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OPERATORS OF PROBABILISTIC TYPE

В данной статье мы изучаем вероятностные операторы $L_n(f; x) = Ef(S_n/n)$, где S_n — сумма n независимых и одинаково распределенных случайных величин со средним x . Мы приводим условия, при которых $L_n(f; x)$ сходится к $f(x)$, и получаем точные скорости сходимости. Примеры показывают, что предлагаемый нами подход делает многие классические результаты более прозрачными и легкими для доказательства. Обсуждается также устойчивость L_n .

Ключевые слова и фразы: операторы, предельные теоремы, скорость сходимости.

1. Introduction. Let f be a function defined on $\mathbb{R}^+$. The Szász–Mirakyan operator is defined as

$$S_n(f; x) = e^{-nx} \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \frac{(nx)^k}{k!}, \quad x > 0, \quad n \in \mathbb{N}.$$

This operator has the following probabilistic interpretation. For $x > 0$, let $X_1, X_2, \dots, X_n$ be independent random variables (r.v.'s) all having the same Poisson(x)-distribution. Then $S_n = X_1 + \dots + X_n$ has a Poisson(nx)-distribution and

$$S_n(f; x) = Ef\left(\frac{S_n}{n}\right). \quad (1.1)$$

This interpretation allows one to obtain most classical results about $S_n(f; x)$, see e.g. [20]. Replacing the Poisson distribution by other distributions, it is clear that many linear operators are of the form (1.1), see [22], [21] and Section 3 below.

In Section 2 below we investigate convergence and rates of convergence of operators of the type (1.1) in $\mathbb{R}$ and in $\mathbb{R}^n$; in Section 3 we show that many results are covered by this general approach. Section 4 is devoted to the stability of the operators.

2. Operators of probabilistic type. Let $X, X_1, X_2, \dots, X_n$ be independent r.v.'s having the same distribution and let $E(X) = x$, $\text{var}(x) = \sigma^2$, $S_n = X_1 + X_2 + \dots + X_n$. For a suitable class of measurable real functions C , a linear operator $L_n: C \rightarrow \mathbb{R}$ can be defined as follows:

$$L_n: f \longrightarrow L_n(f; x) = Ef\left(\frac{S_n}{n}\right).$$

Since the actual construction of L_n often requires infinite sums (cf. (1.1)), which restricts their usefulness from the computational point of view, we shall also consider operators $L_{n,N}$ of the form

$$L_{n,N}(f; x) = E\left(f\left(\frac{S_n}{n}\right) I_{\{S_n \leq N\}}\right),$$

where, as usual, I_A denotes the indicator function, and N is a positive number which may depend on n and x .

In the sequel we investigate some questions related to the asymptotic behavior of L_n and $L_{n,N}$ as n tends to infinity.

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We shall often use the following classes of functions: $C(\mathbf{R})$ = the class of continuous and bounded real functions; $E(\mathbf{R})$ = the class of real, continuous and exponentially bounded functions.

When we use functions f in $E(\mathbf{R})$, we shall also assume that the r.v. X satisfies

$$M(t) = E(e^{t|X|}) < \infty, \quad \text{for } t \leq d, d > 0. \quad (2.1)$$

It is well known that (2.1) implies that $E|X|^\alpha$ is finite for all α , and that (see e.g. [21, p. 273] or [17, p. 279])

$$\limsup_{n \rightarrow \infty} n^{\alpha/2} E \left| \frac{S_n}{n} - x \right|^\alpha = \limsup_{n \rightarrow \infty} n^{\alpha/2} L_n(|t - x|^\alpha; x) < \infty. \quad (2.2)$$

In our first result we consider convergence of L_n . In the sequel we shall always consider $\lim$ or $\limsup$ for n tending to infinity.

Theorem 1. (i) If $f \in C(\mathbf{R})$, then

$$\lim L_n(f; x) = f(x), \quad (2.3)$$

$$\lim E \left| f\left(\frac{S_n}{n}\right) - f(x) \right| = 0. \quad (2.4)$$

(ii) If $f \in E(\mathbf{R})$ and if (2.1) holds, then (2.3) and (2.4) hold.

(iii) If N is such that $\lim(N - nx)/\sqrt{n}\sigma = C$, $C < \infty$, then under the conditions of (i) or (ii), we have

$$\lim L_{n,N}(f; x) = f(x) P\{Z \leq C\}, \quad (2.5)$$

where Z is a $N(0, 1)$ -distributed r.v.

P r o o f. (i) If f is continuous, expressions (2.3) and (2.4) are well-known, see e.g. [8, Ch. VII].

(ii) Since f is exponentially bounded, we have

$$|f(x)| \leq A \exp(m|x|), \quad x \in \mathbf{R}, \quad m > 0. \quad (2.6)$$

Using (2.1) it follows that $|f(S_n/n)| \leq A \exp(m \sum |X_i|/n)$. Hence $L_n(f; x)$ exists for all n sufficiently large. It follows that $|f(S_n/n)|$ is uniform integrable and hence (2.3) and (2.4) follows, see e.g. [2, Theorem 5.4]. See also Theorem 1 in [16].

(iii) To prove (iii) note that

$$\begin{aligned} |L_{n,N}(f; x) - f(x) P\{S_n \leq N\}| &= \left| E \left(f\left(\frac{S_n}{n}\right) - f(x) \right) I_{\{S_n \leq N\}} \right| \\ &\leq E \left| f\left(\frac{S_n}{n}\right) - f(x) \right|. \end{aligned}$$

Now use part (i) or (ii) to see that $\lim L_{n,N}(f; x) = f(x) \lim P\{S_n \leq N\}$. The result now follows from the central limit theorem. Theorem 1 is proved.

In our second result we discuss uniform convergence in (2.3). To state the result we define the modulus of continuity of a function as usual:

$$w(f; d) = \sup \left\{ |f(u) - f(v)|, \forall u, v \text{ such that } |u - v| \leq d \right\}.$$

It is well known that for fixed d and all x, y there holds

$$|f(x) - f(y)| \leq \left(1 + \frac{|x - y|}{d}\right) w(f; d). \quad (2.7)$$

In the next proposition we show that uniform boundedness of $\text{var}(X)$ yields uniform convergence in (2.3).

Proposition 1. *If $\text{var}(X) \leq K$ for $a \leq x \leq b$, then uniformly in $a \leq x \leq b$ we have*

$$|L_n(f; x) - f(x)| \leq (1 + K) w(f; n^{-1/2}).$$

P r o o f. Using (2.7) we have

$$\begin{aligned} |L_n(f; x)| &\leq \mathbb{E} \left(\left| f\left(\frac{S_n}{n}\right) - f(x) \right| \right) \leq \left(1 + \frac{\mathbb{E}|S_n/n - x|}{d} \right) w(f; d) \\ &\leq \left(1 + \frac{\sqrt{\text{var}(S_n/n)}}{d} \right) w(f; d) = \left(1 + \frac{\sigma}{\sqrt{nd}} \right) w(f; d). \end{aligned}$$

Taking $d = n^{-1/2}$ yields the result. Proposition 1 is proved.

Corollary. *If f is Hölder continuous (i.e., $w(f; d) \leq kd^\alpha$), then*

$$|L_n(f; x) - f(x)| \leq (1 + \sigma) kn^{-\alpha/2}.$$

Proposition 1 and its proof can easily be modified to obtain the following multidimensional result.

Proposition 2. *Suppose $X_1, \dots, X_n$ are i.i.d. n -dimensional random vectors with mean x (componentwise) and finite variance. Suppose $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is such that $w(f; d)$ is finite. Then $|L_n(f; x) - f(x)| \leq (1 + K) w(f; n^{-1/2})$, where K is a constant and L_n is defined as before.*

If, for example, X_1 is multinomial, then the operator L_n is the multivariate Bernstein operator, see e.g. [8, VII.9] or [24].

Our next result is devoted to the rate of convergence in (2.3). The result is related to those of Khan [13].

Theorem 2. *Assume $f^{(r)}(x)$, the r th derivative of f , is Hölder continuous, i.e., $f^{(r)}(x)$ satisfies a condition of the form*

$$|f^{(r)}(x) - f^{(r)}(y)| \leq K|x - y|^\alpha. \quad (2.8)$$

Also assume that (2.1) holds. Then

$$\limsup n^{(r+\alpha)/2} \left| L_n(f; x) - \sum_{j=0}^r f^{(j)}(x) \frac{\mathbb{E}(S_n/n - x)^j}{j!} \right| < \infty. \quad (2.9)$$

Moreover, if $\alpha = 1$ and $f^{(r+1)}(x)$ exists, then

$$\lim \frac{n^{(r+1)/2}}{\sigma^{r+1}} \left(L_n(f; x) - \sum_{j=0}^r f^{(j)}(x) \frac{\mathbb{E}(S_n/n - x)^j}{j!} \right) = f^{(r+1)}(x) \mathbb{E}(Z^{r+1}), \quad (2.10)$$

where Z is $\mathcal{N}(0, 1)$ -distributed.

P r o o f. From (2.8) and the Taylor expansion of $f(y)$ around x we obtain

$$\left| f(y) - \sum_{j=0}^r f^{(j)}(x) \frac{(y-x)^j}{j!} \right| \leq K|y-x|^{r+\alpha} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+r)}, \quad (2.11)$$

where $\Gamma(\cdot)$ denotes the Gamma-function.

Replacing y by S_n/n it follows that

$$\mathbb{E} \left| f\left(\frac{S_n}{n}\right) - \sum_{j=0}^r f^j(x) \frac{(S_n/n - x)^j}{j!} \right| \leq K \mathbb{E} \left| \frac{S_n}{n} - x \right|^{r+\alpha} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+r)}. \quad (2.12)$$

Using (2.2) and (2.12) we obtain (2.9).

Now we prove (2.10). Since S_n/n converges to x a.s. and $f^{(r+1)}(x)$ exists, we have

$$Z_n := \frac{f(S_n/n) - \sum_{j=0}^r f^j(x) (S_n/n - x)^j / j!}{(S_n/n - x)^{r+1}} - f^{r+1}(x) \rightarrow 0 \quad \text{a.s.} \quad (2.13)$$

Now we write

$$f\left(\frac{S_n}{n}\right) - \sum_{j=0}^r f^j(x) \frac{(S_n/n - x)^j}{j!} = (Z_n + f^{r+1}(x)) \left(\frac{S_n}{n} - x\right)^{r+1}.$$

Using (2.13) and the central limit theorem, we obtain

$$\frac{n^{(r+1)/2}}{\sigma^{r+1}} \left(f\left(\frac{S_n}{n}\right) - \sum_{j=0}^r f^j(x) \frac{(S_n/n - x)^j}{j!} \right) \Rightarrow f^{r+1}(x) Z^{r+1}. \quad (2.14)$$

Using (2.12) and (2.2) we may and do apply the Lebesgue theorem on dominated convergence and (2.10) follows from (2.14) see [2, Theorem 5.4]. Theorem 2 is proved.

R e m a r k. If $r = 0$, then (2.9) and (2.10) reduce to

$$\limsup n^{\alpha/2} |L_n(f; x) - f(x)| < \infty \quad (2.15)$$

and

$$\lim n^{\alpha/2} (L_n(f; x) - f(x)) = f'(x) \mathbb{E}(Z) = 0.$$

The example $f(y) = |y - x|$ shows that (2.15) is best possible, see also [5, p. 229].

As to the operator $L_{n,N}$ one can combine Theorem 2 and the following proposition to obtain also rates of convergence.

Proposition 3. If $f \in E(\mathbb{R})$ and if N is such that

$$y := \frac{N - nx}{n^{1/2}} \rightarrow \infty,$$

then

$$\limsup e^y |L_{n,N}(f; x) - L_n(f; x)| < \infty.$$

P r o o f. Let A denote the event $\{S_n > N\}$; using (2.1) and (2.6) we obtain

$$|L_{n,N}(f; x) - L_n(f; x)| \leq \mathbb{E} \left| f\left(\frac{S_n}{n}\right) I_A \right| \leq \text{const.} \sqrt{\mathbb{P}\{A\}} M^n \left(\frac{2m}{n}\right). \quad (2.16)$$

Now we choose n so that $y > 0$ and we take an arbitrary $t > 0$. Writing $W_n = (S_n - nx)/n^{1/2}$ we have

$$\mathbb{P}\{A\} = \mathbb{P}\{W_n > y\} \leq \exp(-ty) \mathbb{E}(\exp(tW_n)).$$

Since W_n converges in distribution to $Z \sim \mathcal{N}(0, 1)$, we obtain that $\mathbb{P}\{A\} = O(\exp(-ty))$. Combining this estimate with (2.16) completes the proof of the result.

3. Examples. Depending on the choice of the r.v. X our results can be applied to a variety of classical operators of approximation theory.

E x a m p l e 3.1. If X is $\text{Poisson}(x)$, then L_n is the Szász–Mirakyan operator and $L_{n,N}$ is the modified Szász–Mirakyan operator. In the special case where $n = o(N)$, Theorem 2.1 (ii) is due to Grof [9]. In the case where $N = [n(x + d(n))]$ with $n^{1/2}d(n) \rightarrow \infty$, the result is due to Lehnhof [17]. We also refer to related results in [1], [3], [6], [12], [21].

E x a m p l e 3.2. If X has a Bernoulli(x) distribution, then L_n is the well-known Bernstein operator. Rates of convergence are given e.g. in [5].

E x a m p l e 3.3. A Kantorovic type of operator can be defined as follows. Let f denote an integrable function and define

$$K_n(f; x) = \mathbf{E} \left(\int_0^1 f \left(\frac{S_n + u}{n+1} \right) du \right).$$

If S_n is binomial, this is the operator of Kantorovic (see [14] or [11]). Using (2.11) and the method of proof of Proposition 1 yields

$$|K_n(f; x) - f(x)| \leq (1 + K) w(f; n^{-1/2}).$$

A rate of convergence result (in the binomial case) is given in [11].

E x a m p l e 3.4. If X has a Gamma-distribution, we get for L_n the Gamma-operator of [8, p. 220] which is useful in proving the inversion theorem for the Laplace transform of a function.

E x a m p l e 3.5. The following modified Szász–Mirakyan operator was proposed by Kasana et al. [15] and was studied among others by Gupta and Agrawal [10]:

$$S_n^*(f; x) = n \sum_{k=0}^{\infty} p_k(nx) \int_0^{\infty} p_k(nt) f(t) dt,$$

where $p_k(u) = (u^k/k!) e^{-u}$.

It is not hard to show that this modified operator can be treated as before by noting that

$$S_n^*(f; x) = \mathbf{E} \left(f \left(\frac{S_N}{n} \right) \right), \quad (3.1)$$

where N is $\text{Poisson}(nx)$ and S_N is a random sum of exponential(1) random variables. Since N is equal in distribution to a sum of n independent $\text{Poisson}(x)$ r.v.'s, Theorem 2.1 applies.

R e m a r k. Replacing the $\text{Poisson}(x)$ distribution by other distributions one can obtain a whole variety of modified operators of the type (3.1).

E x a m p l e 3.6. Beta-type operators were introduced by Lupas [18] and further modified and studied among others by Upreti [23], Divis [7] and others. These authors define the Beta-operator as follows:

$$B_n(f; x) = \int_0^1 \frac{t^{nx-1}(1-t)^{n(1-x)-1}}{B(nx, n(1-x))} f(t) dt,$$

where $B(\cdot, \cdot)$ is the Beta-function.

To see this operator is part of our framework, note that $B_n(f; x) = \mathbf{E}(f(Z_n))$, where Z_n is $\text{Beta}(nx, n(1-x))$ -distributed.

It is well known that any $\text{Beta}(u, v)$ -r.v. W can be represented as $W = g(X, Y)$, where $g(x, y) = x/(x + y)$ and $X \sim \text{Gamma}(u, 1)$, $Y \sim \text{Gamma}(v, 1)$ are independent r.v.'s.

It follows that Z_n is equal in distribution to $g(S_n/n, S'_n/n)$, where the sums are sums of independent and i.i.d. Gamma($x, 1$), resp. Gamma($1 - x, 1$) r.v.'s.

Hence $B_n(f; x) = E(f(Z_n)) = E(f(g(S_n/n, S'_n/n)))$.

An application of Proposition 2 gives

$$B_n(f; x) - f(g(x, 1 - x)) = B_n(f; x) - f(x) = O(1) w(f(g); n^{-1/2}).$$

Example 3.7. The next operator we study is the Bernstein–Durmeier operator defined as follows (see e.g. [25], [26]):

$$D_n(f; x) = (n+1) \sum_{k=0}^n p_{n,k}(x) \int_0^1 p_{n,k}(t) f(t) dt,$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$.

Obviously $(n+1)p_{n,k}(t)$ is the density of a Beta($k+1, n-k+1$) r.v. $Z_{n,k}$. As in the previous example we see that $Z_{n,k}$ is equal in distribution to $g(S_{k+1}/n, S'_{n-k+1}/n)$, where the sums are sums of independent and i.i.d. Gamma(1,1) r.v.'s. It follows that

$$D_n(f; x) = E\left(f\left(g\left(\frac{S_{N+1}}{n}, \frac{S'_{n-N+1}}{n}\right)\right)\right),$$

where N is Bin(n, x). For $0 < a \leq x \leq b < 1$ we obtain from Proposition 2 that

$$D_n(f; x) - f(g(x, 1-x)) = D_n(f; x) - f(x) = O(1) w(f(g); n^{-1/2}).$$

4. Stability. In order to discuss the stability of the operators L_n we analyse the consequences of a «small» disturbance in the r.v. X . To formulate our results we use the following semi-metric: for a fixed function f and r.v.'s U and V , we define

$$\epsilon_f(U, V) = \sup_{y \in \mathbb{R}} \left| E(f(U+y) - f(V+y)) \right|.$$

Semi-metrics of this type where f is not fixed have been analysed by several authors, see e.g. [27], [28], [19]. Obviously $\epsilon_f(U, V)$ has the following properties, summarized in the lemma.

Lemma 1. (i) If Z is independent of (U, V) , then $\epsilon_f(U+Z, V+Z) \leq \epsilon_f(U, V)$.

(ii) If (U, V) and (U', V') are independent, then $\epsilon_f(U+U', V+V') \leq \epsilon_f(U, V) + \epsilon_f(U', V')$.

(iii) Always $|E(f(U) - f(V))| \leq \epsilon_f(U, V)$.

Next we suppose that $f^{(r)}$ is Hölder continuous as in Section 2. As in (2.11) we obtain that

$$\left| f(x+y) - \sum_{j=0}^r f^{(j)}(y) \frac{x^j}{j!} \right| \leq K |x|^{r+\alpha} \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+r)}. \quad (4.1)$$

If U and V are r.v.'s with equal first r moments, we obtain by twice using (4.1) that

$$\left| E(f(U+y) - f(V+y)) \right| \leq E \left| f(U+y) - f(V-y) \right| \leq C B_s(U, V), \quad (4.2)$$

where $B_s(U, V) = E|U|^s + E|V|^s$ and $s = r + \alpha$. Here the constant C depends on K, r and α .

Now we state the main result of this section.

Theorem 3. Suppose $f^{(r)}$ is Hölder continuous and suppose that $(X_1, Y_1), (X_2, Y_2), \dots, (X_n, Y_n)$ are i.i.d. random vectors for which the first r moments are equal. Then for each $n \geq 1$,

$$\epsilon_f \left(\frac{\sum_{i=1}^n X_i}{n}, \frac{\sum_{i=1}^n Y_i}{n} \right) \leq K n^{1-s} B_s(X_1, Y_1).$$

P r o o f. Combine Lemma 1 (ii), (iii) and (4.2).

If X_1 generates operator L_n and Y_1 generates operator L'_n , the previous discussion proves

Corollary 1. Under appropriate conditions on f and X_1, Y_1 , we have

$$\left| L_n(f; x) - L'_n(f; x) \right| \leq K n^{1-s} B_s(X_1, Y_1).$$

The previous stability result also opens the way to study operators of the type L_n , where we normalize the sum S_n not by n but by another function of n , see e.g. [4].

Suppose (X, Y) is a r.v. such that Y is h -decomposable, i.e., for each $n > 0$, there exist i.i.d. r.v.'s $Y_1, Y_2, \dots$ such that the distribution of Y is equal to that of $h(n) S'_n$, where S'_n is the sum of the Y_i .

Now we define the operator L as follows:

$$L_X(f; E f(Y)) = E f(h(n) S_n),$$

where S_n is a sum of n i.i.d. copies of X .

Similar to Corollary 1 we now have

Corollary 2. Under appropriate conditions on f and (X, Y) we have

$$\left| L_X(f; E f(Y)) - E f(Y) \right| \leq K (h(n))^{1-s} B_s(X, Y_1).$$

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SZYSZKOWSKI I.*

WEAK INVARIANCE PRINCIPLE FOR INTEGRAL TYPE FUNCTIONALS¹⁾

Приведены условия, при которых интегралы $\int f_n dX_n$, $n \geq 1$, существуют и образуют слабо сходящуюся в пространстве Скорохода $D(0, \infty)$ последовательность случайных процессов.

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