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ON BOUNDS FOR MODERATE DEVIATIONS FOR STUDENT'S STATISTIC¹⁾

Пусть $X_1, X_2, \dots$ — независимые случайные величины с нулевыми средними и конечными дисперсиями. В статье доказываются нижние оценки в теореме типа Крамера о больших отклонениях для самонормированных сумм, позволяющие утверждать, что оценки, полученные в [14], точны.

Ключевые слова и фразы: зоны Линника, самонормированная сумма, t -статистика, умеренные отклонения, неравномерные оценки.

1. Introduction and results. Let $X, X_1, X_2, \dots$ denote independent identically distributed (i.i.d.) random variables such that $EX = 0$ and $0 < \sigma^2 = EX^2 < \infty$. Write

$$S_n = \sum_{j=1}^n X_j \quad \text{and} \quad V_n^2 = \sum_{j=1}^n X_j^2.$$

Define the self-normalized sum as S_n/V_n , if $V_n \neq 0$, and 0 if $V_n = 0$.

Let $\Lambda(n)$ denote a nondecreasing function satisfying $\Lambda(n) \rightarrow \infty$ as $n \rightarrow \infty$. The interval $[0, \Lambda(n)]$ is called zone of normal convergence if

$$\frac{1 - \mathbf{P}\{S_n < x\sigma\sqrt{n}\}}{1 - \Phi(x)} \rightarrow 1, \quad \frac{\mathbf{P}\{S_n < -x\sigma\sqrt{n}\}}{\Phi(-x)} \rightarrow 1 \quad (1.1)$$

uniformly in $x \in [0, \Lambda(n)]$, where Φ denotes the standard Gaussian distribution function. It is well known, assuming that the condition $\beta = \mathbf{E}|X|^3 < \infty$ holds, that the Berry-Esseen bound implies (1.1) with $\Lambda(n) = (1 - \varepsilon)\sqrt{\ln n}$, for $0 < \varepsilon < 1$. In this case the zone of normal convergence is of logarithmic size and is rather narrow. Consider the condition

$$\mathbf{E} \exp \left\{ |X_1|^{4\alpha/(2\alpha+1)} \right\} < \infty \quad (1.2)$$

with some α such that $0 < \alpha \leq \frac{1}{6}$. Yu. V. Linnik [15] (see as well [13]) proved the following result. For any function $\rho(n)$ with $\lim_{n \rightarrow \infty} \rho(n) = \infty$ the relation (1.1) holds with $\Lambda(n) = n^\alpha/\rho(n)$. Furthermore, if (1.1) holds with $\Lambda(n) = n^\alpha/\rho(n)$, then the condition (1.2) is fulfilled. This result of Linnik shows that for deterministically normalized sums the existence of exponential moments is a natural condition for the existence of power type zones of normal convergence.

E. Giné, F. Götze and D. Mason [9] noticed that the tails of S_n/V_n are uniformly sub-Gaussian in cases when the sequence S_n/V_n is stochastically bounded. This observation suggests that Linnik type results for self-normalized sums have to hold under conditions considerably weaker than (1.2). Q. M. Shao [21] established that for self-normalized sums Linnik type results hold under the more natural condition of the existence of the third moment $\beta < \infty$.

Let us discuss some results for self-normalized sums related to the CLT and their refinements. Self-normalized sums satisfy the CLT provided that X belongs to the domain of attraction of a normal law ([17], [5]). P. Griffin and D. Mason [10] proved that the condition is necessary in the symmetric case. E. Giné, F. Götze and D. Mason [9] proved that the condition is necessary for nonsymmetric X as well. V. A. Egorov [7] established

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necessary and sufficient conditions in the non-i.i.d. symmetric case. These results indicate that self-normalization may improve the limiting tail behavior of sums.

Berry–Esseen bounds of increasing generality and accuracy were obtained in [11], [24], [23], [1], [2]. The paper of V. Bentkus and F. Götze [1] contains bounds which imply, in particular, sufficient conditions for the CLT. The bounds extend to the non-i.i.d. case, see [2]. They obtained the following Berry–Esseen bound. Let $X_1, X_2, \dots$ be independent random variables such that $\mathbf{E}X_j = 0$, $0 < \sigma_j^2 = \mathbf{E}X_j^2 < \infty$, and $\beta_j = \mathbf{E}|X_j|^3$ for all positive integer j . Write $B_n^2 = \sum_{j=1}^n \sigma_j^2$, $A_n = \sum_{j=1}^n \beta_j$, $L_n = A_n/B_n^3$. Then

$$\left| \mathbf{P} \left\{ \frac{S_n}{V_n} < x \right\} - \Phi(x) \right| \leq c \left(B_n^{-2} \sum_{j=1}^n \mathbf{E}X_j^2 \mathbf{I}\{|X_j| > B_n\} + B_n^{-3} \sum_{j=1}^n \mathbf{E}|X_j|^3 \mathbf{I}\{|X_j| \leq B_n\} \right), \quad x \in \mathbf{R}, \quad (1.3)$$

where c is an absolute constant. Here and in the sequel we denote by $\mathbf{I}\{S\}$ the indicator function of the set S . In the case $L_n < \infty$ (1.3) has the form $|\mathbf{P}\{S_n/V_n < x\} - \Phi(x)| \leq cL_n$, $x \in \mathbf{R}$.

Sub-Gaussian type behavior of tails related to self-normalized sums was observed in [16]. E. Giné, F. Götze and D. Mason [9] showed that the tails of S_n/V_n , $n \geq 1$, is stochastically bounded. Q. M. Shao [20]–[22] considered large and moderate deviations. Q. Wang and B. Y. Jing [25] proved a bound for symmetric nonidentically distributed random variables. Q. Wang and B. Y. Jing [25] derived exponential nonuniform Berry–Esseen bounds for symmetric non-identically distributed random variables and for identically distributed random variables. G. P. Chistyakov and F. Götze [3] improved their results and considered the even more general situation of independent nonidentically distributed random variables. If $X_1, X_2, \dots$ denote symmetric independent random variables with $L_n < \infty$, they proved

$$\left| \mathbf{P} \left\{ \frac{S_n}{V_n} < x \right\} - \Phi(x) \right| \leq c \min \{1, (x+1)^3 L_n\} (1 - \Phi(x)), \quad x \geq 0, \quad (1.4)$$

where c is an absolute constant. Moreover, they established in [4] that (1.4) is a simple consequence of a more general result which shows that the preceding result may hold for wide classes of nonindependent random variables.

Let $X_1, \dots, X_n$ be random variables such that the distributions of the random vectors $(\delta_1 X_1, \dots, \delta_n X_n)$ coincide for all values of δ_j , where $\delta_j = \pm 1$, $j = 1, \dots, n$, and $L_n < \infty$. Then, for all $x \geq 0$,

$$\left| \mathbf{P} \left\{ \frac{S_n}{V_n} < x \right\} - \Phi(x) \right| \leq c \min \left\{ 1, (x^3 + 1) \left(L_n + \mathbf{P} \left\{ V_n^2 < \frac{B_n^2}{2} \right\} \right) \right\} (1 - \Phi(x)),$$

where c is an absolute constant.

Recently B. Y. Jing, Q. Wang, and Q. M. Shao [14] obtained a bound of type (1.4) for general independent random variables. Assume that $X_1, X_2, \dots$ are independent random variables with $\mathbf{E}X_j = 0$ and $0 < \mathbf{E}X_j^2 < \infty$ for all j . It is proved in [14] that there is an absolute constant c_1 ($c_1 > 1$) such that

$$\frac{\mathbf{P}\{S_n \geq xV_n\}}{1 - \Phi(x)} = e^{O(1)\Delta_{n,x}} \quad \text{and} \quad \frac{\mathbf{P}\{S_n \leq -xV_n\}}{\Phi(-x)} = e^{O(1)\Delta_{n,x}} \quad (1.5)$$

for all $x \geq 0$ satisfying

$$x^2 \max_{1 \leq j \leq n} \mathbf{E}X_j^2 \leq B_n^2 \quad (1.6)$$

and

$$\Delta_{n,x} \leq c_1^{-1} (1+x)^2, \quad (1.7)$$

where, for $x \geq 0$,

$$\Delta_{n,x} = \frac{(1+x)^2}{B_n^2} \sum_{j=1}^n \mathbf{E}X_j^2 \mathbf{I}\left\{|X_j| > \frac{B_n}{1+x}\right\} + \frac{(1+x)^3}{B_n^3} \sum_{j=1}^n \mathbf{E}|X_j|^3 \mathbf{I}\left\{|X_j| \leq \frac{B_n}{1+x}\right\}$$

and $|O(1)| \leq c$.

In the following we shall denote by θ different real-valued quantities such that $|\theta| \leq 1$. The notation $c, c_2, c_3, \dots$ will be used throughout for absolute positive constants. By c we denote different constants in different (or even in the same) formulae. The symbols $c_2, c_3, \dots$ are used for specific constants.

In this note we shall prove that the bound (1.5) is best possible. To prove this statement we need the following Proposition 1.

Proposition 1. *Let $\{\beta_n\}_{n=1}^\infty$ be a sequence of real numbers such that $\beta_n \geq e$ and $\beta_n/\sqrt{n} \rightarrow 0$ as $n \rightarrow \infty$. Then there exist i.i.d. random variables $X = X(n), X_1, \dots, X_n$ and absolute positive constants c, c_2 such that $\mathbf{E}X = 0$, $\mathbf{E}X^2 = 1$, $\mathbf{E}|X|^3 = (1 + \theta/n)5\beta_n/3$, and*

$$\frac{\mathbf{P}\{S_n/V_n > x\}}{1 - \Phi(x)} \leq \exp\left\{-\frac{c\beta_n x^3}{\sqrt{n}}\right\}, \quad 1 \leq x \leq \frac{c_2\sqrt{n}}{\beta_n}, \quad (1.8)$$

$$\frac{\mathbf{P}\{S_n/V_n < -x\}}{\Phi(-x)} \geq \exp\left\{\frac{c\beta_n x^3}{\sqrt{n}}\right\}, \quad 1 \leq x \leq \frac{c_2\sqrt{n}}{\beta_n}, \quad (1.9)$$

$$\mathbf{P}\left\{\frac{S_n}{V_n} > x\right\} \geq \exp\left\{-\frac{cn}{\beta_n^2}\right\}, \quad \frac{c_2\sqrt{n}}{\beta_n} < x \leq \sqrt{n}, \quad (1.10)$$

for sufficiently large $\sqrt{n}/\beta_n \geq c$. In addition, for these random variables $X_1, \dots, X_n$, the following bounds hold

$$\frac{1}{2} \frac{\beta_n(1+x)^3}{\sqrt{n}} \leq \Delta_{n,x} \leq 2 \frac{\beta_n(1+x)^3}{\sqrt{n}}, \quad 0 \leq x \leq \frac{\sqrt{n}}{\beta_n} - 1, \quad (1.11)$$

and

$$\Delta_{n,x} = \left(1 + \frac{\theta}{n}\right)(1+x)^2, \quad \frac{\sqrt{n}}{\beta_n} - 1 \leq x \leq \sqrt{n}. \quad (1.12)$$

We see that for $x \in [0, \sqrt{n}]$ (1.6) obviously holds for the random variables $X_1, \dots, X_n$ of Proposition 1. By (1.11) and (1.12), (1.7) is satisfied for $0 \leq x \leq c\sqrt{n}/\beta_n$ with $c < 1$. Moreover, by (1.8)–(1.12), we obtain for $X_1, \dots, X_n$

$$\frac{\mathbf{P}\{S_n \geq xV_n\}}{1 - \Phi(x)} \leq e^{-c\Delta_{n,x}}, \quad 1 \leq x \leq \frac{c_2\sqrt{n}}{\beta_n},$$

$$\frac{\mathbf{P}\{S_n \geq xV_n\}}{1 - \Phi(x)} \geq e^{c\Delta_{n,x} - cn/\beta_n^2}, \quad \frac{c_2\sqrt{n}}{\beta_n} < x \leq \sqrt{n},$$

and

$$\frac{\mathbf{P}\{S_n \leq -xV_n\}}{\Phi(-x)} \geq e^{c\Delta_{n,x}}, \quad 1 \leq x \leq \frac{c_2\sqrt{n}}{\beta_n}.$$

These bounds show that the result of [14] is sharp.

2. Proof of Proposition 1. Let $\{\beta_n\}_{n=1}^\infty$ denote a sequence of real numbers such that $\beta_n \geq e$ and $\beta_n/\sqrt{n} \rightarrow 0$ as $n \rightarrow \infty$. Consider i.i.d. random variables $X, X_1, \dots, X_n$ such that

$$\mathbf{P}\{X = -x_1\} = q, \quad \mathbf{P}\{X = \varepsilon\} = 1 - (2\beta_n^2)^{-1}, \quad \mathbf{P}\{X = 2x_1\} = p,$$

where

$$x_1 := \frac{\varepsilon_1}{4} + \frac{1}{2} \sqrt{\frac{\varepsilon_1^2}{4} + 4\beta_n^2 \left(1 - \varepsilon^2 \left(1 - \frac{1}{2\beta_n^2}\right)\right)},$$

$$p := \frac{1}{6\beta_n^2} \left(1 - \frac{\varepsilon_1}{x_1}\right), \quad q := \frac{1}{6\beta_n^2} \left(2 + \frac{\varepsilon_1}{x_1}\right), \quad \varepsilon_1 := (2\beta_n^2 - 1)\varepsilon.$$

Here the parameter ε depends on n and is a sufficiently small positive number. We see from the definition of x_1 that $x_1 = \beta_n(1 + c|\theta|\sqrt{\varepsilon})$. It is easy to see that $\mathbf{E}X = 0$, $\mathbf{E}X^2 = 1$, and $\mathbf{E}|X|^3 = (1 + c\theta\sqrt{\varepsilon})5\beta_n/3$. Denote $L_n := \mathbf{E}|X|^3/\sqrt{n}$. We assume that L_n is sufficiently small, i.e., $L_n^{-1} \geq c$.

In the first step let us prove the relations

$$\mathbf{P}\left\{\frac{S_n}{V_n} > x\right\} \leq (1 - cL_n x^3)(1 - \Phi(x)), \quad 1 \leq x \leq c_3 L_n^{-1/3}, \quad (2.1)$$

$$\mathbf{P}\left\{\frac{S_n}{V_n} > x\right\} \leq (1 - \Phi(x))e^{-cL_n x^3}, \quad c_3 L_n^{-1/3} \leq x \leq 2c_2 L_n^{-1}, \quad (2.2)$$

where c_3 is a sufficiently small positive absolute number.

Let $x \in [1, 2c_2/L_n]$, where c_2 is a sufficiently small positive number. Consider the event $A_x = \{S_n/V_n > x\}$ and the event that precisely j of $X_1, \dots, X_n$ assume the value ε , that is, $H_j = \{\mathbf{I}\{X_1 = \varepsilon\} + \dots + \mathbf{I}\{X_n = \varepsilon\} = j\}$. Then

$$\mathbf{P}(A_x) = \sum_{j=0}^n \mathbf{P}\{A_x | H_j\} \mathbf{P}(H_j). \quad (2.3)$$

By the independence of $X_1, \dots, X_n$, we note that

$$\mathbf{P}(H_j) = \binom{n}{j} \left(1 - \frac{1}{2\beta_n^2}\right)^j \left(\frac{1}{2\beta_n^2}\right)^{n-j}, \quad j = 0, \dots, n. \quad (2.4)$$

Consider j such that $j_1 \leq j \leq j_2$, where $j_1 := (1 - p - q - q/100)n$ and $j_2 := (1 - p - q + q/100)n$. By means of an elementary computation, we obtain

$$\mathbf{P}(A_x | H_j) = \sum_{m \in B_{x,j}} \binom{n-j}{m} p_1^m q_1^{n-j-m}, \quad (2.5)$$

where $p_1 = q/(p+q) = 2(1 + \theta\sqrt{\varepsilon})/3$, $q_1 = p/(p+q) = (1 + \theta\sqrt{\varepsilon})/3$, and $B_{x,j} := \{m \in \mathbf{Z}_+ : m \leq n-j, a-3m > x\sqrt{b-3m}\}$, where $a := 2(n-j) + j\varepsilon/x_1$ and $b := 4(n-j) + j\varepsilon^2/x_1^2$. It is easy to see that $B_{x,j} = \{m \in \mathbf{Z}_+ : m \leq (a-r)/3\}$, where $r = (x^2 + x\sqrt{x^2 + 4(b-a)})/2$. Since $r = (b-a)^{1/2}x + x^2/2 + \theta x^3/(b-a)^{1/2}$, we finally have, for sufficiently small $\varepsilon > 0$,

$$B_{x,j} = \left\{m \in \mathbf{Z}_+ : m \leq \frac{2(n-j)}{3} - \frac{\sqrt{2(n-j)}}{3}x'\right\},$$

where $x' := x + x^2/(2\sqrt{2(n-j)}) + \theta x^3/(n-j)$.

In the case $x \in [-2c_2/L_n, -1]$ the event $A_x := \{S_n/V_n < x\}$ and the formulae (2.3) and (2.5) hold with $B_{x,j} := \{m \in \mathbf{Z}_+ : m \leq n-j, a-3m < x\sqrt{b-3m}\}$. It is easy to see that $B_{x,j} = \{m \in \mathbf{Z}_+ : 2(n-j)/3 + \sqrt{2(n-j)}x''/3 \leq m \leq n-j\}$, where $x'' := |x| - x^2/(2\sqrt{2(n-j)}) + \theta x^3/(n-j)$.

Let $S_{n'}$ be the number of successes in n' Bernoulli trials with probability $p_2 = \frac{2}{3}$ for success. The following result is well known (see [8], [18]):

$$\begin{aligned} \mathbf{P}\left\{S_{n'} < \frac{2n'}{3} - \frac{\sqrt{2n'}y}{3}\right\} \\ = (1 - \Phi(y)) \exp\left\{\left(1 + \frac{\theta y}{100\sqrt{n'}}\right) \frac{y^3}{6\sqrt{2n'}}\right\} \left(1 + \theta \frac{y+1}{100\sqrt{n'}}\right), \end{aligned} \quad (2.6)$$

$$\begin{aligned} \mathbf{P}\left\{S_{n'} > \frac{2n'}{3} + \frac{\sqrt{2n'}y}{3}\right\} \\ = (1 - \Phi(y)) \exp\left\{-\left(1 + \frac{\theta y}{100\sqrt{n'}}\right) \frac{y^3}{6\sqrt{2n'}}\right\} \left(1 + \theta \frac{y+1}{100\sqrt{n'}}\right) \end{aligned} \quad (2.7)$$

for all $y \in [0, c_4\sqrt{n'}]$ and $n' \geq c$.

Assuming $x \in [1, c_3 L_n^{-1/3}]$, let us prove (2.1). Using (2.6) with $n' = n-j$ and $y = x'$ and Taylor's formula, we easily obtain

$$\begin{aligned} \mathbf{P}(A_x | H_j) &= \Phi(-x) + \frac{x^3}{6\sqrt{2(n-j)}} \Phi(-x) - \frac{x^2}{2\sqrt{2(n-j)}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \\ &\quad + \theta \frac{x+1}{100\sqrt{n-j}} \Phi(-x) + c\theta \frac{x^5}{n-j} e^{-x^2(1+\theta x/\sqrt{n-j})^2/2} + \theta\eta \end{aligned}$$

(here and in the sequel we denote by $\eta = \eta(\varepsilon)$ a positive quantity which depend on ε and tends to zero as $\varepsilon \rightarrow 0$). Recalling the well-known inequality

$$(x^{-1} - x^{-3})(2\pi)^{-1/2} \exp \left\{ -\frac{x^2}{2} \right\} < \Phi(-x) < x^{-1}(2\pi)^{-1/2} \exp \left\{ -\frac{x^2}{2} \right\} \quad (2.8)$$

for every $x > 0$, we finally get

$$\mathbf{P}(A_x | H_j) - \Phi(-x) < -\frac{x^3}{4\sqrt{2(n-j)}} \Phi(-x) + \eta, \quad 1 \leq x \leq c_3 L_n^{-1/3}, \quad (2.9)$$

for all $j_1 \leq j \leq j_2$.

Now let us estimate $\sigma_1 := \sum_{j=0}^{j_1} \mathbf{P}(H_j)$ and $\sigma_2 := \sum_{j=j_2}^n \mathbf{P}(H_j)$. By a well-known inequality (see [8, v. 1, Ch. 6, Section 3]), we have

$$\sigma_1 \leq c \binom{n}{j_1} \left(1 - \frac{1}{2\beta_n^2}\right)^{j_1} \left(\frac{1}{2\beta_n^2}\right)^{n-j_1}.$$

Hence, using Stirling's formula, we get, by means of a simple calculation,

$$\binom{n}{j_1} \left(1 - \frac{1}{2\beta_n^2}\right)^{j_1} \left(\frac{1}{2\beta_n^2}\right)^{n-j_1} \leq e^{-c/L_n^2}.$$

The two last bounds yield

$$\sigma_1 \leq e^{-c/L_n^2}. \quad (2.10)$$

The same bound holds for σ_2 .

We now return to formula (2.3). Using (2.9) and (2.10), we deduce from (2.3) the following relation:

$$\begin{aligned} \mathbf{P}(A_x) - \Phi(-x) &= \sum_{j=0}^n (\mathbf{P}(A_x | H_j) - \Phi(-x)) \mathbf{P}(H_j) < -x^3 \Phi(-x) \sum_{j=j_1+1}^{j_2-1} \frac{\mathbf{P}(H_j)}{4\sqrt{2(n-j)}} \\ &\quad + e^{-c/L_n^2} + \eta < -\frac{x^3}{4\sqrt{2(n-j_1)}} \Phi(-x) \sum_{j=j_1+1}^{j_2-1} \mathbf{P}(H_j) + e^{-c/L_n^2} + \eta \\ &< -\frac{1}{16} L_n x^3 \Phi(-x) + e^{-c/L_n^2} + \eta, \quad 1 \leq x \leq c_3 L_n^{-1/3}. \end{aligned} \quad (2.11)$$

For sufficiently small ε it follows from (2.11) $\mathbf{P}(A_x) - \Phi(-x) \leq -L_n x^3 \Phi(-x)/32$, $1 \leq x \leq c_3 L_n^{-1/3}$. As an obvious consequence of this inequality we obtain the assertion (2.1).

Let $x \in [c_3 L_n^{-1/3}, 2c_2 L_n^{-1}]$. Let us prove (2.2). We deduce from (2.5) and (2.6) that, for $j_1 \leq j \leq j_2$,

$$\mathbf{P}(A_x | H_j) \leq \left(1 + \frac{x' + 1}{100\sqrt{n-j}}\right) \Phi(-x') e^{x'^3/(5\sqrt{2(n-j)})} + \eta. \quad (2.12)$$

Noting that, for $x \in [2, 2c_2 L_n^{-1}]$ with sufficiently small c_2 and $j_1 \leq j \leq j_2$,

$$\begin{aligned} \frac{1}{2} (x^2 - x'^2) &= \frac{1}{2} (x - x') \cdot (x + x') = -\frac{x^2}{2\sqrt{2(n-j)}} \left(1 + \frac{2\sqrt{2}\theta x}{\sqrt{n-j}}\right) \\ &\quad \times 2x \left(1 + \frac{x}{4\sqrt{2(n-j)}} + \frac{\theta x^2}{2(n-j)}\right) \leq -\frac{x^3}{3\sqrt{2(n-j)}}, \end{aligned}$$

we get, using (2.8), $\Phi(-x')/\Phi(-x) \leq (1-x^{-2})^{-1} \exp\{-x^3/(3\sqrt{2(n-j)})\}$. Then we obtain from (2.12), recalling the definition of x' and choosing $\varepsilon > 0$ to be sufficiently small,

$$\mathbf{P}(A_x | H_j) \leq \left(1 + \frac{2x}{100\sqrt{n-j}} + \frac{2}{x^2}\right) \Phi(-x) e^{-x^3/(15\sqrt{2(n-j)})},$$

$x \in [c_3 L_n^{-1/3}, 2c_2 L_n^{-1}]$. Combining this estimate with (2.10) for σ_k , $k = 1, 2$, we easily obtain from (2.3)

$$\begin{aligned} P(A_x) &\leq \left(1 + \frac{2x}{100\sqrt{n-j_2}} + \frac{2}{x^2}\right) \Phi(-x) e^{-x^3/(15\sqrt{2(n-j_1)})} \sum_{j=j_1+1}^{j_2-1} P(H_j) + e^{-c/L_n^2} \\ &\leq \Phi(-x) e^{-cL_n x^3} \sum_{j=j_1+1}^{j_2-1} P(H_j) + e^{-c/L_n^2} \leq \Phi(-x) e^{-cL_n x^3} \end{aligned}$$

for $c_3 L_n^{-1/3} \leq x \leq 2c_2 L_n^{-1}$, thus proving (2.2).

The assertion (1.8) of Proposition 1 immediately follows from (2.1) and (2.2). In a similar way we prove the bound (1.9).

The assertion (1.10) is proved using the simple chain of inequalities

$$\begin{aligned} P\{S_n V_n^{-1} > x\} &\geq P\{S_n V_n^{-1} = \sqrt{n}\} \geq P\{X_1 = \varepsilon, \dots, X_n = \varepsilon\} \\ &= \left(1 - \frac{1}{2\beta_n^2}\right)^n \geq e^{-n/\beta_n^2}, \quad \frac{c_2 \sqrt{n}}{\beta_n} \leq x \leq \sqrt{n}. \end{aligned}$$

Finally the assertions (1.11) and (1.12) can be verified by a simple calculation, which proves the proposition.

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CONVERGENCE OF THE POINCARÉ CONSTANT

Постоянная Пуанкаре R_Y для случайной величины Y связывает $L^2(Y)$ -норму функции g с ее производной g' . Поскольку разность $R_Y - D_Y$ неотрицательна и равна нулю, тогда и только тогда, когда Y — нормальная случайная величина, то эту разность можно рассматривать как расстояние до нормального распределения. В статье устанавливается наилучшая возможная скорость сходимости этого расстояния в центральной предельной теореме. Кроме того, мы показываем, что R_Y конечна для дискретных смесей нормально распределенных величин, что позволяет получить скорость сходимости в центральной предельной теореме в смысле относительной энтропии.

Ключевые слова и фразы: постоянная Пуанкаре, спектральная щель, центральная предельная теорема, информация по Фишеру.

1. Introduction and results. Poincaré (or spectral gap) inequalities provide a relation between L^2 -norms on functions and their derivatives.

Definition 1.1 ([3]). Given a random variable Y , define the Poincaré constant R_Y :

$$R_Y = \sup_{g \in H_1(Y)} \frac{Dg(Y)}{Eg'(Y)^2},$$

where $H_1(Y)$ is the space of absolutely continuous functions on the real line such that $Dg(Y) > 0$ and $Eg'(Y)^2 < \infty$.

R_Y will not in general be finite, however it will be finite for the normal and other strongly unimodal distributions (see, for example, [10], [8], [6], [5], [11], [3]).

We will exploit various relationships between the Poincaré constant and Fisher information.

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