i\omega_1)^l \sin\left(\frac{\omega_1}{2m}\right) \cos\left(\frac{\omega_2}{2m}\right) \right),$$

$$H_4(\omega) = 2i \sum_{l'=1}^{2h-1} \frac{1}{m^{l'+2}} \frac{B_{l'+1}}{(l'+1)!} \left((i\omega_2)^{l'} \cos\left(\frac{\omega_1}{2m}\right) \sin\left(\frac{\omega_2}{2m}\right) \right).$$

When k is even, the term $H_5(\omega)$ as written above, will have faster rate of convergence than the other terms, and thus will not contribute to the overall asymptotic expression of the mean squared error. In fact it suffices to take

$$H_5(\omega) = 4 \sum_{\substack{l, l' \geq 1, \\ l+l' \leq 2h-2}} \frac{1}{m^{l+l'+2}} \frac{B_{l+1}}{(l+1)!} \frac{B_{l'+1}}{(l'+1)!} (i\omega_1)^l (i\omega_2)^{l'} \sin\left(\frac{\omega_1}{2m}\right) \sin\left(\frac{\omega_2}{2m}\right).$$

Using Taylor's expansions around 0 of the sinus and cosinus functions, we obtain

$$\begin{aligned} H_1(\omega) &= \frac{4}{m^2} \sum_{p, p'=0}^{+\infty} \frac{1}{2^{2p+1}(2p+1)!} \frac{1}{2^{2p'+1}(2p'+1)!} \left(\frac{i\omega_1}{m} \right)^{2p} \left(\frac{i\omega_2}{m} \right)^{2p'} \\ &= \frac{4}{m^2} \sum_{p, p'=0}^{+\infty} \left\{ \frac{1}{(2p+1)!} \left(\frac{i\omega_1}{m} \right)^{2p} \left(C_{2p+1}^0 \frac{1}{2^{2p+1}} B_0 \right) \right\} \\ &\quad \times \left\{ \frac{1}{(2p'+1)!} \left(\frac{i\omega_2}{m} \right)^{2p'} \left(C_{2p'+1}^0 \frac{1}{2^{2p'+1}} B_0 \right) \right\} \\ &= \frac{4}{m^2} \sum_{p, p'=0}^{+\infty} Q(\omega_1; 2p+1, 0) Q(\omega_2; 2p'+1, 0) \end{aligned}$$

by putting for every $r \geq s$

$$Q(\omega_j; r, s) = \frac{1}{r!} \left(\frac{i\omega_j}{m} \right)^{r-1} \left(C_r^s \frac{1}{2^{r-s}} B_s \right).$$

Likewise,

$$\begin{aligned} H_2(\omega) &= -\frac{4}{m^2} \sum_{p, p'=0}^{+\infty} \frac{1}{2^{2p+1}(2p)!} \frac{1}{2^{2p'+1}(2p')!} \left(\frac{i\omega_1}{m} \right)^{2p} \left(\frac{i\omega_2}{m} \right)^{2p'} \\ &= -\frac{4}{m^2} \sum_{p, p'=0}^{+\infty} \left\{ \frac{1}{(2p+1)!} \left(\frac{i\omega_1}{m} \right)^{2p} \left(C_{2p+1}^1 \frac{1}{2^{2p}} B_1 \right) \right\} \\ &\quad \times \left\{ \frac{1}{(2p'+1)!} \left(\frac{i\omega_2}{m} \right)^{2p'} \left(C_{2p'+1}^1 \frac{1}{2^{2p'}} B_1 \right) \right\} \\ &= -\frac{4}{m^2} \sum_{p, p'=0}^{+\infty} Q(\omega_1; 2p+1, 1) Q(\omega_2; 2p'+1, 1), \\ H_3(\omega) &= -\frac{4}{m^2} \sum_{p, p'=0}^{+\infty} \left\{ \sum_{l=1}^{2h-1} \frac{1}{(2p+l+2)!} \left(\frac{i\omega_1}{m} \right)^{2p+l+1} \left(C_{2p+l+2}^{l+1} \frac{1}{2^{2p+l}} B_{l+1} \right) \right\} \\ &\quad \times \left\{ \frac{1}{(2p'+1)!} \left(\frac{i\omega_2}{m} \right)^{2p'} \left(C_{2p'+1}^1 \frac{1}{2^{2p'}} B_1 \right) \right\} \\ &= -\frac{4}{m^2} \sum_{p, p'=0}^{+\infty} \left\{ \sum_{l=1}^{2h-1} Q(\omega_1; (2p+1) + (l+1), l+1) \right\} Q(\omega_2; 2p'+1, 1), \end{aligned}$$

$$H_4(\omega) = -\frac{4}{m^2} \sum_{p,p'=0}^{+\infty} Q(\omega_1; 2p+1, 1) \left\{ \sum_{l'=1}^{2h-1} Q(\omega_2; (2p'+1) + (l'+1), l'+1) \right\},$$

and

$$\begin{aligned} H_5(\omega) &= -\frac{4}{m^2} \sum_{p,p'=0}^{+\infty} \sum_{\substack{l,l' \geq 1, \\ l+l' \leq 2h-2}} \left\{ \frac{1}{(2p+l+2)!} \left(\frac{i\omega_1}{m} \right)^{2p+l+1} \left(C_{2p+l+2}^{l+1} \frac{1}{2^{2p+1}} B_{l+1} \right) \right\} \\ &\quad \times \left\{ \frac{1}{(2p'+l'+2)!} \left(\frac{i\omega_2}{m} \right)^{2p'+l'+1} \left(C_{2p'+l'+2}^{l'+1} \frac{1}{2^{2p'+1}} B_{l'+1} \right) \right\} \\ &= -\frac{4}{m^2} \sum_{p,p'=0}^{+\infty} \sum_{\substack{l,l' \geq 1, \\ l+l' \leq 2h-2}} Q(\omega_1; (2p+1) + (l+1), l+1) \\ &\quad \times Q(\omega_2; (2p'+1) + (l'+1), l'+1). \end{aligned}$$

Finally, we obtain

$$\begin{aligned} H_3(\omega) &= -\frac{4}{m^2} \sum_{p,p'=0}^{+\infty} \left\{ \sum_{1 \leq l \leq h} Q(\omega_1; 2(p+l)+1, 2l) \right\} Q(\omega_2; 2p'+1, 1), \\ H_4(\omega) &= -\frac{4}{m^2} \sum_{p,p'=0}^{+\infty} Q(\omega_1; 2p+1, 1) \left\{ \sum_{1 \leq l' \leq h} Q(\omega_2; 2(p'+l')+1, 2l') \right\}, \\ H_5(\omega) &= -\frac{4}{m^2} \sum_{p,p'=0}^{+\infty} \sum_{\substack{l,l' \geq 1, \\ l+l' \leq h}} Q(\omega_1; 2(p+l)+1, 2l) Q(\omega_2; 2(p'+l')+1, 2l'). \end{aligned}$$

Therefore, $H(\omega)$ can be expressed as follows:

$$H(\omega) = \frac{4}{m^2} \sum_{q,q'=0}^{+\infty} Q(\omega_1; 2q+1, 0) Q(\omega_2; 2q'+1, 0) - G_{q,q'}(\omega),$$

where

$$\begin{aligned} G_{q,q'}(\omega) &= Q(\omega_1; 2q+1, 1) Q(\omega_2; 2q'+1, 1) \\ &\quad + \left\{ \sum_{1 \leq l \leq \min(q,h)} Q(\omega_1; 2q+1, 2l) \right\} Q(\omega_2; 2q'+1, 1) \\ &\quad + Q(\omega_1; 2q+1, 1) \left\{ \sum_{1 \leq l' \leq \min(q',h)} Q(\omega_2; 2q'+1, 2l') \right\} \\ &\quad + \sum_{\substack{1 \leq l \leq q, 1 \leq l' \leq q', \\ l+l' \leq h}} Q(\omega_1; 2q+1, 2l) Q(\omega_2; 2q'+1, 2l'). \end{aligned}$$

But if $q+q' \leq h$, then $l+l' \leq h$, $\min(q, h) = q$ and $\min(q', h) = q'$ and thus we have

$$\begin{aligned} G_{q,q'}(\omega) &= \sum_{\substack{1 \leq l \leq 2q+1, \\ 1 \leq l' \leq 2q'+1}} Q(\omega_1; 2q+1, l) Q(\omega_2; 2q'+1, l') \\ &= Q(\omega_1; 2q+1, 0) Q(\omega_2; 2q'+1, 0), \end{aligned}$$

since

$$\sum_{1 \leq l \leq 2q+1} Q(\omega_j; 2q+1, l) = \frac{1}{(2q+1)!} \left(\frac{i\omega_j}{m} \right)^{2q} \left(\sum_{1 \leq l \leq 2q+1} C_{2q+1}^l \left(\frac{1}{2} \right)^{2q+1-l} B_l \right)$$

$$\begin{aligned}
&= \frac{1}{(2q+1)!} \left(\frac{i\omega_j}{m} \right)^{2q} \left(B_{2q+1} \left(\frac{1}{2} \right) - C_{2q+1}^0 \left(\frac{1}{2} \right)^{2q+1} B_0 \right) \\
&= -\frac{1}{(2q+1)!} \left(\frac{i\omega_j}{m} \right)^{2q} \left(C_{2q+1}^0 \left(\frac{1}{2} \right)^{2q+1} B_0 \right) = -Q(\omega_j; 2q+1, 0).
\end{aligned}$$

If $q + q' = h + 1$, then $l + l' \leq h + 1$, and we have in this case

$$\begin{aligned}
G_{q,q'}(\omega) &= Q(\omega_1; 2q+1, 1) Q(\omega_2; 2q'+1, 1) \\
&\quad + \left\{ \sum_{1 \leq l \leq q} Q(\omega_1; 2q+1, 2l) \right\} Q(\omega_2; 2q'+1, 1) \\
&\quad - Q(\omega_1; 2q+1, 2q) Q(\omega_2; 2q'+1, 1) \delta(q') \\
&\quad + Q(\omega_1; 2q+1, 1) \left\{ \sum_{1 \leq l' \leq q'} Q(\omega_2; 2q'+1, 2l') \right\} \\
&\quad - Q(\omega_1; 2q+1, 1) Q(\omega_2; 2q'+1, 2q') \delta(q) \\
&\quad + \sum_{1 \leq l \leq q, 1 \leq l' \leq q'} Q(\omega_1; 2q+1, 2l) Q(\omega_2; 2q'+1, 2l') \\
&\quad - Q(\omega_1; 2q+1, 2q) Q(\omega_2; 2q'+1, 2q') (1 - \delta(q)) (1 - \delta(q')) \\
&= Q(\omega_1; 2q+1, 0) Q(\omega_2; 2q'+1, 0) - Q(\omega_1; 2q+1, 2q) Q(\omega_2; 2q'+1, 1) \delta_0(q') \\
&\quad - Q(\omega_1; 2q+1, 1) Q(\omega_2; 2q'+1, 2q') \delta(q) \\
&\quad - Q(\omega_1; 2q+1, 2q) Q(\omega_2; 2q'+1, 2q') (1 - \delta(q)) (1 - \delta(q')),
\end{aligned}$$

where δ denotes the usual Dirac function. It follows that

$$\begin{aligned}
H(\omega) &= \frac{4}{m^2} \left\{ \sum_{\substack{q=0, \\ (q'=h+1-q)}}^{h+1} Q(\omega_1; 2q+1, 2q + \delta_0(q)) Q(\omega_2; 2q'+1, 2q' + \delta_0(q')) \right. \\
&\quad \left. + \sum_{\substack{q, q'=0, \\ q+q'>h+1}}^{+\infty} Q(\omega_1; 2q+1, 0) Q(\omega_2; 2q'+1, 0) - G_{q,q'}(\omega) \right\}.
\end{aligned}$$

The terms of the first sum vanish when $q + q' \leq h$ and the second sum is of higher order. Therefore the overall rate of convergence is obtained when $q + q' = h + 1$, that is $\lim_{m \rightarrow +\infty} m^{2h+4} H(\omega) = C(\omega)$ by putting

$$\begin{aligned}
C(\omega) &= \frac{(-1)^{h+1}}{(2h+2)!} \sum_{\substack{q=0, \\ (q'=h+1-q)}}^{h+1} C_{2h+2}^{2q} \left(2^{\delta_0(q)} \omega_1^{2q} B_{2q+\delta_0(q)} \right) \left(2^{\delta_0(q')} \omega_2^{2q'} B_{2q'+\delta_0(q')} \right) \\
&= \frac{(-1)^{h+1}}{(2h+2)!} \sum_{\substack{q=0, \\ (q'=h+1-q)}}^{h+1} C_{2h+2}^{2q} \left((-1)^{\delta_0(q)} \omega_1^{2q} B_{2q} \right) \left((-1)^{\delta_0(q')} \omega_2^{2q'} B_{2q'} \right) \\
&= \frac{(-1)^{h+1}}{(2h+2)!} \sum_{\substack{q=0, \\ (q'=h+1-q)}}^{h+1} C_{2h+2}^{2q} \left((-1)^{q+1} \omega_1^{2q} |B_{2q}| \right) \left((-1)^{q'+1} \omega_2^{2q'} |B_{2q'}| \right) \\
&= \frac{1}{(2h+2)!} \sum_{\substack{q=0, \\ (q'=h+1-q)}}^{h+1} C_{2h+2}^{2q} \left(\omega_1^{2q} |B_{2q}| \right) \left(\omega_2^{2q'} |B_{2q'}| \right)
\end{aligned}$$

using the fact that $B_{2q} = (-1)^{q+1} |B_{2q}|$ if $q \geq 1$ and $(-1)^{\delta_0(q)} B_{2q} = (-1)^{q+1} |B_{2q}|$ if $q = 0$.

Following the same arguments as Stein [15], we set for any integer j, k

$$g_m(\omega; j, k) = \frac{(1 - \cos(\omega_1))(1 - \cos(\omega_2))}{4 \sin^2(\omega_1/(2m)) \sin^2(\omega_2/(2m))} \left| H(\omega_1 + 2m\pi j, \omega_2 + 2m\pi k) \right|^2 \\ \times f_Z(\omega_1 + 2m\pi j, \omega_2 + 2m\pi k) 1_{\{| \omega_1 | \leq m\pi, | \omega_2 | \leq m\pi\}}$$

and we notice that $\mathbf{E}((I - I_m)^2) = \int_{\mathbf{R}^2} g_m(\omega; 0, 0) d\omega + \int_{\mathbf{R}^2} \sum^* g_m(\omega; j, k) d\omega$, where $\sum^*$ corresponds to the sum over $\mathbf{Z}^2 \setminus \{(0, 0)\}$.

Since

$$\lim_{m \rightarrow +\infty} m^{4h+4} g_m(\omega; 0, 0) = \frac{(1 - \cos(\omega_1))(1 - \cos(\omega_2))}{\omega_1^2 \omega_2^2} C(\omega)^2 f_Z(\omega) \stackrel{\text{def}}{=} G(\omega)$$

and $m^{4h+4} g_m(\omega; 0, 0)$ is dominated by some constant times $G(\omega)$ which is absolutely integrable over $\mathbf{R}^2$, we have, by dominated convergence, that $\lim_{m \rightarrow +\infty} m^{4h+4} g_m(\omega; 0, 0) = \int_{\mathbf{R}^2} G(\omega) d\omega$. Finally, Theorem 1 follows by showing that $f_Z(\omega) = o(|\omega|^{-4h-4})$ implies

$$\int_{\mathbf{R}^2} \sum^* g_m(\omega; j, k) d\omega = o(m^{-4h-4}).$$

P r o o f o f T h e o r e m 2. Since $D_{k,Z} = I_m - \bar{I}_m = \iint_{\mathbf{R}^2} D_{k,f\omega} dW_Z(\omega)$, where $f_\omega(x) = e^{i(\omega_1 x_1 + \omega_2 x_2)}$, it follows that

$$\mathbf{E}((I_m - \bar{I}_m)^2) = \mathbf{E}(D_{k,Z}^2) = \int_{\mathbf{R}^2} |D_{k,f\omega}|^2 dF_Z(\omega).$$

Now

$$|D_{k,f\omega}|^2 \leq (8h+4) m \sum_{l=1}^{2h+1} \frac{1}{m^{2(l+2)}} \left(\frac{B_{l+1}}{(l+1)!} \right)^2 \\ \times \sum_{j'=0}^{m-1} \left[\left(D^{(l,0)} f_\omega(t_0, t_{j'}) \right)^2 + \left(D^{(l,0)} f_\omega(t_m, t_{j'}) \right)^2 \right. \\ \left. + \left(D^{(l,0)} f_\omega(t_0, t_{j'+1}) \right)^2 + \left(D^{(l,0)} f_\omega(t_m, t_{j'+1}) \right)^2 \right] \\ + (8h+4) m \sum_{l'=1}^{2h+1} \frac{1}{m^{2(l'+2)}} \left(\frac{B_{l'+1}}{(l'+1)!} \right)^2 \\ \times \sum_{j=0}^{m-1} \left[\left(D^{(0,l')} f_\omega(t_j, t_0) \right)^2 + \left(D^{(0,l')} f_\omega(t_j, t_m) \right)^2 \right. \\ \left. + \left(D^{(0,l')} f_\omega(t_{j+1}, t_0) \right)^2 + \left(D^{(0,l')} f_\omega(t_{j+1}, t_m) \right)^2 \right] \\ + (8h+4)^2 \sum_{\substack{l, l' \geq 1, \\ l+l' \leq 2h+1}} \frac{1}{m^{2(l+l'+2)}} \left(\frac{B_{l+1}}{(l+1)!} \frac{B_{l'+1}}{(l'+1)!} \right)^2 \\ \times \left[\left(D^{(l,l')} f_\omega(t_0, t_0) \right)^2 + \left(D^{(l,l')} f_\omega(t_0, t_m) \right)^2 \right. \\ \left. + \left(D^{(l,l')} f_\omega(t_m, t_0) \right)^2 + \left(D^{(l,l')} f_\omega(t_m, t_m) \right)^2 \right].$$

But, using the forward and backward approximation of f_ω up to order $2h+2$ we have

$$D^{(l,0)} f_\omega(t_0, x) = \frac{1}{(2h+2)! m^{2h+2-l}} w_{2h+2}^{(l)}(0) f_\omega^{(2h+2,0)}(\xi_0, x),$$

where $\xi_0 \in]t_0; t_{2h+1}[$,

$$D^{(l,0)} f_\omega(t_m, x) = \frac{(-1)^{2h+2-l}}{(2h+2)! m^{2h+2-l}} w_{2h+2}^{(l)}(0) f_\omega^{(2h+2,0)}(\xi_m, x),$$

where $\xi_m \in]t_{m-2h-1}; t_m[$.

Therefore, $|D^{(l,0)} f_\omega(t_0, x)|^2 = |D^{(l,0)} f_\omega(t_m, x)|^2 = K_{\omega_1}^l / m^{4h+4-2l}$, where $K_{\omega_j}^l = (w_{2h+2}^{(l)}(0)/(2h+2)!)^2 \omega_j^{4h+4}$. Likewise,

$$\left| D^{(0,l')} f_\omega(x, t_0) \right|^2 = \left| D^{(0,l')} f_\omega(x, t_m) \right|^2 = \frac{K_{\omega_2}^{l'}}{m^{4h+4-2l'}}.$$

First,

$$\begin{aligned} f_\omega^{(l,l')}(t_0, t_0) &= \sum_{\substack{j \geq l, j' \geq l', \\ j+j' \leq 2h+1}} \frac{1}{m^{j+j'-(l+l')}} \frac{w_j^{(l)}(0)}{j!} \frac{w_{j'}^{(l')}(0)}{j'!} \Delta^{j,j'} f_\omega(t_0, t_0) \\ &+ \frac{1}{m^{2h+2-(l+l')}} \sum_{j'=l'}^{2h+1-l} \frac{w_{2h+2-j'}^{(l)}(0)}{(2h+2-j')!} \frac{w_{j'}^{(l')}(0)}{j'!} \Delta^{0,j'} f_\omega^{(2h+2-j',0)}(\xi_0, t_0) \\ &+ \frac{1}{m^{2h+2-(l+l')}} \frac{w_{2h+2-l}^{(l')}(0)}{(2h+2-l)!} f_\omega^{(l,2h+2-l)}(t_0, \xi'_0). \end{aligned}$$

But

$$\begin{aligned} \left| \Delta^{0,j'} f_\omega^{(2h+2-j',0)}(\xi_0, t_0) \right| &= \left| \sum_{r=0}^{j'} (-1)^{j'-r} C_{j'}^r f_\omega^{(2h+2-j',0)}(\xi_0, t_r) \right| \\ &\leq \sum_{r=0}^{j'} C_{j'}^r |\omega_1|^{2h+2-j'} = 2^{j'} |\omega_1|^{2h+2-j'}. \end{aligned}$$

Then, $|D^{(l,l')} f_\omega(t_0, t_0)|^2 \leq K_{\omega}^{l,l'} / m^{4h+4-2(l+l')}$, where

$$\begin{aligned} K_{\omega}^{l,l'} &= (4h+2-2(l+l')) \sum_{j'=l'}^{2h+1-l} \left(\frac{w_{2h+2-j'}^{(l)}(0)}{(2h+2-j')!} \frac{w_{j'}^{(l')}(0)}{j'!} \right)^2 2^{2j'} \omega_1^{4h+4-2j'} \\ &+ 2 \left(\frac{w_{2h+2-l}^{(l')}(0)}{(2h+2-l)!} \right)^2 \omega_1^{2l} \omega_2^{4h+4-2l}. \end{aligned}$$

Likewise,

$$\begin{aligned} \left| D^{(l,l')} f_\omega(t_0, t_m) \right|^2 &\leq \frac{K_{\omega}^{l,l'}}{m^{4h+4-2(l+l')}} \quad \left| D^{(l,l')} f_\omega(t_m, t_0) \right|^2 \leq \frac{K_{\omega}^{l,l'}}{m^{4h+4-2(l+l')}} \\ \left| D^{(l,l')} f_\omega(t_m, t_m) \right|^2 &\leq \frac{K_{\omega}^{l,l'}}{m^{4h+4-2(l+l')}}; \end{aligned}$$

so that

$$\begin{aligned} |D_{k,f_\omega}|^2 &\leq (8h+4) m \sum_{l=1}^{2h+1} \frac{1}{m^{2(l+2)}} \left(\frac{B_{l+1}}{(l+1)!} \right)^2 \frac{4m}{m^{4h+4-2l}} K_{\omega_1}^l \\ &+ (8h+4) m \sum_{l'=1}^{2h+1} \frac{1}{m^{2(l'+2)}} \left(\frac{B_{l'+1}}{(l'+1)!} \right)^2 \frac{4m}{m^{4h+4-2l'}} K_{\omega_2}^{l'} \\ &+ (8h+4)^2 \sum_{\substack{l, l' \geq 1, \\ l+l' \leq 2h+1}} \frac{1}{m^{2(l+l'+2)}} \left(\frac{B_{l+1}}{(l+1)!} \frac{B_{l'+1}}{(l'+1)!} \right)^2 \frac{4K_{\omega}^{l,l'}}{m^{4h+4-2(l+l')}} \\ &= \frac{4}{m^{4h+6}} \left\{ (8h+4) \sum_{l=1}^k \left(\frac{B_{l+1}}{(l+1)!} \right)^2 K_{\omega_1}^l + (8h+4) \sum_{l'=1}^k \left(\frac{B_{l'+1}}{(l'+1)!} \right)^2 K_{\omega_2}^{l'} \right. \\ &\quad \left. + \frac{(8h+4)^2}{m^2} \sum_{\substack{l, l' \geq 1, \\ l+l' \leq 2h+1}} \left(\frac{B_{l+1}}{(l+1)!} \frac{B_{l'+1}}{(l'+1)!} \right)^2 K_{\omega}^{l,l'} \right\}. \end{aligned}$$

From Theorem 1 (ii), we obtain $E((I_m - \tilde{I}_m)^2) = O(m^{-(4h+6)}) = o(m^{-(4h+4)})$. Therefore, by Cauchy-Schwarz inequality,

$$\begin{aligned} E\left((I - I_m)(I_m - \tilde{I}_m)\right) &\leq \sqrt{E\left((I - I_m)^2\right)} \sqrt{E\left((I_m - \tilde{I}_m)^2\right)} \\ &= O(m^{-(2h+2)}) O(m^{-(2h+3)}) = o(m^{-(4h+4)}). \end{aligned}$$

Finally,

$$\begin{aligned} E\left((I - \tilde{I}_m)^2\right) &= E\left((I - I_m)^2\right) + E\left((I_m - \tilde{I}_m)^2\right) + 2E\left((I - I_m)(I_m - \tilde{I}_m)\right) \\ &= E\left((I - I_m)^2\right) + o(m^{-(4h+4)}). \end{aligned}$$

REFERENCES

1. Benhenni K. Approximating integrals of stochastic processes: Extensions. — J. Appl. Probab., 1998, v. 35, № 4, p. 843–855.
2. Benhenni K., Cambanis S. Sampling designs for estimating integrals of stochastic processes. — Ann. Statist., 1992, v. 20, № 1, p. 161–194.
3. Benhenni K., Cambanis S. Sampling designs for estimating integrals of stochastic processes using quadratic mean derivatives. — Approximation Theory. Ed. by G. Anastassiou. New York: Dekker, 1992, p. 93–128.
4. Cressie N. Statistics for Spatial Data. New York: John Wiley, 1991.
5. Eubank R. L., Smith P. L., Smith P. W. A note on optimal and asymptotically optimal designs for certain time series models. — Ann. Statist., 1982, v. 10, № 4, p. 1295–1301.
6. Istas J., Laredo C. Estimating functionals of a stochastic process. — Adv. Appl. Probab., 1997, v. 29, p. 249–270.
7. Journal A. G. New ways of assessing spatial distribution of pollutants. — Environmental Sampling for Hazardous Wastes. Ed. by G. E. Schweitzer, J. A. Santolucito. Washington, D. C.: American Chemical Society, 1984, p. 109–118.
8. Matern B. Spatial Variation (2nd ed.). — Lectures Notes in Statist., 1986, v. 36, Springer, New York.
9. Olea R. A. Sampling design optimization for spatial functions. — Math. Geology, 1984, v. 16, p. 369–392.
10. Ripley B. D. Spatial Statistics. New York: John Wiley, 1981.
11. Ritter K. Asymptotic optimality of regular sequence designs. — Ann. Statist., 1996, v. 24, № 5, p. 2081–2096.
12. Sacks J., Ylvisaker D. Designs for regression problems with correlated errors. — Ann. Math. Statist., 1966, v. 37, p. 68–89.
13. Sacks J., Ylvisaker D. Designs for regression problems with correlated errors. III. — Ann. Math. Statist., 1970, v. 41, p. 2057–2074.
14. Sacks J., Ylvisaker D. Statistical designs and integral approximations. — Proc. Twelfth Biennial Sem. Can. Math. Congr. On Time Series and Stochastic Processes; Convexity and Combinatorics (Vancouver, B.C., 1969), Montreal, Que., 1970, p. 115–136.
15. Stein M. L. Asymptotic properties of centered systematic sampling for predicting integrals of spatial processes. — Ann. Appl. Probab., 1993, v. 3, № 3, p. 874–880.
16. Stein M. L. Predicting integrals of stochastic processes. — Ann. Appl. Probab., 1995, v. 5, p. 158–170.
17. Tubilla A. Error convergence rates for estimates of multidimensional integrals of random functions. — Tech. Rep., 72, Department of Statistics, Stanford University, 1975.
18. Yfantis E. A., Flatman G. T., Baher J. V. Efficiency of kriging estimations for square, triangular and hexagonal grids. — J. Math. Geology, 1987, v. 19, p. 183–205.