Math-Net.Ru

Общероссийский математический портал

R. Jajte, Operators on L^p -spaces determined by filtrations and pointwise ergodic theorems, *Теория вероятн. и ее примен.*, 2009, том 54, выпуск 3, 551–566

DOI: <https://doi.org/10.4213/tvp2809>

Использование Общероссийского математического портала Math-Net.Ru подразумевает, что вы прочитали и согласны с пользовательским соглашением

<http://www.mathnet.ru/rus/agreement>

Параметры загрузки:

IP: 3.80.253.173

26 апреля 2023 г., 14:04:59

© 2009 г.

JAJTE R.*

OPERATORS ON L^p -SPACES DETERMINED BY FILTRATIONS AND POINTWISE ERGODIC THEOREMS

Пусть $(\Omega, \mathcal{F}, \mu)$ — вероятностное пространство и $(\mathcal{F}_t, a \leq t \leq b)$ — фильтрация на $(\Omega, \mathcal{F})$. Мы рассматриваем операторы в $L^p(\Omega, \mu)$, $1 < p < \infty$, заданные рядом $\sum_k f(t_k)(E_{t_k} - E_{t_{k-1}})$ или интегралом $\int_a^b f(t) dE_t$, где E_t — условное математическое ожидание $\mathbf{E}(\cdot | \mathcal{F}_t)$, и доказываем для таких операторов поточечные предельные теоремы. Предлагаются доказательства в духе подхода Гапошкина к сходимости почти всюду эргодических средних в L^2 с применением неравенств Буркхольдера для мартингалов и интерполяции операторов в пространствах L^r .

Ключевые слова и фразы: фильтрация, условное математическое ожидание, спектральное представление, мартингаловое преобразование, интерполяция, поточечная предельная теорема.

1. Introduction. Broadly speaking, there are three ways to obtain Individual Ergodic Theorem for an operator A on L^p -space. The first one, indicated by Birkhoff, Hopf, Ionescu-Tulcea, Akcoglu, and others (see [13], [1], [8]) is based on the contractivity and positivity of A (that is, $f \geq 0$ a.e. implies $Af \geq 0$ a.e.). The second one is the method of Dunford and Schwartz [9] based on the 2-contractivity, i.e., the contractivity of A in L^1 and L^∞ simultaneously. The third way has been indicated and developed by Gaposhkin [11], [12] and later extended by Berkson, Bourgain, and Gillespie [3], Houdré (see [14]) and others, is based on the asymptotic properties of the spectrum σ near the value 1 of an operator in L^p which is, in a sense, similar to a normal operator in L^2 , that is, having a *spectral representation* of the form $A = \int_\sigma z dE(z)$ [7].

In the paper we consider a class of linear operators on L^p -spaces which is a natural and important counterpart of normal operators in L^2 . This class provides, among others, natural nontrivial examples of operators which are not positive or even noncontractive in L^p but satisfy the pointwise ergodic theorem. For such operators the method of Gaposhkin and martingale theory can be successfully applied.

* University of Łódź, Faculty of Mathematics, Banacha 22, PL-90-238 Łódź, Poland;
e-mail: rjajte@math.uni.lodz.pl

It is reasonable to start with $L^2 \ominus [\text{const}]$ over a probability space $(\Omega, \mathcal{F}, \mu)$, say. We are interested in the operators of the form

$$T = \int_{[a,b]} f(t) dE_t, \quad (1)$$

where the spectral family $(E_t)_{a \leq t \leq b}$ is given by a filtration $(\mathcal{F}_t)_{a \leq t \leq b}$, i.e., $E_t(\cdot) = \mathbf{E}(\cdot | \mathcal{F}_t)$ are conditional expectation operators. Here, the projections E_t are defined in any L^p , $1 \leq p \leq \infty$, so formula (1) defines an operator on $\bigcup_{1 \leq p \leq \infty} L^p$. The integral in (1) is taken in the sense of the Riemann–Stieltjes integration (see the next section). Thanks to the additional regularity of the spectral family (E_t) , the operators of the form (1) enjoy nice asymptotic properties. Some of them will be discussed in the paper.

2. Preliminaries. Let us begin with some notation. For a probability space $(\Omega, \mathcal{F}, \mu)$, a family $(\mathcal{F}_t)_{a \leq t \leq b}$ of sub- σ -fields of $\mathcal{F}$ is a filtration if

- (i) $\mathcal{F}_{t'} \subset \mathcal{F}_{t''}$, for $t' < t''$;
- (ii) $\bigcap_{s>t} \mathcal{F}_s = \mathcal{F}_t$, for $a \leq t < b$;
- (iii) $\sigma(\bigcup_{t<b} \mathcal{F}_t) = \mathcal{F}_b = \mathcal{F}$;
- (iv) $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

In the above definition a and b are finite.

Let $p \geq 1$. Put $\hat{L}^p(\Omega, \mathcal{F}, \mu) = \{f \in L^p(\Omega, \mathcal{F}, \mu) : \int_{\Omega} f d\mu = 0\}$. For a filtration $(\mathcal{F}_t)_{a \leq t \leq b}$, we consider a family of conditional expectations

$$E_t = \mathbf{E}(\cdot | \mathcal{F}_t), \quad a \leq t \leq b. \quad (2)$$

Clearly, the family (2) is a resolution of the identity in $\hat{L}^2(\Omega, \mathcal{F}, \mu)$.

For a fixed $p \geq 1$, $(E_t)_{a \leq t \leq b}$ in (2) is a *uniformly bounded family* of projections on $\hat{L}^p$ satisfying the conditions

- (a) $E_s E_t = E_t E_s = E_{\min(s,t)}$, $a \leq s, \quad t \leq b$,
- (b) $E_a = 0, \quad E_b = I$,
- (c) E_t is right continuous in the strong operator topology, for $a \leq t < b$, and E_t has a strong left limit at each point $a < t \leq b$.

A family (E_t) enjoying all the above properties is called a *spectral family of projections*.

In the context of operators in $\hat{L}^p$ of the form (1), a theory of the Riemann–Stieltjes integration with respect to a spectral family (E_t) can be used ([7], summarized in [4]). For convenience we mention here a few main points of this theory (adapted to our context). Let $(E_t)_{a \leq t \leq b}$ be a spectral family of projections in $\hat{L}^p$ as described above. Let $BV[a, b]$ be the Banach algebra (under pointwise operations) consisting of all complex-valued functions f on $[a, b]$ of bounded variation, with the norm

$$\|f\|_{\text{var}} = |f(b)| + \text{var}(f, [a, b]).$$

Given a partition $\pi = (a = t_0 < t_1 < \cdots < t_n = b)$ of the interval $[a, b]$, we put

$$\pi(f) = \sum_{j=1}^n f(t_j)[E_{t_j} - E_{t_{j-1}}]. \quad (4)$$

The sums $\pi(f)$ converge in the strong operator topology as π runs through the partitions of $[a, b]$ directed by refinement. This strong limit of $\pi(f)$ is denoted by

$$\int_a^b f(t) dE_t. \quad (5)$$

In what follows a spectral family (E_t) is always of the form (2). The mapping

$$f \mapsto \int_a^b f(t) dE_t$$

is an identity preserving algebra homomorphisms of $BV[a, b]$ into $B(\widehat{L}^p)$, the algebra of bounded linear operators on $\widehat{L}^p$. Moreover, applying to the Riemann–Stieltjes sums (4) the Abel transformation we get

$$\left\| \int_a^b f(t) dE_t \right\| \leq \|f\|_{\text{var}}, \quad f \in BV[a, b].$$

In the case of discrete parameter t , say, t taking the values $t_1 < t_2 < \cdots$ from a compact interval, instead of integral (1) we have the series

$$Tg = \sum_{i=1}^{\infty} t_i (\mathbf{E}(g | F_i) - \mathbf{E}(g | F_{i-1})), \quad g \in L^p, \quad (6)$$

(F_i) being an increasing sequence of σ -fields.

Thus, the sequence of partial sums of (5) forms a martingale transform.

Note that the value of the operator (4) at the point $g \in \widehat{L}^p$ is the martingale integral [17].

Throughout the paper $\|\cdot\|_p$ denotes the norm in L^p , and C_β denotes a positive constant depending only on the parameter β (which may be different in different places).

3. Series. The Haar system on the unit Lebesgue interval leads to simple and illuminating examples of martingale transforms. That is why we begin with a pointwise ergodic theorem for the Haar series. The following theorem is, in a sense, a crucial result for the whole paper, though it concerns the special case of the Haar functions.

Theorem 1. *Let $(h_0, h_1, \dots)$ be the Haar system over $[0, 1]$, and let $0 \leq t_j \leq 1$, $t_j < t_{j+1}$ ($j = 0, 1, \dots$). We set*

$$A = \sum_{j=0}^{\infty} t_j \widehat{h}_j, \quad (7)$$

where $\widehat{h}_j(g) = (\int_0^1 gh_j)h_j$.

Then the sequence $\{A^n g\}$ converges a.e., for each $g \in L^p$, $1 < p < \infty$.

P r o o f. Clearly, A is L^p -bounded, $1 < p < \infty$, by the Burkholder inequality for martingale transform. Let us fix $1 < p < \infty$ and $g \in L^p[0, 1]$. To avoid trivial cases we assume that $t_j \rightarrow 1$.

We shall prove that the *extended Gaposhkin criterion* [11], [12], [3], which in our context reads as follows:

$$\{A^n g\} \text{ converges a.e. if and only if } r_n = \sum_{1-2^{-n} < t_s < 1} \widehat{h}_s(g) \rightarrow 0 \text{ a.e.} \quad (8)$$

Having (8), we readily get the a.e. convergence of $\{A^n g\}$ since $r_n \rightarrow 0$ a.e. by the martingale convergence theorem.

Let us note that

$$A^n g = \sum_{s=0}^{\infty} t_s^n \widehat{h}_s(g). \quad (9)$$

To prove (8) it suffices to show that

$$\alpha_n = \sum_{1-2^{-n} \leq t_s < 1} (1 - \sigma_{2^n}(t_s)) \widehat{h}_s(g) \rightarrow 0 \quad \text{a.e.}, \quad (10)$$

$$\beta_n = \sum_{0 \leq t_s < 1-2^{-n}} \sigma_{2^n}(t_s) \widehat{h}_s(g) \rightarrow 0, \quad \text{a.e.}, \quad (11)$$

$$\delta_n = \max_{1 \leq m < 2^n} |S_{2^n+m} - S_{2^n}| \rightarrow 0 \quad \text{a.e.} \quad (12)$$

To prove (10) we shall show that

$$\Delta_1 = \left\| \left\{ \sum_{n=1}^{\infty} \alpha_n^2 \right\}^{1/2} \right\|_p < \infty. \quad (13)$$

Writing

$$[1 - 2^{-n}, 1) = \bigcup_{k=0}^{\infty} I_{n,k} \quad \text{with } I_{n,k} = [1 - 2^{-n-k}, 1 - 2^{-n-k-1}),$$

we readily get

$$\Delta_1 \leq \sum_{k=0}^{\infty} \left\| \left\{ \sum_{n=1}^{\infty} \left| \sum_{t_s \in I_{n,k}} (1 - \sigma_{2^n}(t_s)) \widehat{h}_s(g) \right|^2 \right\}^{1/2} \right\|_p. \quad (13')$$

For a fixed k , let us set

$$A_{\nu}^{(k)} = \sum_{s=0}^{\nu} \left(\sum_{n=1}^{\infty} \sum_{t_s \in I_{n,k}} (1 - \sigma_{2^n}(t_s)) \right) \widehat{h}_s(g), \quad \nu = 1, 2, \dots$$

Then $(A_\nu^{(k)}, \nu = 1, 2, \dots)$ is a martingale transform and, by the Burkholder inequality [5],

$$\sup_\nu \|A_\nu^{(k)}\|_p \leq C_p \sup_\nu \|\mathbf{E}(g | \mathcal{F}_\nu)\|_p \leq C_p \|g\|_p, \quad (14)$$

where $\mathcal{F}_\nu = \sigma(h_0, h_1, \dots, h_\nu)$.

For a fixed k , we set $m_n^{(k)}$ to be such that $t_s \in I_{n,k}$ if and only if $m_n^{(k)} < s \leq m_{n+1}^{(k)}$. Let us put

$$B_n^{(k)} = A_{m_n^{(k)}}^{(k)}.$$

Thus, for any k , $(B_n^{(k)})$ is a martingale and its square function equals

$$Q_k = \left\{ \sum_{n=1}^{\infty} \left| \sum_{t_s \in I_{n,k}} (1 - \sigma_{2^n}(t_s)) \widehat{h}_s(g) \right|^2 \right\}^{1/2}. \quad (15)$$

By the Burkholder inequality [5], (14), and (15) we get, for $g \in L^r$,

$$\|Q_k\|_r \leq C_r \|g\|_r, \quad 1 < r < \infty \quad (16)$$

(here and in the sequel C_β denotes a constant depending only on the parameter β , which may be different in different places).

To obtain $\Delta_1 < \infty$ we have to improve estimate (16) to get a good enough order of decrease of $\|Q_k\|_r$ as $k \rightarrow \infty$ (for $1 < r < \infty$). To this end we shall interpolate some operators related to Q_k in (15). We follow the idea indicated in [3]. Let us first notice that, fortunately, for $p = 2$, using the estimate $1 - \sigma_n(t) \leq Cn(1 - t)$, we easily obtain, for $g \in L^2$,

$$\|Q_k\|_2 \leq C_2 \cdot 2^{-k} \|g\|_2. \quad (17)$$

Fix $1 < r < \infty$ such that p lies between 2 and r . To interpolate between (16) and (17) we define (for a fixed k) an operator

$$W_k^{(r)}: L^r[0, 1] \rightarrow L^r([0, 1], l^2),$$

by putting

$$W_k^{(r)}: g \mapsto \left\{ \sum_{t_s \in I_{n,k}} (1 - \sigma_{2^n}(t_s)) \widehat{h}_s(g) \right\}_{n=1}^{\infty}. \quad (18)$$

Then $\|W_k^{(r)}\|_r \leq C_r$ and $\|W_k^{(2)}\|_2 \leq C_2 \cdot 2^{-k}$.

By (15)–(18) and interpolation theorems [2, Chap. 5], we obtain, for $1 < p < \infty$, the estimate

$$\|Q_k\|_p \leq C_p \cdot 2^{-\theta_p k} \|g\|_p, \quad 0 < \theta_p \leq 1.$$

Consequently, after summing up over k on the both sides of the above inequality, by (13'), we get (13), and (10) follows.

We proceed in a similar way to show (11), proving that

$$\Delta_2 = \left\| \left\{ \sum_{n=1}^{\infty} \beta_n^2 \right\}^{1/2} \right\| < \infty.$$

Writing $(0, 1 - 2^{-n}) = \bigcup_{k=1}^n (1 - 2^{-n+k}, 1 - 2^{-n+k-1}] = \bigcup_{k=1}^n J_{n,k}$, we get

$$\Delta_2 \leq \sum_{k=1}^{\infty} \left\| \sum_{n=k}^{\infty} \left| \sum_{t_s \in J_{n,k}} \sigma_{2^n}(t_s) \hat{h}_s(g) \right|^2 \right\|_p.$$

For a fixed k , we set

$$A_{\nu}^{(k)} = \sum_{s=0}^{\nu} \left(\sum_{n=1}^{\infty} \sum_{t_s \in J_{n,k}} \sigma_{2^n}(t_s) \right) \hat{h}_s(g), \quad \nu = 1, 2, \dots$$

Now, we can follow, mutatis mutandis, the argument for Δ_1 , applying the Burkholder inequalities [5], interpolation, and using the estimate $|\sigma_n(t)| \leq Cn^{-1}(1-t)^{-1}$.

The proof of (12) is more complicated, though the main tools will be the same as before. Using the classical dyadic decomposition like in [11, § 2], we find that it suffices to show the estimate

$$\left\| \left\{ \sum_{n=1}^N \delta_n^2 \right\}^{1/2} \right\|_p \leq C_p \sum_{\nu=1}^3 \|A_{\nu}\|_p \quad (\text{uniformly in } N),$$

with

$$A_{\nu} = \left\{ \sum_{n=1}^N \sum_{k=1}^n k^2 \sum_{j=1}^{2^k} \left| \sum_{t_s \in \Gamma_{\nu}} R_{n,k,j}(t_s) \hat{h}_s(g) \right|^2 \right\}^{1/2}, \quad (19)$$

where

$$\Gamma_1 = [1 - 2^{-n}, 1), \quad \Gamma_2 = [1 - 2^{-n+k}, 1 - 2^{-n}), \quad \Gamma_3 = (0, 1 - 2^{-n+k}), \quad (20)$$

and

$$R_{n,k,j}(t) = \sigma_{2^n+j2^{n-k}}(t) - \sigma_{2^n+(j-1)2^{n-k}}(t), \quad k = 1, \dots, n, \quad j = 1, \dots, 2^k. \quad (21)$$

Thus, to show (12) it suffices to prove that

$$\|A_{\nu}\|_p \leq C_p \|g\|_p, \quad \text{for } \nu = 1, 2, 3 \quad (22)$$

(the estimates being independent of N). Writing

$$[1 - 2^{-n}, 1) = \bigcup_{\alpha=0}^{\infty} I_{n,\alpha},$$

where $I_{n,\alpha} = [1 - 2^{-n-\alpha}, 1 - 2^{-n-\alpha-1})$, we readily obtain

$$\|A_1\|_p \leq \sum_{\alpha=\nu}^{\infty} \sum_{k=1}^N k \left\| \left\{ \sum_{n=k}^N \sum_{j=1}^{2^k} \left| \sum_{t_s \in I_{n,\alpha}} R_{n,k,j}(t_s) \widehat{h}_s(g) \right|^2 \right\}^{1/2} \right\|_p. \quad (23)$$

Let us set

$$\psi_{\alpha,k,j}(s) = \sum_{n=k}^N \sum_{t_s \in I_{n,k}} R_{n,k,j}(t_s).$$

Fixing α, k, j , and using the estimate $|t^n - t^m| \leq (n - m)(1 - t)$, for $0 < t < t$ and $1 \leq m \leq n$, we obtain for the martingale transform

$$Y_{\nu}^{(\alpha,k,j)} = \sum_{s=0}^{\nu} \psi_{\alpha,k,j}(s) \widehat{h}_s(g)$$

the inequality

$$\sup_{\nu} \|Y_{\nu}^{(\alpha,k,j)}\|_p \leq C 2^{-k-\alpha} \|g\|_p, \quad 1 < p < \infty. \quad (24)$$

Let $\{m_n^{(\alpha)}\}$ be such that $t_s \in I_{n,\alpha}$ is equivalent to $m_n^{(\alpha)} < s \leq m_{n+1}^{(\alpha)}$. Putting $B_n^{(\alpha,k,j)} = Y_{m_n^{(\alpha)}}^{(\alpha,k,j)}$, and taking for the martingale $(B_n^{(\alpha,k,j)})_{n=k}^N$ its square function

$$Q^{(\alpha,k,j)} = \left\{ \sum_{n=k}^N \left| \sum_{t_s \in I_{n,\alpha}} R_{n,k,j}(t_s) \widehat{h}_s(g) \right|^2 \right\}^{1/2} \quad (25)$$

we get, by (24), (25) and the Burkholder inequality, that

$$\|Q^{(\alpha,k,j)}\|_r \leq C \cdot 2^{-\alpha-k} \|g\|_r, \quad 1 < r < \infty, \quad g \in L^r. \quad (26)$$

For $r = 2$, we readily get

$$\left\| \left\{ \sum_{n=k}^N \sum_{j=1}^{2^k} \left| \sum_{t_s \in I_{n,\alpha}} R_{n,k,j}(t_s) \widehat{h}_s(g) \right|^2 \right\}^{1/2} \right\|_2 \leq C \cdot 2^{-k/2} 2^{-\alpha}, \quad g \in L^2. \quad (27)$$

Like in the previous argument for Δ_1 , thanks to a good estimate of $Q^{(\alpha,k,j)}$ in L^2 -norm (formula (27)) we can improve (26) using interpolation. To interpolate between (26) and (27), we define the operator

$$W_{\alpha,k}^{(r)}: L^r[0, 1] \rightarrow L^r([0, 1], l^2),$$

by putting

$$W_{\alpha,k}^{(r)}: g \mapsto \left\{ \sum_{t \in I_{n,\alpha}} R_{n,k,j}(t_s) \widehat{h}_s(g) \right\}_{\substack{k \leq n \leq N \\ 1 \leq j \leq 2^k}}.$$

By (26) and (27), we have

$$\|W_{\alpha,k}^{(r)}\|_r \leq C \cdot 2^{-\alpha}, \quad \|W_{\alpha,k}^{(2)}\|_2 \leq C \cdot 2^{-\alpha-k/2}.$$

Interpolating between $r < p \leq 2$ or between $2 \leq p < r$, we get

$$\|W_{\alpha,k}^{(p)}\|_p \leq C_p \cdot 2^{-\alpha} 2^{-k\theta_p/2}, \quad \text{with } 0 < \theta_p \leq 1. \quad (28)$$

Summing up over k and α on the both sides of (28), we get by (23) that $\|A_1\|_p \leq C_p \|g\|_p$, for $1 < p < \infty$.

We proceed to show that $\|A_2\|_p \leq C_p \|g\|_p$.

Writing $[1 - 2^{n+k}, 1 - 2^{-n}] = \bigcup_{\alpha=1}^k [1 - 2^{-n+\alpha}, 1 - 2^{-n+(\alpha+1)}] = \bigcup_{\alpha=1}^k J_{n,\alpha}$, we obtain

$$\|A_2\|_p \leq \sum_{k=1}^N \sum_{\alpha=1}^k k \left\| \left\{ \sum_{j=1}^{2^k} \sum_{n=k}^N \left| \sum_{t_s \in J_{n,\alpha}} R_{n,k,j}(t_s) \hat{h}_s(g) \right|^2 \right\}^{1/2} \right\|_p. \quad (29)$$

We set

$$\psi_{k,\alpha,j}(s) = \sum_{n=k}^N \sum_{t_s \in J_{n,\alpha}} R_{n,k,j}(t_s).$$

Let $t_s \in J_{n,\alpha}$ be equivalent to $m_n^{(\alpha)} < s \leq m_{n+1}^{(\alpha)}$. Then, by the Burkholder inequality applied to the martingale transform $Y_\nu^{(k,\alpha,j)} = \sum_{s=0}^\nu \psi_{k,\alpha,j}(s) \hat{h}_s(g)$, we get

$$\sup_\nu \|Y_\nu^{(k,\alpha,j)}\|_p \leq C_p \cdot 2^{-k} \|g\|_p, \quad 1 < p < \infty$$

(since $|\sigma_n(t) - \sigma_m(t)| \leq Cm^{-1}(n - m)$, for $n > m$). Consequently, by the Burkholder inequality for the square function of the martingale $B_n^{(k,\alpha,j)} = Y_{m_n^{(\alpha)}}^{(k,\alpha,j)}$, we obtain

$$\left\| \left\{ \sum_{n=k}^N \left| \sum_{t_s \in J_{n,\alpha}} R_{n,k,j}(t_s) \hat{h}_s(g) \right|^2 \right\}^{1/2} \right\|_p \leq C_p \cdot 2^{-k} \|g\|_p. \quad (30)$$

Define the operator from $L^r[0, 1]$ into $L^r([0, 1], l^2)$ by putting

$$W_{\alpha,k}^{(r)}: g \mapsto \left\{ \sum_{t_s \in J_{n,\alpha}} R_{n,k,j}(t_s) \hat{h}_s(g) \right\}_{\substack{k \leq n \leq N \\ 1 \leq j \leq 2^k}}.$$

By (30), we have

$$\|W_{\alpha,k}^{(r)}(g)\|_r \leq C_r \|g\|_r, \quad 1 < r < \infty. \quad (31)$$

Notice, that for $r = 2$, we easily check that

$$\|W_{\alpha,k}^{(2)}(g)\|_2 \leq C_2 \cdot 2^{-k/2} \|g\|_2. \quad (32)$$

Interpolating between (31) and (32) we obtain

$$\|W_{\alpha,k}^{(p)}\|_p \leq C_p \cdot 2^{-k\theta_p/2}, \quad 0 < \theta_p \leq 1.$$

Summing up on the both sides of the last inequality over α and k , we can readily get

$$\|A_2\|_p \leq C_p \|g\|_p.$$

It remains to show that $\|A_3\|_p \leq C_p \|g\|_p$. To this end, we write

$$(0, 1 - 2^{-n+k}) = \bigcup_{0 \leq \alpha \leq n-k-1} I_{n,\alpha,k},$$

where $I_{n,\alpha,k} = (1 - 2^{-n+k+\alpha+1}, 1 - 2^{-n+k+\alpha})$.

Then

$$\|A_3\|_p \leq \left\| \left\{ \sum_{k=1}^N \sum_{n=k}^N k^2 \sum_{j=1}^{2^k} \left| \sum_{\alpha=0}^{n-k-1} \sum_{t_s \in I_{n,\alpha,k}} R_{n,k,j}(t_s) \hat{h}_s(g) \right|^2 \right\}^{1/2} \right\|_p.$$

Since $t_s \in (0, 1)$, we can take $0 \leq \alpha < \infty$, and write

$$\|A_3\|_p \leq \sum_{\alpha=0}^{\infty} \sum_{k=1}^N k \left\| \left\{ \sum_{n=k}^N \sum_{j=1}^{2^k} \left| \sum_{t_s \in I_{n,\alpha,k}} R_{n,k,j}(t_s) \hat{h}_s(g) \right|^2 \right\}^{1/2} \right\|_p$$

Let us put

$$\psi_{\alpha,k,j}(s) = \sum_{n=k}^N \sum_{t_s \in I_{n,\alpha,k}} R_{n,k,j}(t_s),$$

and take the martingale transform

$$Y_{\nu}^{(\alpha,k,j)} = \sum_{s=0}^{\nu} \psi_{\alpha,k,j}(s) \hat{h}_s(g).$$

In a way similar to the previous cases, using the estimate $t^n \leq n^{-1}(1-t)^{-1}$, applying the Burkholder inequality and interpolation we show that $\|A_3\|_p \leq C_p \|g\|_p$ for each $g \in L^p[0, 1]$, $1 < p < \infty$.

The proof of Theorem 1 is complete.

For the sake of simplicity and clarity, the above theorem has been formulated for the Haar functions. By a slight modification of its proof we can readily obtain a more general result. Namely, we have the following theorem.

Theorem 1^{bis}. *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, and let $(\mathcal{F}_n)_{n \geq 1}$ be a (finite or not) increasing sequence of sub- σ -fields of $\mathcal{F}$. We set $E_n(\cdot) = \mathbf{E}(\cdot | \mathcal{F}_n)$ and, additionally, $E_0 = 0$. Put*

$$A = \sum_{k=1}^{\infty} t_k (E_k - E_{k-1}),$$

with (t_k) being an increasing sequence of numbers in $[0, 1]$. Then the sequence $\{A^n g\}$ converges μ -a.e., for each $g \in L^p$, $1 < p < \infty$.

P r o o f. To get the above result it is enough to replace in the proof of Theorem 1 the operators $\hat{h}_s$ by $E_s - E_{s-1}$.

4. Examples. Passing to continuous parameter t , i.e., to integrals of the form (1) let us begin with some examples.

Consider $\hat{L}^2[0, 1]$ (with the Lebesgue measure) and take the filtration $(\mathcal{F}_t)_{0 \leq t \leq 1}$ with

$$\begin{aligned}\mathcal{F}_t &= \{\text{Borel field in } [0, t]\} \cup \{\emptyset, [t, 1]\}, & 0 < t < 1, \\ \mathcal{F}_0 &= \{\emptyset, [0, 1]\}, \\ \mathcal{F}_1 &= \text{Borel field in } [0, 1].\end{aligned}\tag{33}$$

Put $E_t = \mathbf{E}(\cdot | \mathcal{F}_t)$, $0 \leq t \leq 1$. Then $(E_t)_{0 \leq t \leq 1}$ is the resolution of the identity in $\hat{L}^2[0, 1]$. Let us define the operator in $\hat{L}^p$, by putting

$$A = \int_0^1 t dE_t.$$

We have

$$E_t f = f \chi_{[0, t]} + c_t \chi_{[t, 1]},$$

where

$$c_t = c_t(f) = \frac{1}{1-t} \int_t^1 f(u) du, \quad 0 \leq t < 1.$$

Recall that $\phi_t: f \mapsto \chi_{[0, t]} f$, $0 \leq t \leq 1$, is the spectral family of the «position operator» $Q: f(x) \mapsto x f(x)$ in $L^2[0, 1]$, so we can say that E_t in $\hat{L}^2$ is the spectral projection ϕ_t of Q «centered» by $-c_t \chi_{[t, 1]}$. The operator A is in a sense a counterpart in $\hat{L}^p$ of the «position operator» in $L^2[0, 1]$. More exactly, we shall show that, for $f \in \hat{L}^p[0, 1]$, $p \geq 1$,

$$(Af)(x) = x f(x) - \int_0^x c_u(f) du, \quad 0 \leq x < 1.\tag{34}$$

To show (34), let us write, for $s < t$, $f \in \hat{L}^p$

$$(E_t - E_s)f(x) = \chi_{[s, t]}(f(x) - c_s) + \chi_{[t, 1]}(c_t - c_s).$$

Let $\pi = (0 = t_0 < t_2 < \dots < t_n = 1)$ be a partition of $[0, 1]$. The corresponding Riemann–Stieltjes sum for A is of the form

$$\begin{aligned}\sum_{k=1}^n t_k (E_{t_k} - E_{t_{k-1}}) f(x) &= \sum_{k=1}^n t_k \chi_{[t_{k-1}, t_k]}(x) f(x) + \sum_{k=1}^n t_k (-c_{t_{k-1}}) \chi_{[t_{k-1}, t_k]}(x) \\ &\quad + \sum_{k=1}^n t_k (c_{t_k} - c_{t_{k-1}}) \chi_{[t_k, 1]}(x) = \alpha_\pi + \beta_\pi + \gamma_\pi.\end{aligned}$$

Passing to the limit with π , we get

$$\alpha_\pi \rightarrow xf(x), \quad \beta_\pi \rightarrow -xc_x(f).$$

Write γ_π in the form (using the Abel transformation)

$$\begin{aligned} \gamma_\pi &= \sum_{k=1}^{n-1} c_{t_k} [t_k \chi_{[t_k, 1]}(x) - t_{k+1} \chi_{[t_{k+1}, 1]}(x)] \\ &= \sum_{k=1}^{n-1} c_{t_k} [t_k \chi_{[t_k, 1]}(x) + (t_k - t_{k+1}) \chi_{[t_{k+1}, 1]}(x)] \\ &= \sum_{k=1}^{n-1} c_{t_k} t_k \chi_{[t_k, t_{k+1}]}(x) - \sum_{k=1}^{n-1} c_{t_k} (t_{k+1} - t_k) \chi_{[t_{k+1}, 1]}(x) = \delta_\pi - \kappa_\pi, \end{aligned}$$

since $c_0 = \int_0^1 f(u) du = 0$.

Clearly, $\delta_\pi \rightarrow xc_x(t)$.

Applying the Abel transformation to κ_π , we obtain

$$\kappa_\pi = \sum_{k=1}^{n-2} \left(\sum_{v=1}^k c_{t_v} (t_{v+1} - t_v) \chi_{[t_{k+1}, t_{k+2}]}(x) \right).$$

Passing to the limit with π , we get

$$\kappa_\pi \rightarrow \int_0^x c_u du,$$

which completes the proof of (34).

R e m a r k. Let us note that since $A: \widehat{L}^p \rightarrow \widehat{L}^p$, we have

$$\int_0^1 xf(x) dx = \int_0^1 \int_0^x c_\alpha(f) d\alpha dx, \quad \text{for each } f \in \widehat{L}^p[0, 1].$$

The above formula can also be checked directly via integration by parts.

For the spectral family (E_t) determined by the filtration (33), let us put

$$B = \int_0^1 e^{2\pi it} dE_t.$$

Then, in a similar way, we obtain the formula

$$Bf(x) = 2^{2\pi ix} f(x) - 2\pi i \int_0^x e^{2\pi iu} c_u(f) du, \quad f \in \widehat{L}^p[0, 1].$$

5. Integrals. We now pass to some general results. Let $(\mathcal{F}_t)_{-\pi \leq t \leq \pi}$ be a filtration in a probability space $(\Omega, \mathcal{F}, \mu)$. We set $E_t(\cdot) = \mathbf{E}(\cdot | \mathcal{F}_t)$, the conditional expectation operator, for $t \in [-\pi, \pi]$. Then, for any $1 < p < \infty$, the integral

$$U = \int_{-\pi}^{\pi} e^{it} dE_t \quad (35)$$

defines a power-bounded operator in $\widehat{L}^p$. It means that U is an invertible bounded operator such that $\sup\{\|U^n\|: n = 0, \pm 1, \pm 2, \dots\} < \infty$. Clearly, for $p = 2$, the operator U is unitary.

For the operator U defined by (35) the following individual ergodic theorem holds.

Theorem 2. *For any $1 < p < \infty$, the Cesàro averages*

$$s_n = n^{-1} \sum_{k=0}^{n-1} U^k f \quad (36)$$

and the ergodic Hilbert transform truncates

$$\sigma_n = \sum_{0 < |k| \leq n} k^{-1} U^k f \quad (37)$$

converge μ -a.e., for each $f \in L^p$.

P r o o f. The theorem immediately follows from the general results of Berkson, Bourgain, and Gillespie [3]. They extended to the L^p -space setting the μ -a.e. convergence criteria of Gaposhkin (see [11], [15]) introduced for the averages (36) and (37) in the case of U being a unitary operator on $L^2(\Omega, \mu)$. Namely, for the Cesàro averages (36), with $f \in \widehat{L}^p$, the μ -a.e. convergence of s_n is equivalent to the *extended Gaposhkin condition* [11], [3]

$$(E_{2^{-n}} - E_0)f + (E_0 - E_{-2^{-n}}f) \rightarrow 0 \quad \mu\text{-a.e.}, \quad f \in L^p.$$

But we have that

$$\begin{aligned} (E_{2^{-n}} - E_0)f &= \sum_{s=n}^{\infty} B_s, & \text{with } B_s &= (E_{2^{-s}} - E_{-2^{-s-1}})f, \\ (E_0 - E_{-2^{-n}})f &= \sum_{s=n}^{\infty} B'_s, & \text{with } B'_s &= (E_{-2^{-s-1}} - E_{-2^{-s}})f. \end{aligned}$$

Since the martingales $\beta_n = \sum_{s=0}^n B_s$ and $\beta'_n = \sum_{s=0}^n B'_s$ converge almost everywhere, the *extended Gaposhkin condition* is satisfied, for any $f \in \widehat{L}^p$.

The a.e. convergence of the averages (37) can be proved in a similar way by using the extended criterion for the existence of the ergodic Hilbert transform [3], [15]. It also immediately follows from the fact that the a.e. convergence of (37), for all $f \in \widehat{L}^p$ is equivalent to the a.e. convergence of (36), for all $f \in \widehat{L}^p$ (cf. [3], [15]). This concludes the proof of Theorem 2.

Theorem 3. Let $(\mathcal{F}_t)_{0 \leq t \leq 1}$ be a filtration in a probability space $(\Omega, \mathcal{F}, \mu)$, and let $E_t(\cdot) = \mathbf{E}(\cdot | \mathcal{F}_t)$, $t \in [0, 1]$. Let

$$A = \int_0^1 t dE_t. \quad (38)$$

Then

$$A^n f \rightarrow E\{1\}f, \quad \mu\text{-a.e. for each } f \in \widehat{L}^p, \quad 1 < p < \infty. \quad (39)$$

P r o o f. For $p = 2$ (and, consequently, for $p \geq 2$) the theorem immediately follows from the results of [12], [6] and the martingale convergence theorem.

For $p \in (1, 2)$, our argument is the same as for $p \in (1, \infty)$, so in the sequel we assume that $1 < p < \infty$. Moreover, in the proof, we do not refer to the previous results just mentioned above.

The proof will be reduced to the *discrete parameter* case, that is, to the methods used in Section 3. In the sequel $1 < p < \infty$ and $g \in \widehat{L}^p$ are fixed. We use the following notation. For a function f of bounded variation on $[0, 1]$, and a partition $\pi = (t_0 < t_1 < \dots < t_m)$ of the segment $[0, 1]$, we set

$$\pi(f) = \sum_{k=1}^m f(t_k)(E_{t_k} - E_{t_{k-1}})g \quad \text{and} \quad I(f) = \int_0^1 f(t) dE_t g.$$

Recall (see Section 2) that $I(f)$ is a strong limit in L^p of $\pi_n(f)$, so $\|I(f) - \pi_n(f)\|_p$ may go to zero as fast as we want.

Like in the proof of Theorem 1, to get (39) it suffices to show that

$$\alpha_n = \int_{(1-2^{-n}, 1)} (1 - t^{2^n}) dE_t g \rightarrow 0 \quad \text{a.e.}, \quad (40)$$

$$\beta_n = \int_{(0, 1-2^{-n})} t^{2^n} dE_t g \rightarrow 0 \quad \text{a.e.}, \quad (41)$$

$$\delta_n = \max_{1 \leq m < 2^n} |A^{2^n+m} - A^{2^n}| \rightarrow 0 \quad \text{a.e.} \quad (42)$$

Indeed, the system of conditions (40)–(42) is nothing but the extended Gaposhkin criterion for the a.e. convergence of $A^n g$. Thus, having (40)–(42) and applying the martingale convergence theorem we get (39).

We proceed to prove (40). Take a sequence $\{\pi_N\}$ of partition of $[0, 1]$ such that, for some $D < \infty$,

$$\left\| \left\{ \sum_{n=1}^N \left| I(f_n) - \pi_N(f_n) \right|^2 \right\}^{1/2} \right\|_p \leq D \quad (N = 1, 2, \dots),$$

with $f_n(t) = (1 - t^{2^n})\chi_{(1-2^{-n}, 1)}(t)$. Thus $\alpha_n = I(f_n)$, and we have

$$\begin{aligned} \left\| \left(\sum_{n=1}^N \alpha_n^2 \right)^{1/2} \right\|_p &\leq \left\| \left\{ \sum_{n=1}^N \left| \alpha_n - \pi_N(f_n) \right|^2 \right\}^{1/2} \right\|_p + \left\| \left\{ \sum_{n=1}^N (\pi_N(f_n))^2 \right\}^{1/2} \right\|_p \\ &\leq D + \left\| \left\{ \sum_{n=1}^N (\pi_N(f_n))^2 \right\}^{1/2} \right\|_p, \quad N = 1, 2, \dots \end{aligned}$$

It is enough to show that

$$\gamma_N = \left\| \left\{ \sum_{n=1}^N (\pi_N(f_n))^2 \right\}^{1/2} \right\|_p \leq C_p \|g\|_p, \quad \text{for } N = 1, 2, \dots \quad (43)$$

Let $\pi_N = (0 = t_0 < t_1 < \dots < t_m = 1)$. Writing

$$(1 - 2^{-n}, 1) = \bigcup_{k=0}^{\infty} I_{n,k} \quad \text{with} \quad I_{n,k} = (1 - 2^{-n-k}, 1 - 2^{-n-k-1}],$$

we readily get the estimate

$$\gamma_N \leq \sum_{k=0}^{\infty} \left\| \left\{ \sum_{n=1}^N \sum_{t_s \in I_{n,k}} (1 - \sigma_{2^n}(t_s))(E_{t_s} - E_{t_{s-1}})g \right\}^2 \right\|_p^{1/2}.$$

Now we can continue the proof imitating the procedure described in Section 3 (after formula (13)), the only difference that we consider $1 \leq n \leq N$ instead of $1 \leq n < \infty$, and $(E_{t_s} - E_{t_{s-1}})g$ instead of $\hat{h}_s(g)$. In this way, finally, we obtain (40).

In a similar way we show convergence (41), using the estimate $|\sigma_n(t)| \leq Cn^{-1}(1-t)^{-1}$.

Passing to the proof of (42) we first repeat the classical procedure of dyadic decomposition like in [11, § 2], coming to the conclusion that it suffices to show that

$$\left\| \left\{ \sum_{n=1}^N \delta_n^2 \right\}^{1/2} \right\|_p \leq C_p \sum_{\nu=1}^3 \|A_\nu\|_p \quad (N = 1, 2, \dots), \quad (44)$$

where

$$A_\nu = \left\{ \sum_{n=1}^N \sum_{k=1}^n k^2 \sum_{j=1}^{2^k} \left| \int_{\Gamma_\nu} R_{n,k,j}(t) dE_t g \right|^2 \right\}^{1/2}, \quad \nu = 1, 2, 3, \quad (45)$$

with Γ_ν and $R_{n,k,j}$ defined by (20) and (21) in Section 3. To prove (44) we show that

$$\|A_\nu\| \leq C_p \|g\|_p + D, \quad \nu = 1, 2, 3, \quad \text{for some } 0 < D < \infty \quad (46)$$

(the estimate being then uniform with respect to N).

To this end we first approximate the integrals $\int_{\Gamma_\nu} R_{n,k,j}(t) dE_t g$ by their Riemann–Stieltjes sums by taking the suitable partition π_N of $[0, 1]$. Namely, for any positive integer N , we fix π_N such that

$$\left\| \left\{ \sum_{n=1}^N \sum_{k=1}^n k^2 \sum_{j=1}^{2^k} \left| \pi_N(R_{n,k,j} \chi_{\Gamma_\nu}) - I(R_{n,k,j} \chi_{\Gamma_\nu}) \right|^2 \right\}^{1/2} \right\|_p \leq D < \infty, \quad (47)$$

for $N = 1, 2, \dots$, $\nu = 1, 2, 3$. For a fixed N , let $\pi_N = (t_0 < \dots < t_m)$. Then, for A_ν defined by (45), we have the estimate

$$\|A_\nu\|_p \leq \left\| \left\{ \sum_{n=1}^N \sum_{k=1}^n k^2 \sum_{j=1}^{2^k} \left| \sum_{t_s \in \Gamma_\nu} R_{n,k,j}(t_s) (E_{t_s} - E_{t_{s-1}}) g \right|^2 \right\}^{1/2} \right\|_p + D, \quad (48)$$

for $\nu = 1, 2, 3$, $N = 1, 2, \dots$.

The rest of the proof is in fact a repetition of the argument in Section 3 after formula (21), with the replacing $\widehat{h}_s(g)$ by $(E_{t_s} - E_{t_{s-1}})g$. In this way we finally show that $\{A^n g\}$ converges a.e. Theorem 3 is proved.

6. Final remarks. From Theorem 3 it follows that for the *centered position operator* A defined in (34), the sequence of powers $\{A^n g\}$ converges a.e., for each $g \in L^p[0, 1]$, $1 < p < \infty$. As it has already been noticed Theorem 3 is closely related to the results of Gaposhkin [12] and Burkholder and Chow [6], in case $p = 2$. In connection with Stein's theorem [18] saying that $(P^n f)$ converges a.e. for P a positive and positive definite contraction in L^2 , let us note that the operator A in L^p appearing in Theorem 3 is not positive, and is not a contraction, in general. Elementary examples of this kind are also given by Theorem 1 (and Theorem 1^{bis}).

Acknowledgments. The author wishes to thank very much the referee for his valuable suggestions, which led to essential improvement of the first version of the paper. In particular, the proofs of Theorem 1 and Theorem 3 have been simplified.

REFERENCES

1. Akcoglu M. A. A pointwise theorem in L^p -spaces. — Canad. J. Math., 1975, v. 27, № 5, p. 1075–1082.
2. Берг Й., Лефстрем Й. Интерполяционные пространства. М.: Мир, 1980, 264 с.
3. Berkson E., Bourgain J., Gillespie T. A. On the almost everywhere convergence of ergodic averages for power-bounded operators on L^p -subspaces. — Integral Equations Operator Theory, 1991, v. 14, № 5, p. 678–715.
4. Berkson E., Gillespie T. A. Stečkin's theorem, transference, and spectral decompositions. — J. Funct. Anal., 1987, v. 70, № 1, p. 140–170.

5. *Burkholder D. L.* Martingale transforms. — *Ann. Math. Statist.*, 1966, v. 37, p. 1494–1504.
6. *Burkholder D. L., Chow Y. S.* Iterates of conditional expectation operators. — *Proc. Amer. Math. Soc.*, 1961, v. 12, p. 490–495.
7. *Dowson R. H.* Spectral Theory of Linear Operators. New York: Academic Press, 1978, 422 p. (London Math. Soc. Monogr., v. 12.)
8. *Duncan R.* Some pointwise convergence results in $L^p(\mu)$, $1 < p < \infty$. — *Canad. Math. Bull.*, 1977, v. 20, p. 277–284.
9. *Данфорд Н., Шварц Дж.* Линейные операторы. Ч. 1: Общая теория. М.: ИЛ, 1962, 895 с.
10. *Edwards R. E., Gaudry G. I.* Littlewood–Paley and Multiplier Theory. Berlin: Springer-Verlag, 1977, 212 p. (Ergeb. Math. Grenzgeb., v. 90.)
11. *Гапошкин В. Ф.* Критерии усиленного закона больших чисел для классов стационарных в широком смысле процессов и однородных случайных полей. — *Теория вероятн. и ее примен.*, 1977, т. 22, в. 2, с. 295–319.
12. *Гапошкин В. Ф.* Об индивидуальной эргодической теореме для нормальных операторов в L_2 . — *Функц. анализ и его прил.*, 1981, т. 15, № 1, с. 18–22.
13. *Garsia A. M.* Topics in Almost Everywhere Convergence. Chicago: Markham, 1970, 154 p.
14. *Houdré C.* On the spectral SLLN and pointwise ergodic theorem in L^α . — *Ann. Probab.*, 1992, v. 20, № 4, p. 1731–1753.
15. *Jajte R.* On the existence of the ergodic Hilbert transform. — *Ann. Probab.*, 1987, v. 15, № 2, p. 831–835.
16. *Katznelson Y.* An Introduction to Harmonic Analysis. — New York: Dover, 1976, 264 p.
17. *Millar P. W.* Martingale integrals. — *Trans. Amer. Math. Soc.*, 1968, v. 133, p. 145–166.
18. *Stein E. M.* Topics in Harmonic Analysis Related to the Littlewood–Paley Theory. Princeton: Princeton Univ. Press, 1970, 146 p. (Ann. of Math. Stud., v. 63.)
19. *Зигмунд А.* Тригометрические ряды. Т. 1. М.: Мир, 1965, 615 с.

Поступила в редакцию
21.XI.2005

Исправленный вариант
22.II.2007