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CSÖRGŐ S.^{*,**}, MEGYESI Z.^{*}MERGING TO SEMISTABLE LAWS¹⁾

Показывается, что хотя асимптотические распределения в обычном смысле не существуют, функции распределения подходящим образом центрированных и нормированных частичных сумм независимых одинаково распределенных случайных величин из области геометрического частичного притяжения равномерно «состремляются» с семейством полуустойчивых функций распределения. Более того, даже и соответствующие, возможно, мягко урезанные, суммы состремляются. Аналогичный результат с состремлением также имеет силу для экстремальных наблюдений. Основной результат иллюстрируется на примере обобщенных Петербургских игр.

Ключевые слова и фразы: полуустойчивые законы, области геометрического частичного притяжения, состремление, мягко урезанные суммы, экстремальные наблюдения, обобщенные Петербургские игры.

1. Semistable laws and their domains of geometric partial attraction. We begin with a probabilistic representation of infinitely divisible real random variables and their «trimmed» versions from [7] to be used throughout along with the integral conventions: for a left-continuous function $l(\cdot)$ and a right-continuous function $r(\cdot)$ we write $\int_a^b r(x) dl(x) = \int_{[a,b)} r(x) dl(x)$ and $\int_a^b l(x) dr(x) = \int_{(a,b]} l(x) dr(x)$ whenever these integrals make sense for the finite constants $a < b$ as Lebesgue–Stieltjes integrals, in which case the usual formula holds for integration by parts.

Let Ψ be the class of all nonpositive, nondecreasing, right-continuous functions $\psi(\cdot)$ defined on the positive half-line $(0, \infty)$ such that the integrals $\int_\varepsilon^\infty \psi^2(s) ds < \infty$ for all $\varepsilon > 0$. Let $E_1^{(j)}, E_2^{(j)}, \dots, j = 1, 2$, be two independent sequences of independent exponentially distributed random variables with mean 1, and with the respective partial sums $Y_n^{(j)} = E_1^{(j)} + \dots + E_n^{(j)}$

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as jump points, consider the standard left-continuous independent Poisson processes

$$N_j(u) = \sum_{n=1}^{\infty} I(Y_n^{(j)} < u), \quad 0 \leq u < \infty, \quad j = 1, 2,$$

where $I(\cdot)$ is the indicator function. For a function $\psi \in \Psi$, define the random variables

$$W_j(\psi) = \int_1^{\infty} [N_j(s) - s] d\psi(s) + \int_0^1 N_j(s) d\psi(s) - \psi(1), \quad j = 1, 2,$$

and, also for $j = 1, 2$ and for an integer $l \geq 0$,

$$\begin{aligned} W_j^{(l)}(\psi) &= \int_{Y_{l+1}^{(j)}}^{\infty} [N_j(s) - s] d\psi(s) - \int_1^{Y_{l+1}^{(j)}} s d\psi(s) + l\psi(Y_{l+1}^{(j)}) \\ &\quad - \int_1^{l+1} \psi(s) ds - \psi(1), \end{aligned}$$

where the first integrals are almost surely well defined, by the condition that $\psi \in \Psi$, as improper Riemann integrals. It is easy to see that $W_j^{(0)}(\psi) = W_j(\psi)$, $j = 1, 2$. Let Z be a standard normal random variable such that $N_1(\cdot)$, Z , and $N_2(\cdot)$ are independent, and for $\psi_1 \in \Psi$ and $\psi_2 \in \Psi$ and a finite constant $\sigma \geq 0$ finally introduce the random variables

$$\begin{aligned} V(\psi_1, \psi_2, \sigma) &= -W_1(\psi_1) + \sigma Z + W_2(\psi_2), \\ V_{l,m}(\psi_1, \psi_2, \sigma) &= -W_1^{(l)}(\psi_1) + \sigma Z + W_2^{(m)}(\psi_2), \end{aligned} \quad (1.1)$$

where l and m are nonnegative integers, so that we have $V_{0,0}(\psi_1, \psi_2, \sigma) = V(\psi_1, \psi_2, \sigma)$. Considering then the constant, also for $\psi \in \Psi$,

$$\theta(\psi) = \int_0^1 \frac{\psi(s)}{1 + \psi^2(s)} ds - \int_1^{\infty} \frac{\psi^3(s)}{1 + \psi^2(s)} ds$$

and letting $W(\psi_1, \psi_2, \sigma) = V(\psi_1, \psi_2, \sigma) + \theta(\psi_2) - \theta(\psi_1)$, by Theorem 3 in [7] this variable has characteristic function

$$\begin{aligned} \mathbf{E}(e^{itW(\psi_1, \psi_2, \sigma)}) &= \exp \left\{ -\frac{\sigma^2}{2} t^2 + \int_{-\infty}^0 \left(e^{itx} - 1 - \frac{itx}{1+x^2} \right) dL(x) \right. \\ &\quad \left. + \int_0^{\infty} \left(e^{itx} - 1 - \frac{itx}{1+x^2} \right) dR(x) \right\} \end{aligned} \quad (1.2)$$

for all $t \in \mathbf{R}$, where $L(x) = \inf\{s > 0: \psi_1(s) \geq x\}$, $x < 0$, and $R(x) = -\inf\{s > 0: \psi_2(s) \geq -x\}$, $x > 0$. Here $L(\cdot)$ is left-continuous and nondecreasing on $(-\infty, 0)$ with $L(-\infty) = 0$ and $R(\cdot)$ is right-continuous and nondecreasing on $(0, \infty)$ with $R(\infty) = 0$, and we have $\int_{-\varepsilon}^0 x^2 dL(x) + \int_0^{\varepsilon} x^2 dR(x) < \infty$ for every $\varepsilon > 0$ since $\psi_1, \psi_2 \in \Psi$. Thus $V(\psi_1, \psi_2, \sigma)$ is

infinitely divisible by Lévy's formula; see [14] for example. Conversely, given the right side of (1.2) with $L(\cdot)$ and $R(\cdot)$ having the properties just listed, the variable $W(\psi_1, \psi_2, \sigma)$ has this characteristic function again if we choose $\psi_1(s) = \inf\{x < 0: L(x) > s\}$ and $\psi_2(s) = \inf\{x < 0: -R(-x) > s\}$, $s > 0$, for then $\psi_1, \psi_2 \in \Psi$. The characteristic functions of the «trimmed» versions $V_{l,m}(\psi_1, \psi_2, \sigma)$ are in the proof of Lemma 2 below.

Thus, modulo a location constant, the class $\mathcal{J}$ of all nondegenerate infinitely divisible distributions can be identified with the class of triplets $\{(\psi_1, \psi_2, \sigma) \neq (0, 0, 0): \psi_1, \psi_2 \in \Psi, \sigma \geq 0\}$. Then F being in the domain of partial attraction of a $G = G_{\psi_1, \psi_2, \sigma} \in \mathcal{J}$, written $F \in \mathbb{D}_p(G)$, means that there exist a subsequence $\{k_n\}_{n=1}^\infty \subset \mathbf{N}$ and centering and norming constants $C_{k_n} \in \mathbf{R}$ and $A_{k_n} > 0$ such that

$$\frac{1}{A_{k_n}} \left\{ \sum_{j=1}^{k_n} X_j - C_{k_n} \right\} \xrightarrow{\mathcal{D}} V(\psi_1, \psi_2, \sigma), \quad (1.3)$$

where, with convergence meant to hold as $n \rightarrow \infty$ unless otherwise specified, $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution and $G_{\psi_1, \psi_2, \sigma}$ is the distribution function of the random variable $V(\psi_1, \psi_2, \sigma)$ from (1.1). By classical theory (see in [14] or Corollaries 1 and 3 in [7]) this happens for $\{k_n\} = \{n\} = \mathbf{N}$ if and only if either $(\psi_1, \psi_2, \sigma) = (0, 0, \sigma)$ for some $\sigma > 0$, in which case F is in the domain of attraction of the normal distribution, written $F \in \mathbb{D}(2)$, or $(\psi_1, \psi_2, \sigma) = (m_1\psi^\alpha, m_2\psi^\alpha, 0)$ for some constants $\alpha \in (0, 2)$, $m_1, m_2 \geq 0$, $m_1 + m_2 > 0$, where $\psi^\alpha(s) = -s^{-1/\alpha}$, $s > 0$, in which case F is in the domain of attraction of a stable distribution of exponent α , written $F \in \mathbb{D}(\alpha)$. (The superscript α in ψ^α , and in ψ_1^α and ψ_2^α beginning with (1.4) below, is meant as a label, not as a power exponent.) The normal law being the stable law of exponent 2, let $\mathcal{S}$ denote the class of all stable laws.

Lévy [18] introduced the class $\mathcal{S}_* \subset \mathcal{S}$ of semistable laws by extending a defining property of stable characteristic functions and, as translated into the framework of the present description of infinitely divisible laws, showed that $G_{\psi_1, \psi_2, \sigma} \in \mathcal{S}_*$ if and only if either $(\psi_1, \psi_2, \sigma) = (0, 0, \sigma)$ for some $\sigma > 0$, giving the normal distribution as a semistable distribution of exponent 2, or $(\psi_1, \psi_2, \sigma) = (\psi_1^\alpha, \psi_2^\alpha, 0)$, where

$$\psi_j^\alpha(s) = M_j(s) \psi^\alpha(s) = -M_j(s) s^{-1/\alpha}, \quad s > 0, \quad j = 1, 2, \quad (1.4)$$

for some $\alpha \in (0, 2)$, defining a semistable law of exponent $\alpha \in (0, 2)$, where M_1 and M_2 are nonnegative, right-continuous functions on $(0, \infty)$, either identically zero or bounded away from both zero and infinity, such that $M_1 + M_2$ is not identically zero, the functions $M_j(\cdot)\psi^\alpha(\cdot)$ are nondecreasing and $M_j(cs) = M_j(s)$ for all $s > 0$, $j = 1, 2$, for some constant $c > 1$; the latter property will be referred to as multiplicative periodicity

with period c . For $\alpha \in (0, 2)$, Lévy's original description of the property in (1.4) in terms of L and R in (1.2) is that there exist nonnegative bounded functions $M_L(\cdot)$ on $(-\infty, 0)$ and $M_R(\cdot)$ on $(0, \infty)$, one of which has a strictly positive infimum and the other one either has a strictly positive infimum or is identically zero, such that $L(x) = M_L(x)/|x|^\alpha$, $x < 0$, is left-continuous and nondecreasing on $(-\infty, 0)$ and $R(x) = -M_R(x)/x^\alpha$, $x > 0$, is right-continuous and nondecreasing on $(0, \infty)$, while $M_L(c^{1/\alpha}x) = M_L(x)$ for all $x < 0$ and $M_R(c^{1/\alpha}x) = M_R(x)$ for all $x > 0$, for the same period $c > 1$. Because of the inversions given above, the two descriptions are equivalent.

The realization of an apparent significance of $\mathcal{S}_* \supset \mathcal{S}$ starts with a remark of Doeblin [13], without any elaboration or, for that matter, even a precise statement, to the effect that semistable laws arise in the limit in (1.3) if the normalizing constants A_{k_n} satisfy a geometric growth condition. Thirty years later, Shimizu [23] and Pillai [22] came close while Kruglov [17] and Mejlzler [21] fully achieved that realization, all four of them acting independently of one another. It turned out that the following *Characterization Theorem* is true: If (1.3) holds along a subsequence $\{k_n\} \subset \mathbf{N}$ for which $\liminf_{n \rightarrow \infty} k_{n+1}/k_n = c$ for some $c \in (1, \infty)$, then the distribution $G_{\psi_1, \psi_2, \sigma}$ of $V(\psi_1, \psi_2, \sigma)$ is in $\mathcal{S}_*$ such that, in the case when the exponent of $G_{\psi_1, \psi_2, \sigma} = G_{\psi_1^\alpha, \psi_2^\alpha, 0}$ is $\alpha < 2$, the multiplicative period of the functions M_1 and M_2 in (1.4) is the c from this growth condition on $\{k_n\}$. Conversely, for every $G_{\psi_1, \psi_2, \sigma} \in \mathcal{S}_*$ there exists an F such that if $X_1, X_2, \dots$ are independent random variables with the common distribution function F , then (1.3) holds along a subsequence $\{k_n\} \subset \mathbf{N}$ for which

$$\lim_{n \rightarrow \infty} \frac{k_{n+1}}{k_n} = c \quad \text{for some } c \in [1, \infty). \quad (1.5)$$

An equivalent form of this theorem was proved by Kruglov [17] and Mejlzler [21] in terms of the Lévy type description of $\mathcal{S}_*$, while the present version was obtained by Megyesi [20] with an independent proof within the framework of the «probabilistic» or «quantile-transform» approach of Csörgő, Haeusler and Mason [7]–[8] and Csörgő [5] to domains of attraction and partial attraction.

For $G = G_{\psi_1, \psi_2, \sigma} \in \mathcal{S}_*$, we say that F is in the *domain of geometric partial attraction of G with rank $c \geq 1$* , in short $F \in \mathbb{D}_{\text{gp}}^{(c)}(G)$, if (1.3) holds along a subsequence $\{k_n\} \subset \mathbf{N}$ satisfying (1.5). Letting $\mathbf{Z}$ be the set of all integers and $[y] = \max\{k \in \mathbf{Z} : k \leq y\}$ be the integer part of $y \in \mathbf{R}$, the geometric subsequence $k_n \equiv [c^n]$ of course satisfies the (quasi)geometric growth condition (1.5) if $c > 1$. Recalling that $(\psi_1, \psi_2, \sigma) \neq (0, 0, 0)$ for $G = G_{\psi_1, \psi_2, \sigma} \in \mathcal{S}_*$, define $\mathbf{c} = \mathbf{c}(G_{0,0,\sigma}) = 1$ for any $\sigma > 0$ and $\mathbf{c} = \mathbf{c}(G_{\psi_1^\alpha, \psi_2^\alpha, 0}) = \inf\{c > 1 : M_j(cs) = M_j(s), s > 0, j = 1, 2\}$, the minimal common period c (among those greater than 1) of the factor functions M_1 and M_2 in ψ_1^α and ψ_2^α in (1.4) for $\alpha \in (0, 2)$. It turns out for the whole

domain $\mathbb{D}_{\text{gp}}(G) = \bigcup_{c \geq 1} \mathbb{D}_{\text{gp}}^{(c)}(G)$ of geometric partial attraction of $G \in \mathcal{S}_*$ that $\mathbb{D}_{\text{gp}}(G) = \bigcap_{m \in \mathbf{N}} \mathbb{D}_{\text{gp}}^{(c^m)}(G) = \mathbb{D}_{\text{gp}}^{(c)}(G)$. Also, if $\mathbf{c}(G) = 1$ for $G \in \mathcal{S}_*$, then $G \in \mathcal{S}$ and $\mathbb{D}_{\text{gp}}(G) = \mathbb{D}(G)$, the domain of attraction of the stable G . In other words, if $\mathbb{D}(\mathcal{S}) = \bigcup_{G \in \mathcal{S}} \mathbb{D}(G) = \bigcup_{0 < \alpha \leq 2} \mathbb{D}(\alpha)$ is the classical domain of attraction and $\mathbb{D}_{\text{gp}}(\mathcal{G}) = \bigcup_{G \in \mathcal{G}} \mathbb{D}_{\text{gp}}(G)$ is the domain of geometric partial attraction of a class $\mathcal{G} \subset \mathcal{S}_*$, then $\mathbb{D}_{\text{gp}}(\mathcal{S}) = \mathbb{D}(\mathcal{S})$. Some of these results were first proved by Mejlzer [21], all of them and related other observations are in Megyesi [20].

The first characterization of an $F \in \mathbb{D}_{\text{gp}}(\mathcal{S}_*)$ was obtained by Grinevich and Khokhlov [15]. It is in terms of the norming factors A_{k_n} in (1.3) and the tails of F , and a small oversight in it must be corrected. This is done in [20] and is also given following (2.9) below. However, an alternative characterization, also due to Megyesi [20], in terms of the quantile function

$$Q(s) = \inf \{x \in \mathbf{R}: F(x) \geq s\}, \quad 0 < s < 1,$$

of the underlying distribution serves our aims better here. It goes as follows.

Consider a subsequence $\{k_n\}_{n=1}^\infty \subset \mathbf{N}$ satisfying (1.5). If $c = 1$ in (1.5), then put $\gamma(s) = 1$, $0 < s < 1$. If $c > 1$, then the sequence $\{k_n\}$ is eventually strictly increasing. So for all $s \in (0, s_0)$, with $s_0 \in (0, 1]$ small enough, there exists a unique $k_{n^*(s)}$ such that $k_{n^*(s)}^{-1} \leq s < k_{n^*(s)-1}^{-1}$. We define $\gamma(s) = sk_{n^*(s)}$ for $s \in (0, s_0)$ and $\gamma(s) = 1$ for $s \in [s_0, 1)$, so that $1 \leq \gamma(s) < c + \varepsilon$ for any fixed $\varepsilon > 0$ and all $s \in (0, 1)$ small enough for the limiting $c \geq 1$ from (1.5). Let $Q_+(\cdot)$ denote the right-continuous version of the quantile function $Q(\cdot)$ of $F(\cdot)$. Since $\mathbb{D}_{\text{gp}}(G) = \mathbb{D}(G)$ for a normal $G \in \mathcal{S}_*$, we only have to describe the domain of geometric partial attraction of nonnormal semistable laws, for which the *Domain Theorem* is this: If $G_{\psi_1^\alpha, \psi_2^\alpha, 0} \in \mathcal{S}_*$ is semistable with exponent $\alpha \in (0, 2)$, so that ψ_1^α and ψ_2^α satisfy (1.4), and $F \in \mathbb{D}_{\text{gp}}(G_{\psi_1^\alpha, \psi_2^\alpha, 0})$ such that (1.3) holds for $V(\psi_1^\alpha, \psi_2^\alpha, 0)$ and a subsequence $\{k_n\}_{n=1}^\infty \subset \mathbf{N}$ satisfying (1.5), then for all $s \in (0, 1)$,

$$\begin{aligned} Q_+(s) &= -s^{-1/\alpha} \ell(s) \left[M_1(\gamma(s)) + h_1(s) \right], \\ Q(1-s) &= s^{-1/\alpha} \ell(s) \left[M_2(\gamma(s)) + h_2(s) \right] \end{aligned} \tag{1.6}$$

for the same $\alpha \in (0, 2)$, where $\ell(\cdot)$ is a right-continuous function, slowly varying at zero, and the errors h_1 and h_2 are right-continuous functions such that if M_j is continuous, then $\lim_{s \downarrow 0} h_j(s) = 0$ for the corresponding h_j , while if M_j has discontinuities, then the corresponding $h_j(s)$ may not go to zero as $s \downarrow 0$ but $\lim_{n \rightarrow \infty} h_j(t/k_n) = 0$ for every continuity point $t > 0$ of M_j , $j = 1, 2$. Conversely, if for the quantile function pertaining to F the equations in (1.6) hold with the properties of ℓ and of h_1 and h_2 just described, for some $\alpha \in (0, 2)$ and functions M_1 and M_2 satisfying the properties imparted at (1.4), and for $\gamma(\cdot)$ determined by a given subsequence $\{k_n\}_{n=1}^\infty \subset \mathbf{N}$

satisfying (1.5), then $F \in \mathbb{D}_{\text{gp}}(G_{\psi_1^\alpha, \psi_2^\alpha, 0})$ for the ψ_1^α and ψ_2^α given by (1.4), and, in particular, the relation (1.3) can be specified as

$$S(k_n) = \frac{\sum_{j=1}^{k_n} X_j - k_n \int_{1/k_n}^{1-1/k_n} Q(u) du}{k_n^{1/\alpha} \ell(1/k_n)} \xrightarrow{\mathcal{D}} V(\psi_1^\alpha, \psi_2^\alpha, 0). \quad (1.7)$$

Finally, $F \in \mathbb{D}_{\text{gp}}^{(\mathbf{c})}(G_{\psi_1^\alpha, \psi_2^\alpha, 0})$ for some $\alpha \in (0, 2)$ and $\mathbf{c} = \mathbf{c}(G_{\psi_1^\alpha, \psi_2^\alpha, 0}) > 1$

$$\begin{aligned} &\text{such that (1.3) holds for the sequence } k_n \equiv \lfloor \mathbf{c}^n \rfloor \\ &\text{if and only if } Q_+(s) = -s^{-1/\alpha} \ell(s) [M_1(s) + h_1(s)] \\ &\text{and } Q(1-s) = s^{-1/\alpha} \ell(s) [M_2(s) + h_2(s)], \quad s \in (0, 1), \\ &\text{where } h_1(\cdot), h_2(\cdot) \text{ and } \ell(\cdot) \text{ are as in (1.6),} \end{aligned} \quad (1.8)$$

in which case (1.7) also holds for $k_n \equiv \lfloor \mathbf{c}^n \rfloor$; that is, for $k_n \equiv \lfloor \mathbf{c}^n \rfloor$ we can dispense with $\gamma(s) = s \lfloor \mathbf{c}^{\lceil \log_{\mathbf{c}} \lceil 1/s \rceil \rceil} \rfloor$ in (1.6), where $\log_{\mathbf{c}}$ is the logarithm to the base $\mathbf{c}$.

One cannot in general replace k_n by n in (1.7), the resulting $\{S(n)\}_{n=1}^\infty$ may not converge in distribution. The main goal of this note is to show that the distribution functions of $S(n)$ nevertheless merge together, in fact uniformly, with the distribution functions of members of a family of semistable random variables around $V(\psi_1^\alpha, \psi_2^\alpha, 0)$. This is in fact obtained as a special case of a more general merge result in Theorem 2 in the next section for possibly lightly trimmed sums, following Theorem 1 of related interest on stochastic compactness; note that the effect of light trimming was first analyzed by Darling [12] and by Arov and Bobrov [4] when $F \in \mathbb{D}(\alpha)$ for some $\alpha \in (0, 2)$. Analogous merging phenomena for sample extremes are in Theorem 3. The results are discussed in Section 2, all the proofs are placed in Section 4, following a class of what we believe interesting examples.

Merging together, in the distributional sense, of properly centered and normalized sums and suitably chosen members of their subsequential limits was first discovered by Csörgő [6] for the cumulative gains in a series of St. Petersburg games; the merge theorem there is the basis for resolving the St. Petersburg paradox. The present note is an attempt to understand the general essence of the merging phenomenon.

2. Merging sums and extremes. Retaining the notation of the preceding section completely, for fixed choices of integers $l, m \geq 0$ the sequence of the lightly trimmed or whole (if $l = 0 = m$) sums $\sum_{l+1}^{n-m} X_{j,n}$, $n \in \mathbf{N}$, where $X_1, X_2, \dots$ are independent random variables with a common distribution function F , is said to be stochastically compact, when we write $F \in \text{SC}(l, m)$, if there exist a sequence of normalizing constants $A_n > 0$ and a sequence of centering constants $C_n \in \mathbf{R}$ such that for every subsequence

$\{n_k\}_{k=1}^\infty \subset \mathbf{N}$ there exists a further subsequence $\{n_{k_r}\}_{r=1}^\infty \subset \{n_k\}_{k=1}^\infty$ such that

$$\frac{1}{A_{n_{k_r}}} \left\{ \sum_{j=l+1}^{n_{k_r}-m} X_{j,n_{k_r}} - C_{n_{k_r}} \right\} \xrightarrow{\mathcal{D}} W_{l,m} \quad \text{as } r \rightarrow \infty, \quad (2.1)$$

for a nondegenerate random variable $W_{l,m}$, depending in general on the sequence $\{n_{k_r}\}$. Since $\mathbb{D}_{\text{gp}}(G_{0,0,\sigma}) = \mathbb{D}(2)$ for any $\sigma > 0$ and $\mathbb{D}(2)$ is well known by classical theory, we henceforth assume that $F \notin \mathbb{D}(2)$.

To state the main results, for $\psi \in \Psi$ and $\lambda > 0$ let ${}_\lambda\psi(s) = \psi(\lambda s)$, and put $\psi_j^{\alpha,\lambda}(s) = \lambda^{1/\alpha} {}_\lambda\psi_j^\alpha(s) = -M_j(\lambda s)s^{-1/\alpha}$, $s > 0$, for the $\psi_j^\alpha(\cdot)$ in (1.4), $j = 1, 2$. Then, with the random variables taken from (1.1), define $W_{1,\alpha,\lambda}^{(l)}(M_1) = W_1^{(l)}(\psi_1^{\alpha,\lambda})$, $W_{2,\alpha,\lambda}^{(m)}(M_2) = W_2^{(m)}(\psi_2^{\alpha,\lambda})$ and finally $V_{\alpha,\lambda}^{l,m}(M_1, M_2) = V_{l,m}(\psi_1^{\alpha,\lambda}, \psi_2^{\alpha,\lambda}, 0)$, and notice the identities $V_{\alpha,\lambda}^{l,m}(M_1, 0) = W_{1,\alpha,\lambda}^{(l)}(M_1) = V_{\alpha,\lambda}^{l,0}(M_1, 0)$, $V_{\alpha,\lambda}^{l,m}(0, M_2) = W_{2,\alpha,\lambda}^{(m)}(M_2) = V_{\alpha,\lambda}^{0,m}(0, M_2)$ and

$$V_{\alpha,\lambda}^{l,m}(M_1, M_2) = W_{2,\alpha,\lambda}^{(m)}(M_2) - W_{1,\alpha,\lambda}^{(l)}(M_1) = \lambda^{1/\alpha} V_{l,m}({}_\lambda\psi_1^\alpha, {}_\lambda\psi_2^\alpha, 0) \quad (2.2)$$

for all choices of the integers $l, m \geq 0$. Let $\stackrel{\mathcal{D}}{=}$ denote equality in distribution.

That $\mathbb{D}_{\text{gp}}(\mathcal{S}_*) \subset \bigcap_{l,m \geq 0} \mathbb{SC}(l, m)$, which is the content of Theorem 1(i) below, can be seen by checking from (1.6) the necessary and sufficient condition (1.42a) of Corollary 10 in [7] for $F \in \mathbb{SC}$, from which corollary we also know that $\bigcap_{l,m \geq 0} \mathbb{SC}(l, m) = \mathbb{SC}(0, 0)$. Theorem 1(ii), however, says much more: it is a necessary and sufficient condition for convergence in distribution along any given subsequence, from which part (i) comes in a trivial fashion. For this the full force of the assumption $F \in \mathbb{D}_{\text{gp}}(G_{\psi_1^\alpha, \psi_2^\alpha, 0})$ is needed for some $\alpha \in (0, 2)$ as described by the domain theorem at (1.6) and by the structural details concerning this attracted domain between (1.5) and (1.6). In particular, it is no loss of generality to assume that $F \in \mathbb{D}_{\text{gp}}^{(\mathbf{c})}(G_{\psi_1^\alpha, \psi_2^\alpha, 0})$ for the minimal common period $\mathbf{c} = \mathbf{c}(G_{\psi_1^\alpha, \psi_2^\alpha, 0}) \geq 1$, that is, that (1.7) holds for $\{k_n\}_{n=1}^\infty \subset \mathbf{N}$ for which $k_{n+1}/k_n \rightarrow \mathbf{c}$, and it is then this sequence $\{k_n\}$ through which the function $\gamma(\cdot)$ in (1.6) is defined.

Theorem 1. Suppose that $F \in \mathbb{D}_{\text{gp}}^{(\mathbf{c})}(G_{\psi_1^\alpha, \psi_2^\alpha, 0})$ for a semistable distribution $G_{\psi_1^\alpha, \psi_2^\alpha, 0}$ of exponent $\alpha \in (0, 2)$.

(i) Then $F \in \mathbb{SC}(l, m)$ for every pair of nonnegative integers (l, m) , and the norming and centering constants for (2.1) may be chosen as $A_n = n^{1/\alpha} \ell(1/n)$ and $C_n = n \int_{(l+1)/n}^{1-(m+1)/n} Q(s) ds$, $n \in \mathbf{N}$, where $\ell(\cdot)$ is from (1.6).

(ii) Suppose that for a subsequence $\{n_r\}_{r=1}^\infty \subset \mathbf{N}$ and norming and centering constants $A_{n_r} > 0$ and C_{n_r} ,

$$\frac{1}{A_{n_r}} \left\{ \sum_{j=1}^{n_r} X_{j,n_r} - C_{n_r} \right\} \xrightarrow{\mathcal{D}} W_{0,0}, \quad \text{as } r \rightarrow \infty, \quad (2.3)$$

for a nondegenerate random variable $W_{0,0}$. Then, necessarily, the sequence $\{\gamma(1/n_r)\}$ either converges to some $\kappa \in [1, \mathbf{c}]$, as $r \rightarrow \infty$, or has exactly two limit points, 1 and $\mathbf{c}$ (in which case we define $\kappa = 1$). With $V_{\alpha,\kappa}^{0,0}(M_1, M_2)$ as in (2.2), the limiting random variable necessarily satisfies $W_{0,0} \stackrel{\mathcal{D}}{=} \delta V_{\alpha,\kappa}^{0,0}(M_1, M_2) + d$, where

$$\delta = \lim_{r \rightarrow \infty} \frac{n_r^{1/\alpha} \ell(1/n_r)}{A_{n_r}} > 0 \quad \text{and} \quad d = \lim_{r \rightarrow \infty} \frac{n_r \int_{1/n_r}^{1-1/n_r} Q(s) ds - C_{n_r}}{A_{n_r}}. \quad (2.4)$$

Conversely, if for a subsequence $\{n_r\}_{r=1}^\infty \subset \mathbf{N}$ the sequence $\{\gamma(1/n_r)\}$ either converges to some $\kappa \in [1, \mathbf{c}]$, as $r \rightarrow \infty$, or has exactly two limit points, 1 and $\mathbf{c}$ (in which case we put $\kappa = 1$), then, with $V_{\alpha,\kappa}^{l,m}(M_1, M_2)$ as in (2.2),

$$\frac{1}{n_r^{1/\alpha} \ell(1/n_r)} \left\{ \sum_{j=l+1}^{n_r-m} X_{j,n_r} - n_r \int_{(l+1)/n_r}^{1-(m+1)/n_r} Q(s) ds \right\} \\ \xrightarrow{\mathcal{D}} V_{\alpha,\kappa}^{l,m}(M_1, M_2), \quad (2.5)$$

as $r \rightarrow \infty$, for every pair (l, m) of nonnegative integers.

While true, the theorem is trivial for $\mathbf{c} = 1$ since $\mathbb{D}_{\text{gp}}^{(1)}(G_{\psi_1^\alpha, \psi_2^\alpha, 0}) = \mathbb{D}(\alpha)$ and (2.5) in fact holds by Corollary 3 in [7] along the whole $\mathbf{N}$ for any integers $l, m \geq 0$. For $\mathbf{c} > 1$, (2.5) holds for all pairs (l, m) of nonnegative integers along a given $\{n_r\}_{r=1}^\infty \subset \mathbf{N}$ if and only if $\lim_{r \rightarrow \infty} e^{[2\gamma(1/n_r) - \mathbf{c} - 1]\pi i / [\mathbf{c} - 1]} = e^{[2\kappa - \mathbf{c} - 1]\pi i / [\mathbf{c} - 1]}$ for some $\kappa \in [1, \mathbf{c}]$; convergence on the unit circle, the point $e^{-\pi i} = e^{\pi i}$ representing both 1 and $\mathbf{c}$.

An immediate consequence for the case $(l, m) = (0, 0)$, a version of which was first obtained by Mejlzer [21], is that if $F \in \mathbb{D}_{\text{gp}}(G_{\psi_1^\alpha, \psi_2^\alpha, 0})$ for a semistable $G_{\psi_1^\alpha, \psi_2^\alpha, 0}$ of index $\alpha \in (0, 2)$ given by (1.4), then the class of all nondegenerate subsequential limiting distributions of suitably centered and normalized full sums from F is contained in the family of distributions of the random variables $\{\delta V_{\alpha,\lambda}^{0,0}(M_1, M_2) + d : \lambda \geq 1, \delta > 0, d \in \mathbf{R}\}$. This implies that if $\mathbb{D}_{\text{gp}}(G_1) \cap \mathbb{D}_{\text{gp}}(G_2) \neq \emptyset$ for $G_1, G_2 \in \mathcal{S}_*$, then $\mathbb{D}_{\text{gp}}(G_1) = \mathbb{D}_{\text{gp}}(G_2)$.

The next corollary is also immediate. As an analogue of the classical convergence of types theorem [14, §10], it was recently proved by Meerschaert and Scheffler [19].

Corollary. Suppose that (1.3) holds for a nonstable semistable limit and a subsequence $\{k_n\}$ for which $\lim_{n \rightarrow \infty} k_{n+1}/k_n = c > 1$. If (1.3) also holds, for a nondegenerate limit, along another subsequence $\{k'_n\}$ such that $\lim_{n \rightarrow \infty} k'_{n+1}/k'_n = c$ for the same c , then $\lim_{n \rightarrow \infty} k'_n/k_n = \lambda$ and $\lim_{n \rightarrow \infty} A_{k'_n}/A_{k_n} = \delta$ for some constants $\lambda, \delta \in (0, \infty)$.

In order to state the main result, introduce the distribution functions

$$F_n^{l,m}(x) = \mathbf{P} \left\{ \frac{1}{n^{1/\alpha} \ell(1/n)} \left[\sum_{j=l+1}^{n-m} X_{j,n} - n \int_{(l+1)/n}^{1-(m+1)/n} Q(s) ds \right] \leq x \right\}$$

and

$$G_{\kappa}^{l,m}(x) = \mathbf{P}\left\{V_{\alpha,\kappa}^{l,m}(M_1, M_2) \leq x\right\}, \quad x \in \mathbf{R}, \quad (2.6)$$

for all $n \in \mathbf{N}$. With the same conventions as for Theorem 1, the merge results in the next theorem hold along the whole sequence of natural numbers.

Theorem 2. *If $F \in \mathbb{D}_{\text{gp}}^{(\text{c})}(G_{\psi_1^\alpha, \psi_2^\alpha, 0})$ for a semistable distribution $G_{\psi_1^\alpha, \psi_2^\alpha, 0}$ of exponent $\alpha \in (0, 2)$ and $\mathbf{c} = \mathbf{c}(G_{\psi_1^\alpha, \psi_2^\alpha, 0}) \geq 1$, then for every pair (l, m) of nonnegative integers,*

$$\Delta_n^{l,m} = \sup_{x \in \mathbf{R}} \left| F_n^{l,m}(x) - G_{\gamma(1/n)}^{l,m}(x) \right| \rightarrow 0. \quad (2.7)$$

Furthermore, for all pairs of sequences $\{q_n\}_{n=1}^\infty$ and $\{r_n\}_{n=1}^\infty$ of natural numbers such that $q_n \rightarrow \infty$, $r_n \rightarrow \infty$ and $q_n/n \rightarrow 0$, $r_n/n \rightarrow 0$,

$$\sup_{x \in \mathbf{R}} \left| \mathbf{P} \left\{ \frac{S_n^{l,m}(r_n, q_n) - n\mu_n^{l,m}(r_n, q_n)}{n^{1/\alpha}\ell(1/n)} \leq x \right\} - G_{\gamma(1/n)}^{l,m}(x) \right| \rightarrow 0,$$

where $S_n^{l,m}(r_n, q_n) = \sum_{j=l+1}^{r_n} X_{j,n} + \sum_{j=n-q_n+1}^{n-m} X_{j,n}$ and $\mu_n^{l,m}(r_n, q_n) = \int_{(l+1)/n}^{r_n/n} Q(s) ds + \int_{1-q_n/n}^{1-(m+1)/n} Q(s) ds$, and if $M_1(\cdot) \equiv 0$ and $M_2(\cdot) > 0$, then

$$\sup_{x \in \mathbf{R}} \left| \mathbf{P} \left\{ \frac{\sum_{j=n-q_n+1}^{n-m} X_{j,n} - n \int_{1-q_n/n}^{1-(m+1)/n} Q(s) ds}{n^{1/\alpha}\ell(1/n)} \leq x \right\} - G_{\gamma(1/n)}^{0,m}(x) \right| \rightarrow 0,$$

while if $M_1(\cdot) > 0$ and $M_2(\cdot) \equiv 0$, then

$$\sup_{x \in \mathbf{R}} \left| \mathbf{P} \left\{ \frac{\sum_{j=l+1}^{r_n} X_{j,n} - n \int_{(l+1)/n}^{r_n/n} Q(s) ds}{n^{1/\alpha}\ell(1/n)} \leq x \right\} - G_{\gamma(1/n)}^{l,0}(x) \right| \rightarrow 0.$$

It is of course the case $(l, m) = (0, 0)$ of (2.7) that gives merge with semistable laws for full sums. In general, for all possible choices of (l, m) , the three extra statements show the fine structure in delineating those parts of the possibly lightly trimmed sums which govern the merging approximation: the rest with its share of the centering sequence is asymptotically negligible when normalized by what is needed for the whole sum. In the special case of $(l, m) = (0, 0)$, these three statements together may be viewed as a generalization of Theorem 3 in [9], which is for $F \in \mathbb{D}(\alpha)$, $0 < \alpha < 2$.

The reason that merge takes place uniformly in Theorem 2 is that by Lemma 2 below $G_{\kappa}^{l,m}$ has a smooth density function for each $\kappa > 0$. Due to this fact, the distribution functions $F_{n,r}^{l,m}$ of the variables in (2.5) also converge uniformly to the limiting distribution function $G_{\kappa}^{l,m}$ there, as $r \rightarrow \infty$. In the trivial special case $\mathbf{c} = 1$ of Theorem 2, when

$F \in \mathbb{D}_{\text{gp}}^{(1)}(G_{\psi_1^\alpha, \psi_2^\alpha, 0}) = \mathbb{D}(\alpha)$, the respective sequence $G_{\gamma(1/n)}^{l,m}(x)$ can be replaced in all the statements by the limiting («possibly trimmed stable») distribution function $G_1^{l,m}(x) = \mathbf{P}\{V_{l,m}(m_1\psi^\alpha, m_2\psi^\alpha, 0) \leq x\}$, $x \in \mathbf{R}$, where the ingredients for the corresponding random variable $V_{l,m}(m_1\psi^\alpha, m_2\psi^\alpha, 0)$ in (1.1) are defined between (1.3) and (1.4), and, in comparison with Corollary 3 in [7], the resulting uniformity of convergence is still new in the genuinely trimmed cases when $(l, m) \neq (0, 0)$, even in (2.7).

Observe also that $\dots = G_{1/c^2}^{l,m} = G_{1/c}^{l,m} = G_1^{l,m} = G_c^{l,m} = G_{c^2}^{l,m} = \dots$ for every pair (l, m) of nonnegative integers, the source of which is the joint multiplicative periodicity of the two functions M_1 and M_2 in (1.4), the basic feature of semistability. Finally we point out that if the condition in (1.8) holds, then $\gamma(1/n) = \lfloor c^{\lceil \log_c n \rceil} \rfloor / n$ for all $n \in \mathbf{N}$ in Theorems 1 and 2.

Turning now to extreme values, we shall use the characterization from [20] for an $F \in \mathbb{D}_{\text{gp}}(\mathcal{S})$ in terms of a sequence $\{k_n\}$ satisfying (1.5). Recall (1.6) and the description following (1.4) of the Lévy functions L and R of a semistable law. Set $A_n = n^{1/\alpha} \ell(1/n)$. Obviously $A_{k_{n+1}}/A_{k_n} \rightarrow c^{1/\alpha}$. So, if $c > 1$ then for all $x > 0$ large enough there exists a uniquely determined A_{k_n} , which will be denoted by $a(x)$, such that $A_{k_n} \leq x < A_{k_{n+1}}$. Thus if $c > 1$, we may write x as $x = \delta(x)a(x)$; we put $\delta(x) \equiv 1$ if $c = 1$. Now, $F \in \mathbb{D}_{\text{gp}}(\psi_1^\alpha, \psi_2^\alpha, 0)$ along the subsequence k_n with normalizing constants A_{k_n} if and only if for some $\alpha \in (0, 2)$ and all $x > 0$ large enough,

$$\begin{aligned} F_-(-x) &= x^{-\alpha} \ell^*(x) \left[M_L(-\delta(x)) + h_L(x) \right], \\ 1 - F(x) &= x^{-\alpha} \ell^*(x) \left[M_R(\delta(x)) + h_R(x) \right], \end{aligned} \quad (2.8)$$

where ℓ^* is a right-continuous function, slowly varying at infinity, given by

$$x^{-\alpha} \ell^*(x) = \sup \{t: t^{-1/\alpha} \ell(t) > x\}, \quad x > 0, \quad (2.9)$$

F_- is the left-continuous version of F , and the right-continuous error functions h_R and h_L are such that $h_K(A_{k_n} x_0) \rightarrow 0$, as $n \rightarrow \infty$, for every continuity point x_0 of M_K , $K \in \{L, R\}$. If M_K is continuous, then $h_K(x) \rightarrow 0$ as $x \rightarrow \infty$ for the corresponding $K \in \{L, R\}$. Furthermore, for a geometric subsequence $k_n = \lfloor c^n \rfloor$, $c > 1$, the description of the tails in (2.8) holds with $\delta(x)$ replaced by x in both arguments.

The characterization given here describes the attracted distributions pertaining to fixed sequences k_n satisfying (1.5) and the special norming sequence $A_{k_n} \equiv k_n^{1/\alpha} \ell(1/k_n)$. If one is interested in describing the attracted distributions fixing only k_n , this can be done by the free choice of ℓ . Finally, if we fix only the normalizing sequence A'_n such that $A'_{k_{n+1}}/A'_n \rightarrow c^{1/\alpha}$, then convergence in (1.3) takes place along arbitrary subsequences k_n for which $k_{n+1}/k_n \rightarrow c$ and $A_{k_n} \equiv A'_n$. In this case the function ℓ in (1.6) can be defined as $\ell(s) = A'_n/k_n^{1/\alpha}$, $s \in [1/k_n, 1/k_{n-1})$, and up to asymptotic equivalence

all slowly varying functions ℓ^* may appear in (2.8). This last version is mathematically equivalent to the one given by Grinevich and Khokhlov [15], and the correction of the oversight there is nothing but requiring the behavior of the error functions h_L and h_R as above.

To state our result for maxima, introduce the distribution functions

$$H_n(x) = \mathbf{P} \left\{ \frac{X_{n,n}}{n^{1/\alpha} \ell(1/n)} \leq x \right\}, \quad x \in \mathbf{R},$$

for all $n \in \mathbf{N}$; and for $\kappa > 0$ and $M_R \neq 0$ (which is equivalent to $\psi_2 \neq 0$)

$$K_\kappa(x) = \begin{cases} 0, & x \leq 0, \\ e^{-M_R(x/\kappa^{1/\alpha})/x^\alpha} = e^{R(x/\kappa^{1/\alpha})/\kappa}, & x > 0. \end{cases}$$

We shall also use the Lévy distance

$$\mathcal{L}(F, G) = \inf \{ \varepsilon > 0: G(x - \varepsilon) - \varepsilon \leq F(x) \leq G(x + \varepsilon) + \varepsilon \text{ for all } x \in \mathbf{R} \}$$

of the distribution functions F and G , which is well-known to metrize weak convergence of distribution functions on $\mathbf{R}$, convergence in all continuity points of the limiting distribution function [14, p. 38]. The merge theorem for maxima is now as follows:

Theorem 3. *If $F \in \mathbb{D}_{\text{gp}}^{(\mathbf{c})}(G_{\psi_1^\alpha, \psi_2^\alpha, 0})$ for a semistable distribution $G_{\psi_1^\alpha, \psi_2^\alpha, 0}$ of exponent $\alpha \in (0, 2)$, for which $\mathbf{c} = \mathbf{c}(G_{\psi_1^\alpha, \psi_2^\alpha, 0}) \geq 1$ and $M_2 \neq 0$, then*

$$\mathcal{L}(H_n, K_{\gamma(1/n)}) \longrightarrow 0. \quad (2.10)$$

Furthermore, if M_R is continuous, then (2.10) may be strengthened to

$$\sup_{x \in \mathbf{R}} |H_n(x) - K_{\gamma(1/n)}(x)| \longrightarrow 0.$$

We note that merging together of two sequences of probability measures is discussed in [3] in very general terms. The abstract results there do not have implications for our problems here; the example in [3] nearest to what we do here is the investigation of the «wobbling maxima» of the integer parts of independent standard normal variables.

While the merging of cumulative St. Petersburg gains in [6] and the more general results in Theorem 2 above are the first appearances of (possibly lightly trimmed) sums of independent and identically distributed random variables merging together in distribution with a concretely defined sequence of approximate random variables, Arak's [1] famous theorem may also be viewed as a distributional merge theorem for sums of independent and identically distributed random variables. Greatly improving Kolmogorov's early constructions and some intermediate refinements, it states that the

n -th convolution power F^{*n} of *any* distribution function on $\mathbf{R}$ can be uniformly approximated by a sequence of infinitely divisible distribution functions G_n at the rate of $n^{-2/3}$ and this merge rate is best in general; see also [2] for the proofs and references. However, Arak's deep theorem is practically an existence result: practically nothing more than its infinite divisibility is known about G_n . Naturally, one cannot expect that in the untrimmed case $(l, m) = (0, 0)$ the merge in (2.7) would take place near to Arak's rate, i.e., that $G_{\gamma(1/n)}^{0,0}$ would determine the type of Arak's infinitely divisible approximation to F^{*n} for our special $F \in \mathbb{D}_{\text{gp}}(G_{\psi_1^\alpha, \psi_2^\alpha, 0})$, $\alpha \in (0, 2)$. Nevertheless, it would still be of some interest to see whether the merge rates in (2.7), for $(l, m) = (0, 0)$, achieve at least those rates of convergence to stable distributions that are known for the more restrictive cases when $F \in \mathbb{D}(\alpha)$, $\alpha \in (0, 2)$.

3. Examples: Generalized St. Petersburg games. Let $p \in (0, 1)$ and $\alpha \in (0, 2)$ be fixed, put $q = 1 - p$, consider a generalized St. Petersburg game in which the gain X of Paul, the gambler, is such that $\mathbf{P}\{X = q^{-k/\alpha}\} = q^{k-1}p$, $k \in \mathbf{N}$, and let $X_1, X_2, \dots$ be the gains of Paul in a sequence of independent repetitions of the game. This is the classical St. Petersburg game if $\alpha = 1$ and $p = \frac{1}{2}$; the extended versions were recently considered in the context of the strong law of large numbers in [11]. Restricting the general notation to the present X in this section and setting $\lceil y \rceil = \min\{k \in \mathbf{Z}: k \geq y\} = -\lfloor -y \rfloor$ and $\langle y \rangle = y - \lfloor y \rfloor$ for the fractional part of $y \in \mathbf{R}$, we have $F(x) = 0$ for $x < q^{-1/\alpha}$ and

$$1 - F(x) = q^{\lfloor \alpha \log_{1/q} x \rfloor} = \frac{M_R(x)}{x^\alpha} := \frac{(1/q)^{\langle \alpha \log_{1/q} x \rangle}}{x^\alpha}, \quad x \geq q^{-1/\alpha},$$

where $\log_{1/q}$ stands for the logarithm to the base $1/q$, and for all $s \in (0, 1)$,

$$\begin{aligned} Q(1-s) &= q^{-(1/\alpha)\lceil \log_{1/q}(1/s) \rceil} = \frac{M_2(s)}{s^{1/\alpha}} \\ &:= \frac{(1/q)^{(1/\alpha)\{\lceil \log_{1/q}(1/s) \rceil - \log_{1/q}(1/s)\}}}{s^{1/\alpha}} = \frac{(1/q)^{(1/\alpha)\langle \log_{1/q} s \rangle}}{s^{1/\alpha}} \end{aligned}$$

by elementary calculation, while, computed for the sequence $k_n \equiv \lfloor q^{-n} \rfloor$,

$$\gamma(s) = s \left[q^{-\lceil \log_{1/q} \lceil 1/s \rceil \rfloor} \right], \quad \text{so that} \quad 1 \leq \gamma\left(\frac{1}{n}\right) = \frac{\lfloor (1/q)^{\lceil \log_{1/q} n \rceil} \rfloor}{n} \leq \frac{1}{q}$$

for all $n \in \mathbf{N}$. Extending the definition $M_R(x) = q^{-\langle \alpha \log_{1/q} x \rangle}$ also for $x \in (0, q^{-1/\alpha})$, we have $1 \leq M_R(\mathbf{c}^{1/\alpha} x) = M_R(x) < q^{-1}$ for all $x > 0$ and $1 \leq M_2(\mathbf{c}s) = M_2(s) < q^{-1/\alpha}$ for all $s \in (0, 1)$. Hence, using the statement in (1.8) with $M_1(\cdot) \equiv 0$, we see that $F \in \mathbb{D}_{\text{gp}}^{(\mathbf{c})}(G_{0, \psi^\alpha M_2, 0})$ for $\mathbf{c} = \mathbf{c}(G_{0, \psi^\alpha M_2, 0}) = 1/q$. Introducing now the function

$$0 \leq \delta_p(u) = 1 + \frac{p}{q} \langle \log_{1/q} u \rangle - q^{-\langle \log_{1/q} u \rangle} \leq 1 - \frac{p}{q \ln(1/q)} \left[1 - \ln \frac{p}{q} + \ln \ln \frac{1}{q} \right]$$

for all $u > 0$, where $\ln$ stands for the natural logarithm and the inequalities are by (3.4) in [11], by elementary calculation for every $s \in (0, 1)$ we obtain

$$\int_0^{1-s} Q(u) \, du = \begin{cases} \frac{p}{q^{1/\alpha} - q} + \frac{q^{1/\alpha} - 1}{q^{1/\alpha} - q} \left(\frac{1}{q}\right)^{((1-\alpha)/\alpha)\lceil \log_{1/q}(1/s) \rceil} - s \left(\frac{1}{q}\right)^{(1/\alpha)\lceil \log_{1/q}(1/s) \rceil}, & \alpha \neq 1, \\ \delta_p(s) + \frac{p}{q} \log_{1/q} \frac{1}{s}, & \alpha = 1. \end{cases}$$

Setting $C_{n,m+1}^{\alpha,p} = n \int_0^{1-(m+1)/n} Q(u) \, du$ and understanding $V_{\alpha,\gamma(1/n)}^{0,m}(0, M_2) = W_{2,\alpha,\gamma(1/n)}^{(m)}(M_2)$ and $G_{\gamma(1/n)}^{0,m}(x) = \mathbf{P}\{V_{\alpha,\gamma(1/n)}^{0,m}(0, M_2) \leq x\}$ with the present M_2 and $\gamma(1/n)$, an application of (2.7) gives

$$\lim_{n \rightarrow \infty} \sup_{0 < \beta < 1} \left| \mathbf{P} \left\{ \sum_{j=1}^{n-m} X_{j,n} \leq C_{n,m+1}^{\alpha,p} + n^{1/\alpha} x_{n,m}^{\alpha,p}(\beta) \right\} - \beta \right| = 0 \quad (3.1)$$

for any fixed $m \in \{0, 1, 2, \dots\}$, where $x_{n,m}^{\alpha,p}(\beta)$ is the β -quantile of the merging approximation, that is, $G_{\gamma(1/n)}^{0,m}(x_{n,m}^{\alpha,p}(\beta)) = \beta$ for any $\beta \in (0, 1)$, which exists by Lemma 2 below. If $m = 0$, this refers to the original game. If $m > 0$, it is for a modified game in which Paul renounces his largest m gains in n games and Peter, his opponent, desires a «fair» price accordingly. For both cases, if Peter’s winning probability β in n games is agreed upon between them in advance, for large n the premium formula $C_{n,m+1}^{\alpha,p} + n^{1/\alpha} x_{n,m}^{\alpha,p}(\beta)$ represents Paul’s fair price for n games. The rate of merge, i.e., the rate of convergence in (3.1), will be investigated in a subsequent publication. For $\alpha < 1$ the centering sequence $C_{n,m+1}^{\alpha,p}$ can be changed in (3.1) to 0, while for $\alpha > 1$ to $n\mathbf{E}(X) = np/[q^{1/\alpha} - q]$, but one of the two parties would be unhappy to adopt this: Peter in the first case and Paul in the second. The superficially appealing choice $\beta = \frac{1}{2}$, as something «absolutely fair», will not have this appeal for Peter even with $\alpha > 1$; in practice Peter, representing a bank or insurance company, wants a large β , which the n Pauls forced to play the game grudgingly accept.

4. Proofs. Although Fourier-analytic techniques also enter with Lemma 2 below, the proofs are in the framework of the «probabilistic» or «quantile-transform» approach in [5]–[10] and [20] among our references. To warm up, consider

$$\sigma^2(s, 1-s) = \int_s^{1-s} \int_s^{1-s} [\min(u, v) - uv] \, dQ(u) \, dQ(v), \quad 0 < s < \frac{1}{2},$$

a basic function in this approach, and note that if $F \in \mathbb{D}_{\text{gp}}^{(\text{c})}(G_{\psi_1^\alpha, \psi_2^\alpha, 0})$ for a semistable $G_{\psi_1^\alpha, \psi_2^\alpha, 0}$ of exponent $\alpha \in (0, 2)$, then by Lemma 1(i) in [10],

$$\limsup_{s \downarrow 0} \frac{\sigma(s, 1-s)}{s^{1/2-1/\alpha} \ell(s)} < C \quad (4.1)$$

for some constant $C \in (0, \infty)$, where $\ell(\cdot)$ is the slowly varying function from (1.6), and that, introducing for all $n \in \mathbf{N}$ and $0 < s < n$ the functions

$$\psi_1(n, s) = \frac{Q_+(s/n)}{n^{1/\alpha}\ell(1/n)} \quad \text{and} \quad \psi_2(n, s) = \frac{-Q(1-s/n)}{n^{1/\alpha}\ell(1/n)},$$

the very reason why (1.7) holds along $\{k_n\}$ for which $k_{n+1}/k_n \rightarrow \mathbf{c}$ is that

$$\begin{aligned} \psi_j(k_n, s) &\rightarrow \psi_j^\alpha(s), \quad j = 1, 2, \quad \text{and} \\ \lim_{u \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{\sqrt{k_n} \sigma(u/k_n, 1 - u/k_n)}{k_n^{1/\alpha} \ell(1/k_n)} &= 0 \end{aligned} \quad (4.2)$$

at all the respective continuity points $s > 0$ of the limiting functions ψ_j^α , $j = 1, 2$. Indeed, followed by an application of Theorem 1* in [5], it is to show (4.2) what constitutes the proof in [20] of the sufficiency part of the domain theorem, stated at (1.6) above.

Lemma 1. *If $F \in \mathbb{D}_{\text{gp}}^{(\mathbf{c})}(G_{\psi_1^\alpha, \psi_2^\alpha, 0})$ for a semistable $G_{\psi_1^\alpha, \psi_2^\alpha, 0}$ of exponent $\alpha \in (0, 2)$ and $M_{n,j}^*(\cdot) = M_j(\gamma(\cdot/k_n)) + h_j(\cdot/k_n)$ for the functions in (1.6), where $k_{n+1}/k_n \rightarrow \mathbf{c} = \mathbf{c}(G_{\psi_1^\alpha, \psi_2^\alpha, 0})$, and if $\{\tau_n\}_{n=1}^\infty$ is a sequence of positive numbers such that $\tau_n \rightarrow \tau$, where $\tau > 0$ is a continuity point of $M_j(\cdot)$, then $M_{n,j}^*(\tau_n) \rightarrow M_j(\tau)$, $j = 1, 2$.*

P r o o f. We only have to consider those cases of $j \in \{1, 2\}$ when $M_j \not\equiv 0$. Let $M_n^*(\cdot) = M(\gamma(\cdot/k_n)) + h(\cdot/k_n)$ be any one of $M_{n,j}^*(\cdot)$ for which this is the case, and let $\psi(k_n, \cdot)$ stand for the $\psi_j(k_n, \cdot)$ pertaining to this case. In this notation τ is a continuity point of $M(\cdot)$. We pick further continuity points $0 < s < \tau < t$, as close to τ as we wish. Then, by (4.2) and the fact that the quantile function Q is nondecreasing we see that $-s^{-1/\alpha}M(s) \leq \liminf_{n \rightarrow \infty} \psi(k_n, \tau_n) \leq \limsup_{n \rightarrow \infty} \psi(k_n, \tau_n) \leq -t^{-1/\alpha}M(t)$. This implies that $\lim_{n \rightarrow \infty} \psi(k_n, \tau_n) = -\tau^{-1/\alpha}M(\tau)$. Since $\psi(k_n, \tau_n) = -\tau_n^{-1/\alpha}\ell(\tau_n/k_n)M_n^*(\tau_n)/\ell(1/k_n)$ for all n large enough, the slow variation of $\ell(\cdot)$ then implies $M_n^*(\tau_n) \rightarrow M(\tau)$, proving the lemma.

P r o o f o f T h e o r e m 1. Since $1 \leq \gamma(1/n) \leq \mathbf{c} + \varepsilon$ for any $\varepsilon > 0$ if n is large enough, part (i) follows through the Bolzano–Weierstrass theorem from the sufficiency half of part (ii).

To prove that half first, consider any subsequence $\{n_r\}_{r=1}^\infty \subset \mathbf{N}$ for which the condition on $\{\gamma(1/n_r)\}$ is satisfied, with a κ as defined in the theorem. By Theorem 1* in [5], it suffices to show that the three relations in (4.2) hold for the sequence $\{n_r\}_{r=1}^\infty$ replacing $\{k_n\}$ and for the limiting functions $\psi_j^{\alpha, \kappa}(s) = -M_j(\kappa s)s^{-1/\alpha}$ for every $s > 0$ for which κs is a continuity point of $M_j(\cdot)$, $j = 1, 2$.

As to the third, by (4.1) and the slow variation of $\ell(\cdot)$ we see that

$$\limsup_{r \rightarrow \infty} \frac{\sqrt{n_r} \sigma(u/n_r, 1 - u/n_r)}{n_r^{1/\alpha} \ell(1/n_r)} \leq \frac{C}{u^{(2-\alpha)/(2\alpha)}},$$

whence indeed

$$\lim_{u \rightarrow \infty} \limsup_{r \rightarrow \infty} \frac{\sqrt{n_r} \sigma(u/n_r, 1 - u/n_r)}{n_r^{1/\alpha} \ell(1/n_r)} = 0.$$

For the first two relations, noticing that $s/n_r = \gamma(1/n_r)s/k_{n^*(1/n_r)}$, by (1.6) we have

$$\psi_j(s, n_r) = -s^{-1/\alpha} \frac{\ell(s/n_r)}{\ell(1/n_r)} M_{n^*(1/n_r), j}^*(\gamma(1/n_r)s), \quad s > 0, \quad j = 1, 2,$$

in the notation of Lemma 1. Since $\lim_{r \rightarrow \infty} n^*(1/n_r) = \infty$, using the periodicity of M_j in the case of two limit points and the slow variation of $\ell(\cdot)$ again, Lemma 1 implies that $\lim_{r \rightarrow \infty} \psi_j(s, n_r) = -M_j(\kappa s)s^{-1/\alpha}$ whenever κs is a continuity point of $M_j(\cdot)$, $j = 1, 2$.

Conversely, suppose now (2.3) and choose any subsequence $\{n'_r\} \subset \{n_r\}$. Again by the Bolzano–Weierstrass theorem, there is a further subsequence $\{n''_r\} \subset \{n'_r\}$ such that $\gamma(1/n''_r) \rightarrow \kappa$ as $r \rightarrow \infty$ for some $\kappa \in [1, \mathbf{c}]$. But then by the sufficiency half we have (2.5) along $\{n''_r\}$ for $(l, m) = (0, 0)$, with the limiting random variable $V_{\alpha, \kappa}^{0,0}(M_1, M_2)$ for the given (α, M_1, M_2) determined by the condition that $F \in \mathbb{D}_{\text{gp}}^{(\mathbf{c})}(G_{\psi_1^\alpha, \psi_2^\alpha, 0})$. We know from [7] and [5] that by the uniqueness property of Lévy’s formula in (1.2) the representation of an infinitely divisible random variable in the first equation of (1.1) is also unique. This implies that the distributions of the variables $V_{\alpha, \kappa_1}^{0,0}(M_1, M_2)$ and $V_{\alpha, \kappa_2}^{0,0}(M_1, M_2)$ are of different type if $\kappa_1 \neq \kappa_2$, except for the case when $\kappa_1 = 1$ and $\kappa_2 = \mathbf{c}$, because then $V_{\alpha, 1}^{0,0}(M_1, M_2) = V_{\alpha, \mathbf{c}}^{0,0}(M_1, M_2)$ even algebraically. Since we have an asymptotic distribution along the whole original $\{n_r\}$ in (2.3), it follows by the convergence of types theorem that either the subsequential limits of the sequence $\{\gamma(1/n_r)\}$ are the same, and then $\{\gamma(1/n_r)\}$ itself converges to some $\kappa \in [1, \mathbf{c}]$ as $r \rightarrow \infty$, or all the subsequential limits are either 1 or $\mathbf{c}$, in which case the sequence $\{\gamma(1/n_r)\}$ has exactly these two limit points. Thus we have both (2.3) and (2.5) along $\{n_r\}$, and so the two convergence relations in (2.4) also come from the convergence of types theorem.

Kruglov [17] proved that the distribution functions $G_{\psi_1^\alpha, \psi_2^\alpha, 0}$ of all semistable random variables $V(\psi_1^\alpha, \psi_2^\alpha, 0)$ are infinitely many times differentiable. Following from a special case of the following lemma, the first step towards Theorem 2 is an extension of this property to the distributional limits of all possibly lightly trimmed sums.

Lemma 2. *Let $\psi(s) = -M(s)s^{-1/\alpha}$, $0 < s < \infty$, where $\alpha \in (0, 2)$ and $M(\cdot)$ is any right-continuous function such that $C \leq M(s) \leq K$ for all $s > 0$ for some constants $0 < C \leq K < \infty$ and that $\psi(\cdot)$ is nondecreasing on $(0, \infty)$. Then for every nonnegative integer r the common distribution function of the random variables $W_1^{(r)}(\psi)$ and $W_2^{(r)}(\psi)$ in (1.1) is infinitely*

many times differentiable on the whole line $\mathbf{R}$ and each of its derivatives has vanishing limits at $\pm\infty$.

P r o o f. By Theorem 4 in [7], for the common characteristic function $\phi_{r,\psi}(t) = \mathbf{E}(\exp\{itW_j^{(r)}(\psi)\})$, $j = 1, 2$, we have

$$\begin{aligned} \phi_{r,\psi}(t) = \int_0^\infty \exp\left\{\int_x^\infty \left[e^{it\psi(s)} - 1 - \frac{it\psi(s)}{1+\psi^2(s)}\right] ds\right\} \\ \times e^{it\rho_{r,\psi}(x)} \frac{x^r}{r!} e^{-x} dx, \end{aligned} \quad (4.3)$$

where

$$\rho_{r,\psi}(x) = \psi(x) + \int_{x+1}^{r+1} \psi(s) ds + \int_x^{x+1} \frac{\psi(s)}{1+\psi^2(s)} ds - \int_{x+1}^\infty \frac{\psi^3(s)}{1+\psi^2(s)} ds,$$

and hence

$$|\phi_{r,\psi}(t)| \leq \int_0^\infty \exp\left\{-\int_x^\infty [1 - \cos|t\psi(s)|] ds\right\} \frac{x^r}{r!} e^{-x} dx, \quad t \in \mathbf{R}.$$

If, for $x, t \geq 0$,

$$\frac{\pi x^{1/\alpha}}{2K} \leq t \quad \text{or, equivalently, } x \leq \left(\frac{2Kt}{\pi}\right)^\alpha,$$

then

$$\begin{aligned} \int_x^\infty [1 - \cos|t\psi(s)|] ds &\geq \int_{(2Kt/\pi)^\alpha}^\infty [1 - \cos|t\psi(s)|] ds \\ &\geq \frac{4}{\pi^2} t^2 \int_{(2Kt/\pi)^\alpha}^\infty \psi^2(s) ds \geq \frac{4C^2}{\pi^2} t^2 \int_{(2Kt/\pi)^\alpha}^\infty s^{-2/\alpha} ds \\ &= \frac{4C^2}{\pi^2} \frac{\alpha}{2-\alpha} \left(\frac{\pi}{2K}\right)^{2-\alpha} t^\alpha =: K_\alpha t^\alpha \end{aligned}$$

because if $s \geq (2Kt/\pi)^\alpha$, then

$$0 \leq \frac{|t|C}{s^{1/\alpha}} \leq |t\psi(s)| \leq \frac{|t|K}{s^{1/\alpha}} \leq \frac{\pi}{2},$$

and, following for example from the well-known product expansion

$$\cos y = \prod_{k=1}^\infty \left(1 - \frac{4y^2}{(2k-1)^2\pi^2}\right), \quad y \in \mathbf{R},$$

we have

$$1 - \cos y \geq \frac{4y^2}{\pi^2} \quad \text{if } 0 \leq y \leq \frac{\pi}{2}.$$

Therefore,

$$\begin{aligned} & \int_0^\infty t^k |\phi_{r,\psi}(t)| dt \\ & \leq \int_0^\infty \left[\int_0^\infty t^k \exp \left\{ - \int_x^\infty [1 - \cos |t\psi(s)|] ds \right\} dt \right] \frac{x^r}{r!} e^{-x} dx \\ & \leq \int_0^\infty \left[\int_0^{\pi x^{1/\alpha}/(2K)} t^k dt + \int_{\pi x^{1/\alpha}/(2K)}^\infty t^k e^{-K_\alpha t^\alpha} dt \right] \frac{x^r}{r!} e^{-x} dx < \infty \end{aligned}$$

for any nonnegative integer k , which by a standard Fourier-analytic result (see in [16, p. 438], for example) implies that the distribution function pertaining to $\phi_{r,\psi}(\cdot)$ is $k+1$ times differentiable with derivatives vanishing at $\pm\infty$. Since $k \geq 0$ is arbitrary, the lemma follows.

P R O O F O F T H E O R E M 2. Notice first that for any fixed integers $l, m \geq 0$ the parametric family $\{G_\kappa^{l,m}: \kappa > 0\}$ of distribution functions defined in (2.6) is continuous with respect to weak convergence: $\mathcal{L}(G_{\kappa_n}^{l,m}, G_\kappa^{l,m}) \rightarrow 0$ for any $\kappa > 0$ whenever $\kappa_n \rightarrow \kappa$. This can be seen in two different ways. One is, with reference to the continuity theorem of characteristic functions, to check that for each fixed $t \in \mathbf{R}$ the characteristic function

$$\int_{\mathbf{R}} e^{itx} dG_\kappa^{l,m}(x) = \mathbf{E}(\exp \{itV_{\alpha,\kappa}^{l,m}(M_1, M_2)\}) = \phi_{l,\psi_1^{\alpha,\kappa}}(-t) \phi_{m,\psi_2^{\alpha,\kappa}}(t)$$

is continuous in the parameter $\kappa > 0$. Substituting $\psi_j^{\alpha,\kappa}(s) = -s^{-1/\alpha} M_j(\kappa s)$, $s > 0$, $j = 1, 2$, given at (2.2), into the formula in (4.3), this can be seen readily for both factors. The other way is to check that the stochastic process $\{V_{\alpha,\kappa}^{l,m}(M_1, M_2): \kappa > 0\}$ does not have fixed discontinuities, that is, $V_{\alpha,\kappa_n}^{l,m}(M_1, M_2) \rightarrow V_{\alpha,\kappa}^{l,m}(M_1, M_2)$ almost surely whenever $\kappa_n \rightarrow \kappa > 0$, which is elementary using (2.2) and the representation in (1.1).

Now, to prove (2.7), pick any subsequence $\{n_j\}_{j=1}^\infty \subset \mathbf{N}$. Then by the Bolzano–Weierstrass theorem this contains a further subsequence $\{n_{j_r}\}_{r=1}^\infty$ such that $\lim_{r \rightarrow \infty} \gamma(1/n_{j_r}) = \kappa$ for some $\kappa \in [1, c]$. Then by Theorem 1 and by what we just proved,

$$\mathcal{L}(F_{n_{j_r}}^{l,m}, G_\kappa^{l,m}) \rightarrow 0 \quad \text{and} \quad \mathcal{L}(G_{\gamma(1/n_{j_r})}^{l,m}, G_\kappa^{l,m}) \rightarrow 0 \quad \text{as } r \rightarrow \infty.$$

By the domain theorem at (1.6) and by an application of Lemma 2 we see that the distribution function $G_\kappa^{l,m}$ is uniformly continuous for every $\kappa \in [1, c]$, in fact the family $\{G_\kappa^{l,m}: \kappa > 0\}$ is uniformly equicontinuous as can be seen by a simple variant of Lemma 2, and hence the last two relations can be strengthened to

$$\sup_{x \in \mathbf{R}} |F_{n_{j_r}}^{l,m}(x) - G_\kappa^{l,m}(x)| \rightarrow 0 \quad \text{and} \quad \sup_{x \in \mathbf{R}} |G_{\gamma(1/n_{j_r})}^{l,m}(x) - G_\kappa^{l,m}(x)| \rightarrow 0,$$

as $r \rightarrow \infty$, which by the triangle inequality for the uniform distance implies that $\lim_{r \rightarrow \infty} \Delta_{n_{j_r}}^{l,m} = 0$. Since the subsequence $\{n_j\} \subset \mathbf{N}$ was arbitrary, we have $\lim_{n \rightarrow \infty} \Delta_n^{l,m} = 0$, proving (2.7).

Now let $\xrightarrow{\mathbf{P}}$ denote convergence in probability. To prove the three extra statements, we only have to point out that we showed in [10] (in the proof of Theorem 3 there) that

$$\frac{1}{n^{1/\alpha} \ell(1/n)} \left\{ \sum_{j=r_n+1}^{n-q_n} X_{j,n} - n \int_{r_n/n}^{1-q_n/n} Q(s) ds \right\} \xrightarrow{\mathbf{P}} 0,$$

always, and under the respective two conditions on M_1 and M_2 also

$$\begin{aligned} \frac{1}{n^{1/\alpha} \ell(1/n)} \left\{ \sum_{j=1}^{n-q_n} X_{j,n} - n \int_{1/n}^{1-q_n/n} Q(s) ds \right\} &\xrightarrow{\mathbf{P}} 0, \\ \frac{1}{n^{1/\alpha} \ell(1/n)} \left\{ \sum_{j=r_n+1}^n X_{j,n} - n \int_{r_n/n}^{1-1/n} Q(s) ds \right\} &\xrightarrow{\mathbf{P}} 0. \end{aligned}$$

Using these along $\{n_{j_r}\}$, obvious variants of the proof above yield the three extra claims.

P r o o f o f T h e o r e m 3. Since $H_n(0) \rightarrow 0 = K_{\gamma(1/n)}(0)$, to prove (2.10) it clearly suffices to show that for every $\varepsilon > 0$,

$$H_n(x - \varepsilon) - \varepsilon \leq K_{\gamma(1/n)}(x) \leq H_n(x + \varepsilon) + \varepsilon \quad \text{for all } x > 0, \quad (4.4)$$

for all n large enough.

Considering any $x > 0$ and setting $A_n = n^{1/\alpha} \ell(1/n)$, by (2.8) we have

$$H_n(x) = F^n(A_n x) = \left[1 - \frac{x^{-\alpha} (M_R(\delta(A_n x)) + h_R(A_n x))}{n} \frac{\ell^*(x n^{1/\alpha} \ell(1/n))}{\ell^\alpha(1/n)} \right]^n,$$

where $\delta(\cdot)$, and $\gamma(\cdot)$ above, are defined in terms of the basic $\{k_n\}$ sequence for which $k_{n+1}/k_n \rightarrow \mathbf{c}$. The inversion formula in (2.9) implies that $\ell^*(n^{1/\alpha} \ell(1/n))/\ell^\alpha(1/n) \rightarrow 1$, whence $\ell^*(x n^{1/\alpha} \ell(1/n))/\ell^\alpha(1/n) \rightarrow 1$. Now, pick any unbounded subsequence $\{n_j\}_{j=1}^\infty \subset \mathbf{N}$. Then, since the sequence $\{\gamma(1/n_j)\}$ is bounded with all of its limit points in the interval $[1, \mathbf{c}]$, by the Bolzano–Weierstrass theorem there exists a further subsequence $\{n_{j_r}\}_{r=1}^\infty$ such that $\{\gamma(1/n_{j_r})\}$ converges to some $\kappa \in [1, \mathbf{c}]$, as $r \rightarrow \infty$. Then we have the asymptotic equality $M_R(\delta(A_{n_{j_r}} x)) = M_R(A_{n_{j_r}} x/a(A_{n_{j_r}} x)) \sim M_R(x/\gamma^{1/\alpha}(1/n_{j_r}))$ as $r \rightarrow \infty$, provided that $x/\kappa^{1/\alpha}$ is a continuity point of M_R , in the proof of which we use the equation $M_R(\mathbf{c}^{1/\alpha} t) = M_R(t)$ for various choices of $t > 0$. Indeed, by a suitable analogue of Lemma 1, presently based on the classical fact that

$k_n[1 - F(A_{k_n}t)] \rightarrow M_R(t)$ at every continuity point $t > 0$ of M_R , we even see that the error term $h_R(A_{n_{j_r}}x)$ is negligible, so that

$$M_R(\delta(A_{n_{j_r}}x)) + h_R(A_{n_{j_r}}x) \rightarrow M_R\left(\frac{x}{\kappa^{1/\alpha}}\right) \quad \text{as } r \rightarrow \infty.$$

Whence

$$H_{n_{j_r}}(x) \rightarrow K_\kappa(x) = e^{-M_R(x/\kappa^{1/\alpha})/x^\alpha} \quad \text{as } r \rightarrow \infty \quad (4.5)$$

at every $x \in \mathbf{R}$ for which $x/\kappa^{1/\alpha}$ is a continuity point of M_R .

We infer from this property that (4.4) holds. Suppose the contrary. Then there exist a bad $\varepsilon > 0$, a bad $x_\varepsilon > 0$ and an unbounded sequence $\{n_j\}_{j=1}^\infty \subset \mathbf{N}$ such that either

$$H_{n_j}(x_\varepsilon - \varepsilon) - \varepsilon > K_{\gamma(1/n_j)}(x_\varepsilon) \quad \text{for all } j \in \mathbf{N},$$

or

$$H_{n_j}(x_\varepsilon + \varepsilon) + \varepsilon < K_{\gamma(1/n_j)}(x_\varepsilon) \quad \text{for all } j \in \mathbf{N},$$

or both. Let $\{n_{j_r}\}$ be a subsequence for which $\lim_{r \rightarrow \infty} \gamma(1/n_{j_r}) = \kappa \in [1, c]$, so that (4.5) holds. If the first case in (4.6) is not empty, choose any $x_- \in (x_\varepsilon - \varepsilon, x_\varepsilon)$ such that $x_-/\kappa^{1/\alpha}$ is a continuity point of M_R . Then

$$\begin{aligned} \limsup_{r \rightarrow \infty} H_{n_{j_r}}(x_\varepsilon - \varepsilon) - \varepsilon &\leq \lim_{r \rightarrow \infty} H_{n_{j_r}}(x_-) - \varepsilon \\ &= K_\kappa(x_-) - \varepsilon < \liminf_{r \rightarrow \infty} K_{\gamma(1/n_{j_r})}(x_\varepsilon), \end{aligned}$$

which contradicts (4.6). Choosing any $x_+ \in (x_\varepsilon, x_\varepsilon + \varepsilon)$ for a nonempty second case in (4.6), the contradiction arises in the same way. Thus (4.4) and hence (2.10) follow. Using the uniform version of (4.5), the proof of the second statement for the sup-distance is analogous.

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