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WEAK INVARIANCE PRINCIPLE FOR INTEGRAL TYPE FUNCTIONALS¹⁾

Приведены условия, при которых интегралы $\int f_n dX_n$, $n \geq 1$, существуют и образуют слабо сходящуюся в пространстве Скорохода $D(0, \infty)$ последовательность случайных процессов.

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Assuming that $\{(f_n, X_n), n \geq 1\}$ is a sequence of càdlàg processes converging weakly to (f, X) in the Skorokhod topology, conditions are given under which the sequence $\{\int f_n dX_n, n \geq 1\}$ converges weakly in $D[0, \infty)$ to the process $\int f dX$. The problem has been extensively investigated by many authors lately and it seems to be conclusively solved for semimartingales, i.e., for the biggest possible class of stochastic processes for which a notion of stochastic integral can be «sensibly» defined. Recent papers by Jakubowski, Memin and Pages [1], and by Kurtz and Protter [2] treat the problem mentioned by employing advanced methods of stochastic analysis. The main aim of this paper is to show that by exploring some «convenient» definition of stochastic integral one can state and prove the key result by applying standard and well-known methods essentially contained in a book by Billingsley [3]. Even such basic notions of stochastic calculus like stopping time, semimartingale and its local characteristics can be ignored as far as the statement and the proof of the main result is concerned. It makes this paper accessible to researchers without any expertise in the theory of semimartingales and stochastic integration. The main theorem of the paper generalizes some results of Janson and Wichura, Strasser, Yoshihara and Szyszkowski.

1. Notation and definitions. Let E be a real separable Banach space. We denote by $D(\mathbf{R}_+; E) = D(E)$ the space of functions f on $[0, \infty)$ into E that are right continuous and have left hand limits. We give the Skorokhod topology in the space $D(E)$ (cf. [4] and [5]). Many of the statement presented in [3], for the space $D([0, 1]; \mathbf{R})$, are also true in an arbitrary separable Banach space. Thus, in the sequel, we shall use some well-known generalizations of results from [3] without proving them.

The symbol $\mathcal{D}(\mathcal{F}, E)$ (abbreviated $\mathcal{D}(E)$ or $\mathcal{D}$) will denote the set of all stochastic processes on $(\Omega, \mathcal{F}, \mathbf{P})$ with paths in the space $D(E)$. We endow $\mathcal{D}$ with the topology of uniform convergence on compact sets in probability, abbreviated ucp. Note that this topology is metrizable, and the metric

$$d(X, Y) = \sum_{n=1}^{\infty} 2^{-n} \mathbf{E} \left(1 \wedge \sup_{0 \leq t \leq n} \|X(t) - Y(t)\| \right), \quad X, Y \in \mathcal{D},$$

makes $\mathcal{D}$ into a complete metric space compatible with ucp.

Γ_T will denote the set of all finite nonrandom partitions $\gamma: 0 = t_0 < t_1 < \dots < t_k = T$ of the interval $[0, T]$ and $|\gamma| = \max_{1 \leq i \leq k} (t_i - t_{i-1})$.

A sequence of partitions $\{\gamma_n \in \Gamma_T, n \geq 1\}$ of the set $[0, T]$ is called a normal sequence of partitions (NSP) if $|\gamma_n| \rightarrow 0$ as $n \rightarrow \infty$.

Definition 1. We say that a process $f \in \mathcal{D}(\mathbf{R})$ is integrable with respect to a process $X \in \mathcal{D}(E)$ if there exists, in the space $\mathcal{D}(E)$, a unique (up to indistinguishability) process, denoted by $\{\int_0^t f dX: t \geq 0\}$ such that for each $T > 0$

$$f(0)X(0) + \sum_{t_i \in \gamma} f(t_i) \left(X(t_{i+1} \wedge t) - X(t_i \wedge t) \right) \longrightarrow \int_0^t f dX$$

uniformly in probability on $[0, T]$, as $|\gamma| \rightarrow 0, \gamma \in \Gamma_T$.

The limit process $\int f dX := \{\int_0^t f dX: t \geq 0\}$ (if it exists) will be called the stochastic integral of f with respect to X .

Let $\mathcal{X}(\mathcal{F}_t) = \mathcal{X}$ be the set of processes $X \in \mathcal{D}$ satisfying the following condition: for every $T > 0$ and any NSP $\{\gamma_n \in \Gamma_T, n \geq 1\}$

$$\liminf_{L \rightarrow \infty} \lim_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{P} \left\{ M^L(\gamma_n, X) > M \right\} = 0, \quad (1)$$

where

$$M^L(\gamma, X) = \sum_{t_i \in \gamma} \left\| \mathbb{E} \left((R(t_i))_L \mid \mathcal{F}_{t_i} \right) \right\|, \quad R(t_i) = X(t_{i+1}) - X(t_i),$$

and for a random element X we put $(X)_r = X I(\|X\| \leq r)$, $r > 0$.

R e m a r k 1. (i) Let $f \in \mathcal{D}(\mathcal{F}, \mathbf{R})$ have the representation

$$f(s) = \sum_{i=0}^{n-1} g_i I_{[\tau_i, \tau_{i+1}]}(s), \quad s \geq 0,$$

where $0 = \tau_0 < \tau_1 < \dots < \tau_n < \infty$ and $\{g_i: 0 \leq i \leq n\}$ are random variables. Then, as is easy to check, f is integrable with respect to an arbitrary $X \in \mathcal{D}(\mathcal{F}, E)$ and

$$\int_0^t f dX = \sum_{i=0}^{n-1} g_i \left(X_n(\tau_{i+1} \wedge t) - X_n(\tau_i \wedge t) \right), \quad t \geq 0.$$

(ii) One can prove (it is beyond the scope of this paper) that every semimartingale belongs to the class $\mathfrak{X}$. Hence in the case of a semimartingale X being integrator, the concept of stochastic integral $\int f dX$ described by Definition 1 (which is not claimed by this author to be original) coincides with the classical definition of stochastic integral

$$\int f_- dX, \quad \text{where } f_-(t) = f(0) I_{\{t=0\}} + \lim_{s \nearrow t} f(s) I_{\{t>0\}}, \quad f \in \mathcal{D}(\mathbf{R}),$$

(cf. [6, Ch. 11] and [7, Definition 3.3, Theorem 7.12, and Theorem 10.1]). It is only for simplicity of notation that we choose to write $\int f dX$ instead of $\int f_- dX$.

2. Existence of stochastic integral. In this section we shall prove the following theorem.

Theorem 1. Let $f \in \mathcal{D}(\mathcal{F}, \mathbf{R})$ and $X \in \mathcal{X}(\mathcal{F}_t, E)$ be two $(\mathcal{F}_t)$ -adapted stochastic processes. Then f is integrable with respect to X .

At first we state several auxiliary lemmas which are needed in the sequel. Some of them will be used in other sections.

Lemma 1. Let B be a real separable Banach space and $\{X, X_n, n \geq 1\}$ — a sequence of stochastic processes with paths in the space $D(\mathbf{R}_+, B)$ and defined on probability spaces $(\Omega, \mathcal{F}, \mathbf{P})$, $(\Omega^n, \mathcal{F}^n, \mathbf{P}^n)$, respectively. Assume $X_n \rightarrow X$ weakly, in $D(\mathbf{R}_+, B)$, as $n \rightarrow \infty$. Then for each $\varepsilon > 0$ and $T > 0$ there exists a NSP $\{\gamma_n\}$

$$\lim_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{P}^n \left\{ h_\varepsilon^T(X_n^{\gamma_n}) > M \right\} = 0, \quad (2)$$

where

$$X_n^{\gamma}(t) = \sum_{t_i \leq t, t_i \in \gamma} (X_n(t_i) - X_n(t_{i-1})), \quad t \geq 0,$$

$$h_\varepsilon^T(Y) = \sum_{0 \leq s \leq T} \|Y(s) - Y(s-)\| I \left(\|Y(s) - Y(s-)\| \right), \quad Y \in \mathcal{D},$$

$$\Gamma_T(X) = \left\{ (t_i) \in \Gamma_T: \mathbf{P} \left\{ X(t_i-) = X(t_i) \right\} = 1, \text{ for all } i \right\}.$$

P r o o f. Let us note that for every $\varepsilon > 0$ and $T > 0$, $h_\varepsilon^T(X) < \infty$ P-a.s. and $X^\gamma \rightarrow X$, P-a.s., as $|\gamma| \rightarrow 0$, $\gamma \in \Gamma_T(X)$. Hence (2) is a simple consequence of the assumptions and continuity (in the $D(\mathbf{R}_+, B)$ -Skorokhod topology) of the function $h_\varepsilon^T: D(\mathbf{R}_+, B) \rightarrow \mathbf{R}$, at each point $x(\cdot)$ for which $x(T) = x(T-)$ (cf. [8, Ch. 6, § 2]).

Lemma 2 (maximal inequality). Let $\{(S(t), \mathcal{F}_t); 0 \leq t \leq T < \infty\}$ be a martingale which belongs to the space $\mathcal{D}(\mathcal{F}, E)$. Then for every $\varepsilon > 0$

$$\mathbf{P}\left\{\sup_{0 \leq t \leq T} \|S(t)\| > \varepsilon\right\} \leq \varepsilon^{-1} \sup_{0 \leq t \leq T} \mathbf{E}\|S(t)\|.$$

Proof of Theorem 1. Let $f \in \mathcal{D}(\mathcal{F}, \mathbf{R})$ and $X \in \mathcal{X}(\mathcal{F}_t)$ be two $(\mathcal{F}_t)$ -adapted stochastic processes. It is well known that investigations of the $D(\mathbf{R}_+, E)$ -weak convergence can be reduced to considerations of the $D([0, T]; E)$ -weak convergence for every $T > 0$ (cf. [4]). Thus, in the sequel, we restrict our considerations to the interval $[0, 1]$, using notation and some basic facts contained in the well-known Billingsley's book [3, Ch. 2 and 3]. Without loss of generality we also assume that $T = 1$ is a continuity point of the limit process X .

Let us fix $\varepsilon, \rho \in (0, 1)$. Note that by the assumptions and Lemma 1 ($X_n = X, n \geq 1$) we can successively choose positive numbers L, M, n_0 and δ satisfying the following conditions:

$$P(A_1) := \mathbf{P}\{M^L(\gamma_n, X) + N^L(\gamma_n, X) > M\} < \rho, \quad (3)$$

$$P(A_2) := \mathbf{P}\{h_{\varepsilon/2M^2}^1((f, X)^{\gamma_n}) > K\} < \rho, \quad (4)$$

$$P(A_3) := \mathbf{P}\{w'((f, X), 2\delta) > \varepsilon(KM^2)^{-1}\} < \rho, \quad (5)$$

for some NSP $\{\gamma_n \in \Gamma_1, n \geq 1\}$ and every $n \geq n_0$, where

$$N^L(\gamma_n, X) = \sup_{0 \leq t \leq 1} \left\| \sum_{t_i \in \gamma_n, t_i \leq t} ((R(t_i))_L - \mathbf{E}((R(t_i))_L | \mathcal{F}_{t_i})) \right\|.$$

Without loss of generality we may and do assume that $L \geq 1, K > 4$ and $M^{-2}(M+2L) < \rho$. Put

$$\begin{aligned} \Delta_i(s, \gamma) &= X(s_{i+1} \wedge s) - X(s_i \wedge s), \\ I^\gamma(s) &= \sum_{s_i \in \gamma} f(s_i) \Delta_i(s, \gamma), \quad s \in [0, 1], \quad \gamma = (s_i) \in \Gamma. \end{aligned}$$

By (3)–(5), for every pair of partitions $\gamma^i \subset \cup_{n=n_0}^\infty \gamma_n, |\gamma^i| < \delta, i = 1, 2$, we have

$$\begin{aligned} P(A) &:= \mathbf{P}\left\{\sup_{0 \leq s \leq 1} \|I^{\gamma^1}(s) - I^{\gamma^2}(s)\| > 6\varepsilon\right\} \\ &= \mathbf{P}\left\{\sup_{0 \leq s \leq 1} \left\| \sum_{s_i \in \gamma = \gamma^1 \cup \gamma^2} V(s_i) \Delta_i(s, \gamma) \right\| > 6\varepsilon\right\}, \end{aligned}$$

where $V(s_i)$ denotes a $\mathcal{F}_{s_i}$ -measurable random variable such that on the event $A_1^c \cap A_2^c \cap A_3^c \cap A$

$$\begin{aligned} \sup_{0 \leq s \leq 1} \left| \sum_{s_i \in \gamma} (V(s_i))_{\varepsilon/M^2} \left(\sum_{t_k \in [s_i, s_{i+1})} (\Delta_k(s, \gamma_n))_L \right) \right| &\geq 3\varepsilon, \\ \sup_{0 \leq s \leq 1} \sum_{s_i \in \gamma} \left| (V(s_i))_{\varepsilon/M^2} \right| \sum_{t_k \in [s_i, s_{i+1})} \left\| \mathbf{E}((\Delta_k(s, \gamma_n))_L | \mathcal{F}_{t_k}) \right\| &\leq 2\varepsilon. \end{aligned}$$

Consequently, we obtain

$$P(A) \leq 3\rho + P\left\{A_1^c \cap \sup_{0 \leq s \leq 1} \left\| \sum_{s_i \in \gamma} (V(s_i))_{\varepsilon/M^2} \left(\sum_{t_k \in [s_i, s_{i+1})} [(\Delta_k(s, \gamma_n))_L - E((\Delta_k(s, \gamma_n))_L | \mathcal{F}_{s_k})] \right) \right\| > \varepsilon \right\} = 3\rho + P(B) \quad (\text{say}).$$

But, by Lemma 2, $P(B) \leq \varepsilon^{-1}(\varepsilon/M^2)(M+2L) < \rho$. Hence

$$P\left\{\sup_{0 \leq t \leq 1} \|I^{\gamma^1}(t) - I^{\gamma^2}(t)\| > 6\varepsilon\right\} \leq 4\rho,$$

provided $|\gamma^i| \leq \delta$, $\gamma^i \subset \bigcup_{n=n_0}^{\infty} \gamma_n$, $i = 1, 2$.

This completes the proof of Theorem 1, because the metrizable space $(\mathcal{D}, d)$ is complete and $\bigcup_{n=n_0}^{\infty} \gamma_n$ is dense in $[0, 1]$.

3. Weak convergence of stochastic integrals. Let $(f, X) \in \mathcal{D}(\mathcal{F}, \mathbf{R}) \times \mathcal{X}(\mathcal{F}_t, E)$, $(f_n, X_n) \in \mathcal{D}(\mathcal{F}^n, \mathbf{R}) \times \mathcal{X}(\mathcal{F}_t^n, E)$, $n \geq 1$, be a sequence of random processes defined on probability spaces $(\Omega, \mathcal{F}, \mathbf{P})$, $(\Omega^n, \mathcal{F}^n, \mathbf{P}^n)$ and adapted to $(\mathcal{F}_t)$, $(\mathcal{F}_t^n)$, respectively. For simplicity of consideration, from now on, we will always assume that $f(0) = f_n(0) = 0$, $X(0) = X_n(0) = 0$ a.s., $n \geq 1$.

Now we shall consider sufficient conditions for the weak convergence, in the space $D(\mathbf{R}_+; E)$, of the sequence of stochastic integrals

$$I_n(t) = \int_0^t f_n dX_n, \quad t \geq 0, \quad n \geq 1,$$

to the integral

$$I(t) = \int_0^t f dX, \quad t \geq 0.$$

The theorems given below generalize recent results presented by Strasser [9], Janson and Wichura [10], Yoshihara [11], [12], and Szyszkowski [13]–[16].

Theorem 2. Assume

$$(f_n, X_n) \rightarrow (f, X) \text{ weakly, in the space } D(\mathbf{R} \times E), \quad \text{as } n \rightarrow \infty, \quad (6)$$

and for every $T > 0$ and any NSP $\{\gamma_n \in \Gamma_T, n \geq 1\}$

$$\liminf_{L \rightarrow \infty} \lim_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{P}^n \left\{ M^L(\gamma_n, X_n) > M \right\} = 0. \quad (7)$$

Then

$$I_n \rightarrow I \text{ weakly, in the space } D(E), \quad \text{as } n \rightarrow \infty. \quad (8)$$

P r o o f. Following step by step reasoning presented in the proof of Theorem 1 one can prove that

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \sup_{\substack{|\gamma| \leq \delta \\ \gamma \in \Gamma_T}} \mathbf{P}^n \left\{ \sup_{0 \leq t \leq T} \|I_n^\gamma(t) - I_n(t)\| > \varepsilon \right\} = 0, \quad \varepsilon > 0. \quad (9)$$

By (6), for fixed $\gamma \in \Gamma$, we get

$$I_n^\gamma \xrightarrow{\text{f.d.}} I^\gamma \text{ weakly in } D(E), \quad \text{as } n \rightarrow \infty, \quad (10)$$

(«f.d.» — convergence of finite dimensional distributions) provided at the points of γ the limit process (f, X) is $\mathbf{P}$ -a.s. continuous, i.e.,

$$\mathbf{P}\{(f(t_i), X(t_i)) = (f(t_i-), X(t_i-))\} = 1, \quad t_i \in \gamma.$$

Now, on account of (9) and (10), we get

$$\int f_n dX_n \xrightarrow{\text{f.d.}} \int f dX \quad \text{in } D(E), \quad n \rightarrow \infty.$$

The proof of tightness of the sequence $\{I_n, n \geq 1\}$ is quite elementary and may be found in [14, p. 275]. Uniform tightness of the sequence $\{\sup_{0 \leq t \leq 1} \|I_n(t)\|, n \geq 1\}$ follows from (9) and (10). Hence the proof of Theorem 2 is completed.

By slight and obvious modification of the proof of Theorem 2 we derive three important variants of the last theorem. Two of them (Theorem 4 and 5) concerning stochastic arrays, will be stated in the next section.

Theorem 3. Let $\{X, X_n, n \geq 1\}$ be a sequence of L^1 -integrable stochastic processes such that

$$\limsup_{n \rightarrow \infty} \mathbf{E} \left(\sup_{0 \leq t \leq T} \|X_n(t)\| \right) < \infty, \quad T > 0, \quad (11)$$

$$\lim_{M \rightarrow \infty} \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \sup_{\substack{|\gamma| \leq \delta \\ \gamma \in \Gamma_T}} \mathbf{P}^n \{M^\infty(\gamma, X_n) > M\} = 0, \quad T > 0. \quad (12)$$

If condition (6) is satisfied then (8) holds.

Corollary 1. Let $X \in \mathcal{X}(\mathcal{F}_t, E)$, $X_n \in \mathcal{X}(\mathcal{F}_t^n, E)$, $n \geq 1$, be a sequence of martingales satisfying (11) and (6). Then (8) holds.

R e m a r k s 2. (i) Under the same assumptions the assertions of Theorems 2 and 3 can easily be strengthened as follows:

$$(f_n, X_n, I_n) \rightarrow (f, X, I) \quad \text{weakly, in the space } D(\mathbf{R} \times E \times E). \quad (13)$$

Although (6) is necessary for (13) to hold, it is far from being necessary for (8). For example, if $X_n(t) = X(t) \equiv 0$, then (7) and (8) hold irrespective of the sequence $\{f_n, n \geq 1\}$.

(ii) Let us note that $D(\mathbf{R} \times E)$ and $D(\mathbf{R}) \times D(E)$ are two essentially different topological spaces. In general, the theorems presented in this paper are not true for the product topology of the space $D(\mathbf{R}) \times D(E)$. As an illustration we give the following simple example. Put

$$f_n(t) = f(t) = -I_{[1, \infty)}(t), \quad X_n(t) = -I_{[1+1/n, \infty)}(t), \quad t \geq 0, \quad n \geq 1.$$

It is obvious that $(f_n, X_n) \rightarrow (f, X)$ in $D(\mathbf{R}) \times D(\mathbf{R})$ with $f = X$, but I_n does not converge to I in $D(\mathbf{R})$, because for $s > 1$, $I_n(s) = 1$ and $I(s) = 0$. Of course $\{X_n, n \geq 1\}$ satisfies (7), but (6) does not hold.

(iii) In some special cases condition (6) is implied by appropriate convergence in the «product sense». For example, if $\{X_n, n \geq 1\}$ is asymptotically continuous (see [9] for the definition) or $f_n = \Phi(X_n)$, $n \geq 1$, for some continuous function $\Phi: \mathbf{R} \rightarrow \mathbf{R}$, then (6) is implied by its « $D(\mathbf{R}) \times D(E)$ -version».

(iv) Condition (6) in Theorem 2 can be substituted by the following two assumptions (the proof needs only slight and obvious modifications):

$$(a) \quad (f_n, X_n) \rightarrow (f, X) \quad \text{weakly, in the space } D(\mathbf{R}) \times D(E),$$

and, for each $T > 0$ and $\varepsilon > 0$

$$(b) \quad \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} P^n \left\{ r_T(f_n, X_n, \delta) > \varepsilon \right\} = 0,$$

where $r_T(f_n, X_n, \delta) = \sup\{\|f_n(t_2) - f_n(t_1)\| \|X_n(t_4) - X_n(t_3)\| : 0 \leq t_1 < t_2 \leq t_3 < t_4 \leq T, t_4 - t_1 \leq \delta\}$. Let us note that these conditions are essentially weaker than (6). If, for example, $f_n(t) = I_{[1+1/n, 2+1/n)}(t)$, and $X_n(t) = I_{[1+1/(2n), 2+1/(2n))}(t)$, then (6) does not hold but both (a) and (b) are satisfied with $f = X = I_{[1, 2)}$.

Remark 3. Condition (7) is implied by the following condition: for each positive T and L , $\{M^L(\gamma, X_n) \mid \gamma \in \Gamma_T, n \geq 1\}$ is tight. It is also worth noticing that (7) holds if $\{X_n, n \geq 1\}$ is a sequence of quasimartingales with

$$\sup \left\{ \sum_{t_i \in \gamma} \mathbb{E} \left| \mathbb{E}(X_n(t_{i+1}) \mid \mathfrak{F}_n(t_i)) - X_n(t_i) \right| : \gamma \in \Gamma_T, n \geq 1 \right\} < \infty$$

for each $T > 0$, where $\mathfrak{F}_n(t) = \sigma(X_n(s) : s \leq t)$.

4. Stochastic arrays. Let $(\Omega, \mathcal{F}, P)$, $(\Omega^n, \mathcal{F}^n, P^n)$, $n \geq 1$, be probability spaces and $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$, $\mathbb{F}_n = (\mathcal{F}_{nj})_{j \geq 0}$, $n \geq 1$, — filtrations of $\mathcal{F}, \mathcal{F}^n$, $n \geq 1$, respectively.

A doubly indexed array $X(E) = \{X_{nj}, n \geq 1, j \geq 1\}$ of E -valued random elements is said to be adapted if X_{nj} is $\mathcal{F}_{nj}$ -measurable, $j \geq 1, n \geq 1$.

Let $\tau_n = \{\tau_n(t), t \geq 0\}$, $n \geq 1$, be a time scale adapted sequence, i.e., the processes $\{\tau_n, n \geq 1\}$ are of nondecreasing right continuous trajectories, and for each $n \geq 1$ and every $t \geq 0$, $\tau_n(t)$ is a stopping time with respect to $\{\mathcal{F}_{nj}, j \geq 0\}$. We will also assume that for every $n \geq 1$ and $\omega \in \Omega^n$ the function $t \mapsto \tau_n(t)(\omega)$, $t \geq 0$, takes all values between zero and $\tau_n(\infty)(\omega)$.

In this section we are concerned with two doubly indexed adapted (to the same filtrations) arrays $X(E)$ and $Y(\mathbf{R})$, for the brevity denoted by X and Y . Let us consider the partial sum processes: $S_n(t) = S_{n, \tau_n(t)}$, $R_n(t) = R_{n, \tau_n(t)}$, $t \geq 0$, where $S_{nk} = \sum_{i=1}^k X_{ni}$, $R_{nk} = \sum_{i=1}^k Y_{ni}$. Now consider the following transforms

$$I_n(t) = \sum_{i=1}^{\tau_n(t)} R_{n, i-1} X_{ni}, \quad t \geq 0, \quad n \geq 1 \quad (R_{n0} = S_{n0} = 0 \text{ a.s.}).$$

We shall investigate the limit behaviour, in the sense of $D(E)$ -weak convergence, of the sequence $\{I_n, n \geq 1\}$.

The following theorems are analogous of Theorems 2 and 3.

Theorem 4. Let X and Y be as above. Assume

$$(R_n, S_n) \longrightarrow (R, S) \text{ weakly, in } D(\mathbf{R} \times E), \quad (14)$$

where $(R, S) \in \mathcal{D}(\mathcal{F}, \mathbf{R}) \times \mathcal{X}(\mathcal{F}_t, E)$ is a pair of $\mathbb{F}$ -adapted stochastic processes, and for every $t \geq 0$

$$\liminf_{L \rightarrow \infty} \lim_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} P^n \left\{ \sum_{i=1}^{\tau_n(t)} \left\| \mathbb{E}(X_{ni} I(\|X_{ni}\| \leq L) \mid \mathcal{F}_{n, i-1}) \right\| > M \right\} = 0. \quad (15)$$

Then

$$(R_n, S_n, I_n) \longrightarrow (R, S, I) \text{ weakly, in } D(\mathbf{R} \times E \times E), \quad (16)$$

where $I(t) = \int_0^t R ds$, $t \geq 0$.

P r o o f. This is done by an obvious modification of the proof of Theorem 2.

Theorem 5. Let X be a doubly indexed adapted array of L^1 -integrable E -valued random elements such that

$$\limsup_{n \rightarrow \infty} \mathbf{E} \left(\sup_{0 \leq i \leq r_n(T)} \|X_{ni}\| \right) < \infty, \quad T > 0, \quad (17)$$

and

$$\lim_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{P}^n \left\{ \sum_{i=1}^{r_n(t)} \left\| \mathbf{E}(X_{ni} | \mathcal{F}_{n,i-1}) \right\| > M \right\} = 0, \quad t > 0. \quad (18)$$

If (14) is satisfied, then (16) holds.

Some applications of Theorem 4 can be produced in the same way as those devoted to the martingale case and given in [9] and [10].

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