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распределений. Для того чтобы величина (5) давала для нерешетчатой функции распределения $F(x)$ асимптотическое разложение $F_n(x)$ с точностью $O(1/n^{m/2})$, $n \rightarrow \infty$, достаточно выполнения условия Крамера

$$\limsup_{t \rightarrow \infty} |f(t)| < 1.$$

Для нечетных m асимптотическое разложение и оценка его остаточной части выглядят аналогично, лишь в некоторых местах следует заменить m на $m - 1$. Мы не будем останавливаться на этом.

В теоремах 2–6 присутствуют ограничения $\theta_4 < 6$ и $\|\theta T\|_{3,m+1} < 3$. Эти ограничения можно обойти, но мы также не останавливаемся на этом.

Один из промежуточных результатов, полученных при доказательстве теоремы 6, позволяет доказать следующее утверждение.

Теорема 7. Пусть F — решетчатая функция распределения с шагом $h > 0$, нулевым средним, единичной дисперсией и конечным четвертым моментом. Для любого $n \geq 2$, любого $x \in D_n + h/(2\sqrt{n})$ и любого числа $0 < \delta < \frac{1}{3}$ справедливо неравенство

$$\begin{aligned} |F_n(x) - \Phi(x)| \leq & \frac{|\alpha_3|}{6\sqrt{2\pi n}} \left(\frac{n}{n-1}\right)^{3/2} (1-\delta)^{-3/2} + \frac{\beta_4 + 3}{12\pi n} \left(\frac{n}{n-1}\right)^2 (1-\delta)^{-2} \\ & + \left(\frac{h}{\sqrt{n}}\right)^2 \left(\frac{(1-\delta)^{-2}}{3\pi} + \frac{1}{24\sqrt{2\pi e}}\right) + \alpha^n \left(\frac{1}{\pi} \log\left(\frac{\pi}{3} \frac{\beta_3}{\delta h}\right) + \frac{\pi}{6}\right) \\ & + \frac{1}{\pi} \left(\frac{\beta_3}{3\delta\sqrt{n}}\right)^2 e^{-9\delta^2 n/(2\beta_3^2)} + \frac{1}{\pi^3} \left(\frac{h}{\sqrt{n}}\right)^2 e^{-\pi^2 n/(2h^2)}, \end{aligned}$$

где α_3 — третий момент функции распределения F , $\alpha = \max\{|f(t)| : 3\delta/\beta_3 \leq t \leq \pi/h\} < 1$, а f — характеристическая функция распределения F .

Следствием теоремы 7 является неравенство

$$\sup_{-\infty < x < \infty} |F_n(x) - \Phi(x)| \leq \frac{|\alpha_3|}{6\sqrt{2\pi n}} + \frac{1}{\sqrt{2\pi}} \frac{h}{2\sqrt{n}} + O\left(\frac{1}{n}\right), \quad n \rightarrow \infty.$$

Для величины $O(1/n)$ легко получить явную оценку. Отметим, что оба слагаемых в правой части последнего неравенства, убывающие при росте n как $1/\sqrt{n}$, являются правильными.

Отметим также, что если в левой части последнего неравенства снять знак верхней грани, то во второе слагаемое правой части можно добавить множитель, асимптотически (при $n \rightarrow \infty$) равный $e^{-x^2/2}$.

Перечисленные результаты и ряд других, вместе с их доказательствами, содержатся в монографии «Центральная предельная теорема: точность аппроксимации и асимптотические разложения», готовящейся к печати в издательстве «ТВП».

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A NONCLASSICAL CHUNG-TYPE LAW OF THE ITERATED LOGARITHM FOR I.I.D. RANDOM VARIABLES¹⁾

Пусть $\{X, X_n; n \geq 1\}$ — последовательность независимых одинаково распределенных случайных величин; положим $S_n = \sum_{i=1}^n X_i$. Мы определяем последовательность положительных констант $\{d(n), n \geq 1\}$,

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асимптотическое поведение которой отлично от $\ln \ln n$, но для которой $\liminf_{n \rightarrow \infty} (\max_{1 \leq i \leq n} |S_i|) / \sqrt{n/d(n)} = \pi/\sqrt{8}$ почти наверное, что равносильно равенствам $\mathbf{E}X = 0$ и $\mathbf{E}X^2 = 1$.

Ключевые слова и фразы: закон повторного логарифма типа Чжуна, теорема о малых отклонениях, независимые одинаково распределенные случайные величины.

1. Introduction and main results. It is well known that the law of the iterated logarithm (LIL) and Chung-type law of the iterated logarithm (Chung-type LIL) are two striking and fundamental results of probability theory. The results describe the almost sure (a.s.) asymptotic fluctuation behavior of the partial sum, i.e.,

$$\limsup_{n \rightarrow \infty} \frac{|S_n|}{\sqrt{2n \ln \ln n}} = 1 \quad \text{a.s.} \quad (1.1)$$

and

$$\liminf_{n \rightarrow \infty} \frac{\max_{1 \leq i \leq n} |S_i|}{\sqrt{n/\ln \ln n}} = \frac{\pi}{\sqrt{8}} \quad \text{a.s.}, \quad (1.2)$$

where $S_n = \sum_{i=1}^n X_i$ and $\{X_n, n \geq 1\}$ is a sequence of i.i.d. random variables with $\mathbf{E}X = 0$ and $\mathbf{E}X^2 = 1$. Klesov and Rosalsky [4] considered the nonclassical LIL. They first defined a sequence of constant $\{b(n), n \geq 1\}$ as follows: Let $1 < M < \infty$ and

$$N_0 = 1 \quad \text{and} \quad N_k = \min\{n \geq 1: \ln \ln n \geq M \ln \ln N_{k-1}\}, \quad k \geq 1. \quad (1.3)$$

Define

$$b(n) = M \ln \ln N_k, \quad N_k \leq n < N_{k+1}, \quad k \geq 0. \quad (1.4)$$

It is clear that

$$\ln \ln n \leq b(n) \leq M \ln \ln n, \quad n \geq 1,$$

and

$$\limsup_{n \rightarrow \infty} \frac{b(n)}{\ln \ln n} = M, \quad \liminf_{n \rightarrow \infty} \frac{b(n)}{\ln \ln n} = 1.$$

Then they obtained the following result.

Theorem A. Let $1 < M < \infty$, and let $\{b(n), n \geq 1\}$ be the sequence defined in (1.4). Then for all sequence of i.i.d. random variables $\{X_n, n \geq 1\}$,

$$\limsup_{n \rightarrow \infty} \frac{|S_n|}{\sqrt{2nb(n)}} = 1 \quad \text{a.s.} \quad (1.5)$$

is equivalent to

$$\mathbf{E}X = 0 \quad \text{and} \quad \mathbf{E}X^2 = 1. \quad (1.6)$$

It is natural for us to ask whether there is a similar result for Chung-type LIL. In the present paper, we will give the positive answer and get the following result. First, we need to define a new sequence of constants.

Let N_0 and N_k be defined in (1.3), we define

$$d(n) = \frac{1}{M} \ln \ln N_{k+1}, \quad N_k \leq n < N_{k+1}, \quad k \geq 0. \quad (1.7)$$

Obviously,

$$\frac{1}{M} \ln \ln n \leq d(n) \leq \ln \ln n, \quad n \geq 1, \quad (1.8)$$

and

$$\liminf_{n \rightarrow \infty} \frac{d(n)}{\ln \ln n} = \frac{1}{M}, \quad \limsup_{n \rightarrow \infty} \frac{d(n)}{\ln \ln n} = 1. \quad (1.9)$$

Theorem 1.1. Let $1 < M < \infty$, and let $\{d(n), n \geq 1\}$ be the sequence defined in (1.7). Then for all sequence of i.i.d. random variables $\{X_n, n \geq 1\}$,

$$\liminf_{n \rightarrow \infty} \frac{\max_{1 \leq i \leq n} |S_i|}{\sqrt{n/d(n)}} = \frac{\pi}{\sqrt{8}} \quad \text{a.s.} \quad (1.10)$$

is equivalent to

$$\mathbf{E}X = 0 \quad \text{and} \quad \mathbf{E}X^2 = 1. \quad (1.11)$$

Remark 1. As was mentioned in [4], we can easily see that

$$\ln \ln N_{k+1} \sim M \ln \ln N_k, \quad \ln N_{k+1} \geq (\ln N_k)^M, \quad \ln \ln N_k \geq M^k, \quad k \geq 1.$$

Remark 2. Obviously, (1.10) would be proved provided we are able to choose an integer sequence $\{L_k, k \geq 1\}$ such that $d(L_k) \sim \ln \ln L_k$ and (1.10) holds. It would be natural to choose $\{L_k, k \geq 1\}$ in such a way that $d(L_k)$ is as close as possible to $\ln \ln L_k$, $k \geq 1$. The difference $\ln \ln n - d(n)$ attains its minimum on each set $\mathcal{C}_k = \{N_k, \dots, N_{k+1} - 1\}$ at the point $n = N_k$ and $d(n)$ is almost equal to $\ln \ln n$ for such values of n . At the same time, we know that the difference $\ln \ln n - d(n)$ is increasing on each set $\mathcal{C}_k = \{N_k, \dots, N_{k+1} - 1\}$, so $\ln \ln n - d(n)$ attains its maximum at the point $n = N_{k+1} - 1$. But, when n «cross» the border between $\mathcal{C}_k$ and $\mathcal{C}_{k+1}$, i.e., when $n = N_{k+1}$, the difference $\ln \ln n - d(n)$ again becomes small. So it is natural to choose the sequence $L_k = N_{k+1}$, $k \geq 1$. Then $d(L_k) \sim \ln \ln L_k$ obviously. On the other hand, for any large $M' > 0$, by using Lemma 2.1 below and Remark 1, we have for any small $\varepsilon > 0$,

$$\begin{aligned} \sum_{k=1}^{\infty} \mathbf{P} \left\{ \frac{\max_{1 \leq i \leq L_k} |S_i|}{\sqrt{L_k / \ln \ln L_k}} \leq M' \right\} &\leq C \sum_{k=1}^{\infty} \exp \left\{ - \frac{(1-\varepsilon) \pi^2 L_k \ln \ln L_k}{8M'^2 L_k} \right\} \\ &= C \sum_{k=1}^{\infty} (\ln L_k)^{-(1-\varepsilon) \pi^2 / (8M'^2)} = C \sum_{k=1}^{\infty} (\ln N_{k+1})^{-(1-\varepsilon) \pi^2 / (8M'^2)} \\ &\leq C \sum_{k=1}^{\infty} (e^{M^k})^{-(1-\varepsilon) \pi^2 / (8M'^2)} < \infty. \end{aligned} \quad (1.12)$$

Here and in the sequel, C denotes a constant whose value can differ from line to line. So, by the Borel–Cantelli lemma,

$$\lim_{k \rightarrow \infty} \frac{\max_{1 \leq i \leq L_k} |S_i|}{\sqrt{L_k / \ln \ln L_k}} = \infty \quad \text{a.s.} \quad (1.13)$$

This shows that the Chung-type LIL cannot be applied in a simple way to obtain (1.10).

Remark 3. Klesov and Rosalsky [5] gave a very simple proof of Theorem A by using Strassen strong invariance principle, but it seems that this way does not work for the proof of Theorem 1.1.

2. Proofs. We first state the small deviation theorem obtained by Shao [7], [8].

Lemma 2.1. Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables with $\mathbf{E}X = 0$, $\mathbf{E}X^2 < \infty$, and $\sigma_n^2 = \sum_{i=1}^n \mathbf{E}X_i^2$, and let $\{x_n\}$ be a sequence of positive numbers. Assume that the following conditions are satisfied:

$$x_n \rightarrow \infty, \quad \frac{\sigma_n}{x_n} \rightarrow \infty, \quad \max_{1 \leq j \leq n} \frac{\mathbf{E}X_j^2}{x_n^2} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

and for any $\varepsilon > 0$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{\sigma_n^2} \sum_{j=1}^n \mathbf{E}X_j^2 I\{|X_j| \geq \varepsilon \sigma_n\} &= 0, \\ \lim_{n \rightarrow \infty} \max_{\substack{1 \leq i, i+j \leq n \\ \sigma_j^2 - \sigma_i^2 \geq \varepsilon x_n^2}} \frac{\sum_{k=i+1}^{i+j} \mathbf{E}X_k^2 I\{|X_k| \geq \varepsilon \sigma_n\}}{\sigma_j^2} &= 0. \end{aligned}$$

Then we have

$$\ln \mathbf{P} \left\{ \max_{1 \leq i \leq n} |S_i| \leq x_n \right\} \sim - \frac{\pi^2 \sigma_n^2}{8x_n^2} \quad \text{as } n \rightarrow \infty. \quad (2.1)$$

Proof of Theorem 1.1. First, we will give the proof of (1.11) $\Rightarrow$ (1.10). By (1.8) and the classical Chung-type LIL, we have immediately

$$\liminf_{n \rightarrow \infty} \frac{\max_{1 \leq i \leq n} |S_i|}{\sqrt{n/d(n)}} \leq \frac{\pi}{\sqrt{8}} \quad \text{a.s.} \quad (2.2)$$

So we only need to show

$$\liminf_{n \rightarrow \infty} \frac{\max_{1 \leq i \leq n} |S_i|}{\sqrt{n/d(n)}} \geq \frac{\pi}{\sqrt{8}} \quad \text{a.s.} \quad (2.3)$$

Define $n_k = \exp\{k^{M^2}\}$ and $m_k = \sum_{j=0}^k n_j$. It is easy to see that

$$1 \leq \frac{m_k}{n_k} \leq C \sum_{j=0}^{k-1} \frac{1}{k^2} + 1 = \frac{C}{k} + 1. \quad (2.4)$$

By Lemma 2.1, for any $\varepsilon > 0$ small enough,

$$\begin{aligned} \sum_{k=1}^{\infty} \mathbf{P} \left\{ \frac{\max_{m_{k-1} < i \leq m_k} |S_i - S_{m_{k-1}}|}{\sqrt{n_k/d(n_k)}} \leq (1-\varepsilon) \frac{\pi}{\sqrt{8}} \right\} &\leq \sum_{k=1}^{\infty} \exp \left\{ -\frac{(1-\varepsilon)d(n_k)}{(1-\varepsilon)^2} \right\} \\ &\leq \sum_{k=1}^{\infty} \exp \left\{ -\frac{\ln \ln n_k}{(1-\varepsilon)M} \right\} = \sum_{k=1}^{\infty} k^{-M^2/((1-\varepsilon)M)} < \infty. \end{aligned} \quad (2.5)$$

So by the Borel–Cantelli lemma,

$$\liminf_{k \rightarrow \infty} \frac{\max_{m_{k-1} < i \leq m_k} |S_i - S_{m_{k-1}}|}{\sqrt{n_k/d(n_k)}} \geq \frac{\pi}{\sqrt{8}} \quad \text{a.s.}, \quad (2.6)$$

which together with (2.4) implies

$$\liminf_{k \rightarrow \infty} \frac{\max_{m_{k-1} < i \leq m_k} |S_i - S_{m_{k-1}}|}{\sqrt{m_k/d(n_k)}} \geq \frac{\pi}{\sqrt{8}} \quad \text{a.s.} \quad (2.7)$$

For each $k \geq 1$, let $j(k)$ be a unique index $j(k) \geq 0$ such that

$$N_{j(k)} \leq n_k < N_{j(k)+1}.$$

Define

$$\mathcal{K} = \{k \geq 1: m_k \geq N_{j(k)+1}\}. \quad (2.8)$$

It is clear that $n_{k+1} > m_k$ for large k . Thus for all large $k, k' \in \mathcal{K}$,

$$k' > k \Rightarrow j(k') > j(k).$$

Now for any large $M' > 0$, by Lemma 2.1, (2.8) and Remark 1, for any $0 < \varepsilon < \frac{1}{2}$,

$$\begin{aligned} \sum_{k \in \mathcal{K}} \mathbf{P} \left\{ \frac{\max_{m_{k-1} < i \leq m_k} |S_i - S_{m_{k-1}}|}{\sqrt{m_k/\ln \ln m_k}} < M' \right\} &\leq \sum_{k \in \mathcal{K}} \exp \left\{ -\frac{(1-\varepsilon)\pi^2 n_k \ln \ln m_k}{8M'^2 m_k} \right\} \\ &\leq C \sum_{k \in \mathcal{K}} \exp \left\{ -\frac{\pi^2 \ln \ln m_k}{32M'^2} \right\} \leq C \sum_{k \in \mathcal{K}} \exp \left\{ -\frac{\pi^2 \ln \ln N_{j(k)+1}}{32M'^2} \right\} \\ &\leq C \sum_{k=1}^{\infty} (\ln N_k)^{-\pi^2/(32M'^2)} \leq C \sum_{k=1}^{\infty} (e^{M^k})^{-\pi^2/(32M'^2)} < \infty. \end{aligned} \quad (2.9)$$

So by the Borel–Cantelli lemma,

$$\lim_{k \rightarrow \infty, k \in \mathcal{K}} \frac{\max_{m_{k-1} < i \leq m_k} |S_i - S_{m_{k-1}}|}{\sqrt{m_k/\ln \ln m_k}} = \infty \quad \text{a.s.}, \quad (2.10)$$

which together with $d(n) \geq (\ln \ln n)/M$ imply that

$$\begin{aligned} & \lim_{k \rightarrow \infty, k \in \mathcal{K}} \frac{\max_{m_{k-1} < i \leq m_k} |S_i - S_{m_{k-1}}|}{\sqrt{m_k/d(m_k)}} \\ & \geq \frac{1}{\sqrt{M}} \lim_{k \rightarrow \infty, k \in \mathcal{K}} \frac{\max_{m_{k-1} < i \leq m_k} |S_i - S_{m_{k-1}}|}{\sqrt{m_k/\ln \ln m_k}} = \infty \quad \text{a.s.} \end{aligned} \quad (2.11)$$

Note that for all $k \notin \mathcal{K}$, we have $N_{j(k)} \leq n_k < m_k < N_{j(k)+1}$ and hence $d(n_k) = d(m_k)$ due to (1.7). Thus

$$\begin{aligned} & \liminf_{k \rightarrow \infty} \frac{\max_{m_{k-1} < i \leq m_k} |S_i - S_{m_{k-1}}|}{\sqrt{m_k/d(m_k)}} \\ & = \min \left\{ \liminf_{\substack{k \rightarrow \infty \\ k \in \mathcal{K}}} \frac{\max_{m_{k-1} < i \leq m_k} |S_i - S_{m_{k-1}}|}{\sqrt{m_k/d(m_k)}}, \liminf_{\substack{k \rightarrow \infty \\ k \notin \mathcal{K}}} \frac{\max_{m_{k-1} < i \leq m_k} |S_i - S_{m_{k-1}}|}{\sqrt{m_k/d(m_k)}} \right\} \\ & = \min \left\{ \infty, \liminf_{\substack{k \rightarrow \infty \\ k \notin \mathcal{K}}} \frac{\max_{m_{k-1} < i \leq m_k} |S_i - S_{m_{k-1}}|}{\sqrt{m_k/d(n_k)}} \right\} \geq \frac{\pi}{\sqrt{8}} \quad \text{a.s.} \end{aligned} \quad (2.12)$$

by (2.7). At the same time, by the classical LIL and the definitions of n_k and m_k , we can get

$$\begin{aligned} & \limsup_{k \rightarrow \infty} \frac{\max_{1 \leq i \leq m_{k-1}} |S_i|}{\sqrt{m_k/d(m_k)}} \\ & = \limsup_{k \rightarrow \infty} \max_{1 \leq i \leq m_{k-1}} \frac{|S_i|}{\sqrt{m_{k-1} \ln \ln m_{k-1}}} \cdot \sqrt{\frac{d(m_k) m_{k-1} \ln \ln m_{k-1}}{m_k}} \\ & \leq \sqrt{2} \limsup_{k \rightarrow \infty} \sqrt{\frac{m_{k-1}}{m_k} \ln \ln m_k} = 0. \end{aligned} \quad (2.13)$$

Then we can obtain

$$\begin{aligned} \liminf_{k \rightarrow \infty} \frac{\max_{1 \leq i \leq m_k} |S_i|}{\sqrt{m_k/d(m_k)}} & \geq \liminf_{k \rightarrow \infty} \frac{\max_{m_{k-1} < i \leq m_k} |S_i - S_{m_{k-1}}|}{\sqrt{m_k/d(m_k)}} \\ & \quad - 2 \limsup_{k \rightarrow \infty} \frac{\max_{1 \leq i \leq m_{k-1}} |S_i|}{\sqrt{m_k/d(m_k)}} \geq \frac{\pi}{\sqrt{8}} \quad \text{a.s.} \end{aligned} \quad (2.14)$$

by (2.12) and (2.13). Obviously, (2.3) is proved.

Now we will give the proof of (1.10) $\Rightarrow$ (1.11). We first show that $\mathbf{E}X^2 < \infty$. If $\mathbf{E}X^2 = \infty$, then by (1.8) and Theorem 3.2 in [1], we have

$$\mathbf{P} \left\{ \lim_{n \rightarrow \infty} \frac{\max_{1 \leq i \leq n} |S_i|}{\sqrt{n/d(n)}} = \infty \right\} = 1, \quad (2.15)$$

which is contradictory to (1.10). Then we will show that $\mathbf{E}X = 0$. If $\mathbf{E}X^2 < \infty$ and $\mathbf{E}X = \mu \neq 0$, then by the strong law of large numbers, we have $\lim_{n \rightarrow \infty} |S_n|/n = |\mu| > 0$ a.s. It follows that

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{\max_{1 \leq i \leq n} |S_i|}{\sqrt{n/d(n)}} & \geq \sqrt{\frac{1}{M}} \liminf_{n \rightarrow \infty} \frac{|S_n|}{\sqrt{n/\ln \ln n}} \\ & = \sqrt{\frac{1}{M}} \liminf_{n \rightarrow \infty} \frac{|S_n|}{n} \frac{n}{\sqrt{n/\ln \ln n}} = \infty \quad \text{a.s.,} \end{aligned} \quad (2.16)$$

which is also contradictory to (1.10). Next we will show that $\mathbf{E}X^2 = 1$ under the conditions $\mathbf{E}X^2 < \infty$ and $\mathbf{E}X = 0$. By the proof of the direct part,

$$\liminf_{n \rightarrow \infty} \frac{\max_{1 \leq i \leq n} |S_i|}{\sqrt{n/d(n)}} = \frac{\pi}{\sqrt{8}} \mathbf{E}X^2 \quad \text{a.s.} \quad (2.17)$$

Obviously, $\mathbf{E}X^2 = 1$, otherwise (1.10) and (2.17) are contradictory.

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