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THE LARGE DEVIATION PRINCIPLE FOR STOCHASTIC PROCESSES. II

В части I были приведены необходимые и достаточные условия выполнения принципа больших уклонений для случайных процессов. Здесь приводятся результаты, позволяющие получить регулярные функции скорости для принципа больших уклонений. Обсуждается принцип больших уклонений для гауссовских и пуассоновских процессов. В качестве применения устанавливается интегрируемость итерированного дробного броуновского движения.

Ключевые слова и фразы: большие уклонения, случайные процессы, гауссовские процессы, итерированное броуновское движение, процесс Пуассона.

4. The rate function of the LDP for stochastic processes. In many situations, the large deviations for the finite dimensional distributions can be obtained from the following theorem.

Theorem 4.1 ([17, Theorem II.2]). *Let $(U_n(1), \dots, U_n(m))$ be a sequence of r.v.'s with values in $\mathbf{R}^m$. Let ε_n be a sequence of positive numbers converging to zero. Suppose that:*

(i) *For each $\lambda_1, \dots, \lambda_m$, the following limit exists (the limit could be infinity)*

$$\lim_{n \rightarrow \infty} \varepsilon_n \ln \left(\mathbf{E} \left[\exp \left(\varepsilon_n^{-1} \sum_{j=1}^m \lambda_j U_n(j) \right) \right] \right) =: l(\lambda),$$

where $\lambda = (\lambda_1, \dots, \lambda_m)$;

(ii) *zero is in the interior of $\mathcal{D}(l) := \{\lambda \in \mathbf{R}^m: l(\lambda) < \infty\}$;*

(iii) *l is a lower semicontinuous convex function on $\mathbf{R}^m$;*

(iv) *$l(\lambda)$ is differentiable in the interior of $\mathcal{D}(l)$;*

(v) *if λ_n is a sequence in the interior of $\mathcal{D}(l)$ converging to a boundary point of $\mathcal{D}(l)$, then $\|\text{grad } l(\lambda_n)\| \rightarrow \infty$.*

Then $(U_n(1), \dots, U_n(m))$ satisfies the large deviation principle in $\mathbf{R}^m$ with speed ε_n^{-1} and rate function

$$I(u_1, \dots, u_m) = \sup \left\{ \sum_{j=1}^m \lambda_j u_j - l(\lambda_1, \dots, \lambda_m): \lambda_1, \dots, \lambda_m \in \mathbf{R} \right\}.$$

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In many cases, the function l in Theorem 4.1 can be written as

$$l(\lambda_1, \dots, \lambda_m) = \int_S \Phi \left(\sum_{j=1}^m \lambda_j f_j(x) \right) d\mu(x), \quad (4.1)$$

where $(S, \mathcal{S})$ is a measurable space, $f_1, \dots, f_m$ are measurable functions, μ is a measure on S and $\Phi: \mathbf{R} \rightarrow (-\infty, \infty]$ is a convex function. We will take either $\Phi(x) = e^x - 1$ or $\Phi(x) = p^{-1}|x|^p$ for some $p > 1$. In this section, we study the rate function in $l_\infty(T)$, when (4.1) holds for the finite dimensional distributions.

Lemma 4.1. *Let Φ be a convex function. Let $(S, \mathcal{S})$ be a measurable space. Let μ be a measure on S . Let $f_1, \dots, f_m$ be measurable functions in S . Let*

$$\begin{aligned} I^{(1)}(u_1, \dots, u_m) \\ = \sup \left\{ \sum_{j=1}^m \lambda_j u_j - \int \Phi \left(\sum_{j=1}^m \lambda_j f_j(x) \right) d\mu(x) : \lambda_1, \dots, \lambda_m \in \mathbf{R} \right\} \end{aligned}$$

and let

$$\begin{aligned} I^{(2)}(u_1, \dots, u_m) = \inf \left\{ \int \Psi(\gamma(x)) d\mu(x) : \right. \\ \left. \int f_j(x) \gamma(x) d\mu(x) = u_j \text{ for each } 1 \leq j \leq m \right\}, \end{aligned}$$

where Ψ is the conjugate convex function of Φ defined by

$$\Psi(y) = \sup_x (xy - \Phi(x)). \quad (4.2)$$

Then, for each $u_1, \dots, u_m \in \mathbf{R}$, $I^{(1)}(u_1, \dots, u_m) = I^{(2)}(u_1, \dots, u_m)$.

P r o o f. If $\int f_j(x) \gamma(x) d\mu(x) = u_j$, for each $1 \leq j \leq m$, by (4.2),

$$\begin{aligned} \sum_{j=1}^m \lambda_j u_j - \int \Phi \left(\sum_{j=1}^m \lambda_j f_j(x) \right) d\mu(x) \\ = \int \left(\sum_{j=1}^m \lambda_j f_j(x) \gamma(x) - \Phi \left(\sum_{j=1}^m \lambda_j f_j(x) \right) \right) d\mu(x) \leq \int \Psi(\gamma(x)) d\mu(x). \end{aligned}$$

Thus, $I^{(1)}(u_1, \dots, u_m) \leq I^{(2)}(u_1, \dots, u_m)$. Now, we may assume that $I^{(1)}(u_1, \dots, u_m) < \infty$. Since Φ is convex, it has a left and a right derivative (see for example Chapter I in [26]). Let φ be the right derivative of Φ . For the $\lambda_1, \dots, \lambda_m$ attaining the sup in $I^{(1)}(u_1, \dots, u_m)$, for each $1 \leq j \leq m$,

$$u_i = \int \varphi \left(\sum_{j=1}^m \lambda_j f_j(x) \right) f_i(x) d\mu(x).$$

Let $\gamma(x) = \varphi(\sum_{j=1}^m \lambda_j f_j(x))$. Since $x\varphi(x) = \Phi(x) + \Psi(\varphi(x))$ (see for example Theorem I.3.3 in [26]),

$$\begin{aligned} & \sum_{j=1}^m \lambda_j u_j - \int \Phi\left(\sum_{j=1}^m \lambda_j f_j(x)\right) d\mu(x) \\ &= \int \left(\sum_{j=1}^m \lambda_j f_j(x) \varphi\left(\sum_{j=1}^m \lambda_j f_j(x)\right) - \Phi\left(\sum_{j=1}^m \lambda_j f_j(x)\right) \right) d\mu(x) \\ &= \int \Psi(\gamma(x)) d\mu(x). \end{aligned}$$

Thus, $I^{(2)}(u_1, \dots, u_m) \leq I^{(1)}(u_1, \dots, u_m)$. Lemma 4.2 is proved.

Assuming that $\{U_n(t): t \in T\}$ satisfies the large deviation principle (LDP) and the conditions in the previous theorem hold, by Theorem 3.1 we have that for each $k \geq 1$, (T, ρ_k) is totally bounded, where ρ_k is as in (3.1). It is easy to see that this condition is equivalent to the following one: for each $k \geq 1$, (T, d_k) is totally bounded, where

$$d_k(s, t) = \sup \left\{ \left| \int (f(x, s) - f(x, t)) \gamma(x) d\mu(x) \right| : \int \Psi(\gamma(x)) d\mu(x) \leq k \right\}. \quad (4.3)$$

In some cases, previous pseudometric is an Orlicz norm. We recall some notation in Orlicz spaces from [26]. A function $\Phi_1: \mathbf{R} \rightarrow [0, \infty]$ is said to be a Young function if it is convex, $\Phi_1(0) = 0$, $\Phi_1(x) = \Phi_1(-x)$, and $\lim_{x \rightarrow \infty} \Phi_1(x) = \infty$. Given a measurable space $(S, \mathcal{S})$ and a measure μ on S , the Orlicz space $\mathcal{L}^{\Phi_1}(\mu)$ associated with the Young function Φ_1 is the class of measurable functions f on $(S, \mathcal{S})$ such that for some $\lambda > 0$ $\int \Phi_1(\lambda f) d\mu < \infty$. Define the Orlicz norm by

$$\|f\|_{\Phi_1} = \sup \left\{ \left| \int f g d\mu \right| : \int \Psi_1(|g|) d\mu \leq 1 \right\}, \quad (4.4)$$

and the gauge norm of the Orlicz space $\mathcal{L}^{\Phi_1}(\mu)$ by

$$N_{\Phi_1}(f) = \inf \left\{ t > 0: \int \Phi_1\left(\frac{f}{t}\right) d\mu \leq \Phi(1) \right\},$$

where Ψ_1 is the conjugate function of Φ_1 in the sense of (4.2). Assuming that $\Phi(1) < 1$, we have that

$$N_{\Phi_1}(f) \leq \|f\|_{\Phi_1} \leq 2N_{\Phi_1}(f)$$

(see Proposition III.3.4 in [26]). It is well known that the linear space $\mathcal{L}^{\Phi_1}(\mu)$ with the norm N_{Φ_1} is a Banach space. If the convex function Ψ is a Young

function, we have that the distance d_k in (4.3) is an Orlicz norm. Given a Young function Φ_1 , $\mathcal{M}^{\Phi_1}$ denotes the Banach space consisting with the class of functions f such the fact that for each $\lambda > 0$, $\int \Phi_1(\lambda|f|) d\mu < \infty$, with the norm N_{Φ_1} . We will use that $(\mathcal{M}^{\Phi_1})^* = \mathcal{L}^{\Psi_1}$ (see Theorem IV.1.7 in [26]). We will say that a sequence of functions γ_n in $\mathcal{L}^{\Psi_1}$ converges weakly to γ_0 in $\sigma(\mathcal{L}^{\Psi_1}, \mathcal{M}^{\Phi_1})$ if

$$\int \gamma_n f d\mu \longrightarrow \int \gamma_0 f d\mu$$

for each $f \in \mathcal{M}^{\Phi_1}$. A function Φ is called an N -function (a nice Young function) if Φ is a continuous Young function such that $\Phi(x) > 0$ for $x \neq 0$, $\lim_{x \rightarrow 0} x^{-1}\Phi(x) = 0$ and $\lim_{x \rightarrow \infty} x^{-1}\Phi(x) = \infty$. We will use that if Φ is an N -function, then a bounded set in $\mathcal{L}^{\Psi_1}$ is $\sigma(\mathcal{L}^{\Psi_1}, \mathcal{M}^{\Phi_1})$ -sequentially relatively compact (see Corollary IV.5.5 in [26]).

We will need the following lemma.

Lemma 4.2. *Let $\Psi: \mathbf{R} \rightarrow [0, \infty]$ be a convex function. Let $(S, \mathcal{S})$ be a measurable space. Let μ be a measure on S . Let γ be a function on S . Then*

$$\int \Psi(\gamma(x)) d\mu(x) = \sup \left\{ \sum_{j=1}^m \mu(B_j) \Psi\left(\frac{1}{\mu(B_j)} \int_{B_j} \gamma(x) d\mu(x)\right) : \right. \\ \left. B_1, \dots, B_m \text{ are disjoint sets and } 0 < \mu(B_j) < \infty \right\}.$$

The proof of the previous lemma is omitted since it is trivial.

Theorem 4.2. *Let $\Phi: \mathbf{R} \rightarrow [0, \infty)$ be a convex function such that $\Phi(0) = 0$, $\Phi'(0) = a$ exists, $\max(\Phi(x) - ax, \Phi(-x) + ax) > 0$ for each $x \neq 0$, and $\lim_{x \rightarrow \infty} x^{-1} \max(\Phi(x), \Phi(-x)) = \infty$. Let Ψ be the conjugate function of Φ in the sense of (4.2). Let $(S, \mathcal{S})$ be a measurable space. Let μ be a measure on S . Let $\{f(x, t): t \in T\}$ be a class of measurable functions. Suppose that*

(i) *for each $t \in T$ and each $\lambda > 0$,*

$$\int \Phi_1(\lambda f(x, t)) d\mu(x) < \infty,$$

where $\Phi_1(x) = \max(\Phi(x) - ax, \Phi(-x) + ax)$;

(ii) *(T, d) is totally bounded, where $d(s, t) = \sum_{k=1}^{\infty} k^{-2} \min(d_k(s, t), 1)$ and $d_k(s, t)$ is defined in (4.3);*

(iii) *if $a \neq 0$ suppose also that $\mu(S) < \infty$.*

Then

$$I(z) = \sup \left\{ I_{t_1, \dots, t_m}(z(t_1), \dots, z(t_m)): t_1, \dots, t_m \in T, m \geq 1 \right\},$$

where

$$I_{t_1, \dots, t_m}(u_1, \dots, u_m) = \inf \left\{ \int \Psi(\gamma(x)) d\mu(x) : \int f(x, t_j) \gamma(x) d\mu(x) = u_j, \right. \\ \left. \text{for each } 1 \leq j \leq m \right\}, \quad (4.5)$$

$$I(z) = \inf \left\{ \int \Psi(\gamma(x)) d\mu(x) : \int f(x, t) \gamma(x) d\mu(x) = z(t) \right. \\ \left. \text{for each } t \in T \right\}. \quad (4.6)$$

P r o o f. We have that $\Phi_1(x)$ is an N -function with conjugate $\Psi_1(x) = \min(\Psi(a+x), \Psi(a-x))$. Let

$$I^{(1)}(z) = \sup \left\{ I_{t_1, \dots, t_m}(z(t_1), \dots, z(t_m)) : t_1, \dots, t_m \in T, m \geq 1 \right\}.$$

Obviously, $I^{(1)}(z) \leq I(z)$. We need to prove that if $I^{(1)}(z) < \infty$, then $I^{(1)}(z) \geq I(z)$. Take $\{s_n\}$ such that $\{s_n\}$ is dense in (T, d) and $I^{(1)}(z) = \lim_{n \rightarrow \infty} I_{s_1, \dots, s_n}(z(s_1), \dots, z(s_n))$. Take γ_n such that $\int \gamma_n(x) f(x, s_j) d\mu(x) = z(s_j)$ for each $1 \leq j \leq n$ and

$$\int \Psi(\gamma_n(x)) d\mu(x) \leq I_{s_1, \dots, s_n}(z(s_1), \dots, z(s_n)) + n^{-1}.$$

Let $k_0 > I^{(1)}(z)$. We have that for n large enough $\int \Psi_1(\gamma_n(x) + a) d\mu(x) \leq k_0$. So, by Corollary IV.5.5 in [26], $\{\gamma_n + a\}$ is weakly compact in $(\mathcal{M}^{\Phi_1})^* = \mathcal{L}^{\Psi_1}$. Hence there exist a subsequence n_k and $\gamma_0 + a \in \mathcal{L}^{\Psi_1}$ such that $\gamma_{n_k} + a$ converges weakly to $\gamma_0 + a$ in $\sigma(\mathcal{L}^{\Psi_1}, \mathcal{M}^{\Phi_1})$. This implies that $\int \gamma_0(x) f(x, s_j) d\mu(x) = z(s_j)$ for each $j \geq 1$. By Lemma 4.2, $\int \Psi(\gamma_0(x)) d\mu(x) \leq I^{(1)}(z)$. Since z and $\int f(x, t) \gamma(x) d\mu(x)$ are d -uniformly continuous functions, $\int \gamma_0(x) f(x, t) d\mu(x) = z(t)$ for each $t \in T$. Theorem 4.2 is proved.

Unless $\lim_{x \rightarrow \infty} |x|^{-1} \max(\Phi(x), \Phi(-x)) = \infty$, the rate function does not have the form in the previous theorem (see [24]).

The results in this section translate to a Banach space in an usual way. Let B be a separable Banach space. Let $\{U_n\}$ be a sequence of r.v.'s with values in B . Suppose that for each $f \in B^*$,

$$\lim_{n \rightarrow \infty} \varepsilon_n \ln \left(\mathbf{E}[\exp(\varepsilon_n^{-1} f(U_n))] \right) = \int \Phi(f(x)) d\mu(x),$$

where μ is a measure on B and Φ is a convex function. Under the conditions in Theorem 4.2, the rate function for the LDP for $\{U_n\}$ with speed ε_n^{-1} is

$$I(z) = \inf \left\{ \int \Psi(\gamma(x)) d\mu(x) : \int x \gamma(x) d\mu(x) = z, \right. \\ \left. \gamma: B \rightarrow \mathbf{R} \text{ is a measurable function} \right\}.$$

Next, we consider the simplest case to which the previous lemmas apply.

Theorem 4.3. *Let $\{U_n(t): 0 \leq t \leq M\}$ be a sequence of stochastic processes. Let $\{\varepsilon_n\}$ be a sequence of positive numbers that converges to zero. Let Φ be a nonnegative convex function. Suppose that*

(i) *for each $0 \leq t_1 \leq \dots \leq t_m \leq M$ and each $\lambda_1, \dots, \lambda_m \in \mathbf{R}$,*

$$\lim_{n \rightarrow \infty} \varepsilon_n \ln \left(\mathbf{E} \left[\exp \left(\varepsilon_n^{-1} \sum_{j=1}^m \lambda_j U_n(t_j) \right) \right] \right) = \sum_{j=1}^m \Phi \left(\sum_{i=j}^m \lambda_i \right) (t_j - t_{j-1});$$

(ii) *for each $\eta > 0$,*

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \varepsilon_n \ln \left(\mathbf{P} \left\{ \sup_{\substack{|s-t| \leq \delta, \\ 0 \leq s, t \leq M}} |U_n(s) - U_n(t)| \geq \eta \right\} \right) = -\infty;$$

(iii) $\Phi(0) = 0$, $\Phi'(0) = a$ exists, $\max(\Phi(x) - ax, \Phi(-x) + ax) > 0$ for each $x \neq 0$, and $\lim_{x \rightarrow \infty} x^{-1} \max(\Phi(x), \Phi(-x)) = \infty$.

Then $\{U_n(t): 0 \leq t \leq M\}$ satisfies the LDP in $l_\infty[0, M]$ with speed ε_n^{-1} and rate function

$$I(z) = \begin{cases} \int_0^M \Psi(z'(t)) dt, & \text{if } z(0) = 0 \text{ and } z \text{ is absolutely continuous,} \\ \infty, & \text{otherwise.} \end{cases}$$

P r o o f. It follows from Theorems 4.1 and 4.2, and Lemma 4.1 with $f(x, t) = \mathbf{1}(0 \leq x \leq t)$ and μ equal to the Lebesgue measure. Observe that to have condition (iii) in Theorem 4.2, we need that for each $k \geq 1$, $([0, M], d_k)$ is totally bounded, where

$$d_k(s, t) = \sup \left\{ \left| \int_s^t \gamma(x) dx \right| : \int_0^M \Psi(\gamma(x)) dx \leq k \right\}.$$

It suffices to show that $\lim_{\eta \rightarrow 0} \sup_{0 \leq s, t \leq M, |s-t| \leq \eta} d_k(s, t) = 0$. But given $\lambda > 0$, $0 \leq s, t \leq M$ and γ with $\int_0^M \Psi(\gamma(x)) dx \leq k$,

$$\int_s^t \gamma(x) dx \leq \int_s^t \lambda^{-1} (\Psi(\gamma(x)) + \Phi(\lambda)) dx \leq \lambda^{-1} k + \lambda^{-1} |s - t| \Phi(\lambda),$$

and

$$-\int_s^t \gamma(x) dx \leq \int_s^t \lambda^{-1} (\Psi(\gamma(x)) + \Phi(-\lambda)) dx \leq \lambda^{-1} k + \lambda^{-1} |s - t| \Phi(-\lambda).$$

Hence

$$\sup_{\substack{0 \leq s, t \leq M, \\ |s-t| \leq \eta}} d_k(s, t) \leq \inf_{\lambda > 0} \left(\lambda^{-1} k + \lambda^{-1} \eta \max(\Phi(\lambda), \Phi(-\lambda)) \right),$$

which implies condition (iii) in Theorem 4.2. Theorem 4.3 is proved.

An analog of the previous theorem holds for processes defined on $[-M_1, M_2]$, where $M_1, M_2 > 0$.

The previous theorem can be used to give compositions of processes.

Theorem 4.4. Let $\{U_n(t): t \in \mathbf{R}\}$ and $\{V_n(t): 0 \leq t \leq M_0\}$ be two sequences of stochastic processes, where $M_0 > 0$. Let $\{\varepsilon_n\}$ be a sequence of positive numbers that converges to zero. Suppose that

(i) for each $0 \leq t_1 \leq \dots \leq t_m$, each $0 \leq s_1 \leq \dots \leq s_m$, each $0 \leq r_1 \leq \dots \leq r_m \leq M_0$, each $\lambda_1, \dots, \lambda_m \in \mathbf{R}$, each $\tau_1, \dots, \tau_m \in \mathbf{R}$ and each $\nu_1, \dots, \nu_m \in \mathbf{R}$,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \varepsilon_n \ln \left(\mathbf{E} \left[\exp \left(\varepsilon_n^{-1} \left(\sum_{j=1}^m \lambda_j U_n(t_j) + \sum_{j=1}^m \tau_j U_n(-s_j) + \sum_{j=1}^m \nu_j V_n(r_j) \right) \right) \right] \right) \\ &= \sum_{j=1}^m \Phi_1 \left(\sum_{i=j}^m \lambda_i \right) (t_j - t_{j-1}) + \sum_{j=1}^m \Phi_1 \left(\sum_{i=j}^m \tau_i \right) (s_j - s_{j-1}) \\ &+ \sum_{j=1}^m \Phi_2 \left(\sum_{i=j}^m \nu_i \right) (r_j - r_{j-1}), \end{aligned}$$

where Φ_1 and Φ_2 are two nonnegative convex functions;

(ii) for each $\eta > 0$ and each $0 < M < \infty$,

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \varepsilon_n \ln \left(\mathbf{P} \left\{ \sup_{\substack{|s-t| \leq \delta, \\ |s|, |t| \leq M}} |U_n(s) - U_n(t)| \geq \eta \right\} \right) = -\infty;$$

(iii) for each $\eta > 0$,

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \varepsilon_n \ln \left(\mathbf{P} \left\{ \sup_{\substack{|s-t| \leq \delta, \\ 0 \leq s, t \leq M_0}} |V_n(s) - V_n(t)| \geq \eta \right\} \right) = -\infty;$$

(iv) for $i = 1, 2$, $\Phi_i(0) = 0$, $\Phi'_i(0) = a_i$ exists, $\max(\Phi_i(x) - a_i x, \Phi_i(-x) + a_i x) > 0$ for $x > 0$ and $\lim_{x \rightarrow \infty} x^{-1} \max(\Phi_i(x), \Phi_i(-x)) = \infty$;

(v) $\max(\Phi_2(\Phi_1(x)) - a_1 a_2 x, \Phi_2(\Phi_1(-x)) + a_1 a_2 x) > 0$ for $x > 0$ and $\lim_{x \rightarrow \infty} x^{-1} \max(\Phi_2(\Phi_1(x)), \Phi_2(\Phi_1(-x))) = \infty$.

Then $\{U_n(V_n(t)): 0 \leq t \leq M_0\}$ satisfies the LDP in $l_\infty[0, M_0]$ with speed ε_n^{-1} and the rate function

$$I(z) = \begin{cases} \int_0^{M_0} \Psi_{2,1}(z'(t)) dt, & \text{if } z(0) = 0 \text{ and } z \text{ is absolutely continuous,} \\ \infty, & \text{otherwise,} \end{cases}$$

where $\Psi_{2,1}$ is the conjugate of $\Phi_2 \circ \Phi_1$.

P r o o f. By Theorems 3.3 and 4.3, $\{U_n(V_n(t)): 0 \leq t \leq M_0\}$ satisfies the LDP in $l_\infty[0, M_0]$ with speed ε_n^{-1} and the rate function

$$\begin{aligned} I(z) &= \inf \left\{ \int_0^\infty \Psi_1(\alpha'(t)) dt + \int_0^{M_0} \Psi_2(\beta'(t)) dt : \right. \\ &\quad \left. z = \alpha \circ \beta, \alpha(0) = 0 \text{ and } \beta(0) = 0 \right\}. \end{aligned}$$

So, the rate function for the LDP for the finite dimensional distributions is

$$\begin{aligned}
 I_{t_1, \dots, t_m}(u_1, \dots, u_m) &= \inf \left\{ \int_0^\infty \Psi_1(\alpha'(t)) dt + \int_0^{M_0} \Psi_2(\beta'(t)) dt : \right. \\
 &\quad \left. z = \alpha \circ \beta, \gamma(t_j) = u_j, 1 \leq j \leq m \right\} \\
 &= \inf \left\{ \sum_{j=1}^m \int_{v_{j-1}}^{v_j} \Psi_1(\alpha'(t)) dt + \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \Psi_2(\beta'(t)) dt : \right. \\
 &\quad \left. \beta(t_j) = v_j, \alpha(v_j) = u_j, 1 \leq j \leq m \right\}.
 \end{aligned}$$

By Jensen's inequality

$$\begin{aligned}
 \int_{v_{j-1}}^{v_j} \Psi_1(\alpha'(t)) dt &\geq (v_j - v_{j-1}) \Psi_1 \left((v_j - v_{j-1})^{-1} \int_{v_{j-1}}^{v_j} \alpha'(t) dt \right) \\
 &= (v_j - v_{j-1}) \Psi_1 \left(\frac{u_j - u_{j-1}}{v_j - v_{j-1}} \right),
 \end{aligned}$$

where we have equality if α' is a constant. A similar inequality holds for Ψ_2 . So,

$$\begin{aligned}
 I_{t_1, \dots, t_m}(u_1, \dots, u_m) &= \inf \left\{ \sum_{j=1}^m (v_j - v_{j-1}) \Psi_1 \left(\frac{u_{j-1} - u_j}{v_j - v_{j-1}} \right) \right. \\
 &\quad \left. + \sum_{j=1}^m (t_j - t_{j-1}) \Psi_2 \left(\frac{v_j - v_{j-1}}{t_j - t_{j-1}} \right) : v_1, \dots, v_m \right\} \\
 &= \inf \left\{ \sum_{j=1}^m (v_j - v_{j-1}) \Psi_1 \left(\frac{u_{j-1} - u_j}{v_j - v_{j-1}} \right) \right. \\
 &\quad \left. + \sum_{j=1}^m (t_j - t_{j-1}) \Psi_2 \left(\frac{v_j - v_{j-1}}{t_j - t_{j-1}} \right) : v_1, \dots, v_m \right\} \\
 &= \sum_{j=1}^m \inf \left\{ v \Psi_1 \left(\frac{u_{j-1} - u_j}{v} \right) + (t_j - t_{j-1}) \Psi_2 \left(\frac{v}{t_j - t_{j-1}} \right) : v \right\} \\
 &= \sum_{j=1}^m (t_j - t_{j-1}) \inf \left\{ v \Psi_1 \left(\frac{u_{j-1} - u_j}{v(t_j - t_{j-1})} \right) + \Psi_2(v) : v \right\} \\
 &= \sum_{j=1}^m (t_j - t_{j-1}) \Psi_{2,1} \left(\frac{u_{j-1} - u_j}{t_j - t_{j-1}} \right). \tag{4.7}
 \end{aligned}$$

Observe that by the minimax theorem (see for example Theorem 3.6.3 in [27])

$$\begin{aligned} \inf_v \{v\Psi_1(v^{-1}x) + \Psi_2(v)\} &= \inf_v \sup_b \{v(v^{-1}xb - \Phi_1(b)) + \Psi_2(v)\} \\ &= \sup_b \inf_v \{xb - v\Phi_1(b) + \Psi_2(v)\} = \sup_b \left\{xb - \sup_v \{v\Phi_1(b) - \Psi_2(v)\}\right\} \\ &= \sup_b \{xb - \Phi_2(\Phi_1(b))\} = \Psi_{2,1}(x). \end{aligned}$$

Since the rate function for the finite dimensional distributions is given by (4.7), Theorem 4.2 implies that the rate function is as claimed. Theorem 4.4 is proved.

In Theorem 4.4, we do not require $\Phi_2 \circ \Phi_1$ to be a convex function. The previous result is related with Lemma 2.1 on the Strassen class appearing in the law of the iterated logarithm in [10].

5. The LDP for Gaussian processes. In this section, we consider the large deviation principle for Gaussian processes. In the case of a sequence of Gaussian r.v.'s, we have the following theorem.

Theorem 5.1. *Let $\{X_n\}_{n=1}^\infty$ be a sequence of Gaussian r.v.'s with mean μ_n and variance σ_n^2 . Let $\{\varepsilon_n\}$ be a sequence of positive numbers converging to zero. Then the following conditions are equivalent:*

- (a) *there are $\mu \in \mathbf{R}$ and $0 \leq a < \infty$ such that $\lim_{n \rightarrow \infty} \mu_n = \mu$ and $\lim_{n \rightarrow \infty} \varepsilon_n^{-1} \sigma_n^2 = a$;*
- (b) *$\{X_n\}_{n=1}^\infty$ satisfies the LDP with speed ε_n^{-1} .*

Moreover, if either (a) or (b) holds, the rate function is $I(t) = (t - \mu)^2/(2a)$, if $a > 0$; $I(t) = 0$ if $t = \mu$ and $a = 0$; and $I(t) = \infty$ if $t \neq \mu$ and $a = 0$.

The proof of this theorem is omitted, since it follows from well-known estimations on the tail of a standard normal r.v.

To obtain the rate function in the LDP for Gaussian processes, we use the results of Section 4 with $\Phi(x) = \Psi(x) = 2^{-1}x^2$. Suppose that there exist a measurable space $(S, \mathcal{S})$, a positive measure μ on S and a class of measurable functions $\{f(x, t): t \in T\}$ such that the rate function for the finite dimensional distributions is

$$\begin{aligned} I_{t_1, \dots, t_m}(u_1, \dots, u_m) &= \inf \left\{ \int 2^{-1} \gamma^2(x) d\mu(x): \int f(x, t_j) \gamma(x) d\mu(x) = u_j, \right. \\ &\quad \left. \text{for each } 1 \leq j \leq m \right\}. \end{aligned}$$

The rate function for the stochastic process is

$$\begin{aligned} I(z) &= \inf \left\{ \int 2^{-1} \gamma^2(x) d\mu(x): \int f(x, t) \gamma(x) d\mu(x) = z(t), \right. \\ &\quad \left. \text{for each } t \in T \right\}, \end{aligned} \tag{5.1}$$

where $z \in l_\infty(T)$. Sometimes it is preferable to write this rate function using reproducing kernel Hilbert spaces. In the previous situation,

$$R(s, t) = \int f(x, t) f(x, s) d\mu(x)$$

is a covariance function, i.e., for each $\lambda_1, \dots, \lambda_m \in \mathbf{R}$ and each $t_1, \dots, t_m \in T$,

$$\sum_{i,j=1}^m \lambda_i \lambda_j R(t_i, t_j) \geq 0.$$

Hence there exists a mean-zero Gaussian process $\{Z(t): t \in T\}$ such that $\mathbf{E}[Z(s)Z(t)] = R(s, t)$ for each $s, t \in T$. Let $\mathcal{L}$ be the closed linear subspace of L_2 , generated by $\{Z(t): t \in T\}$. Let $\phi: \mathcal{L} \rightarrow l_\infty(T)$ be defined by $\phi(\xi)(t) = \mathbf{E}[Z(t)\xi]$. The reproducing kernel Hilbert space of the covariance function $R(s, t)$ is the Hilbert space $\{\phi(\xi): \xi \in \mathcal{L}\}$ with respect to the inner product $\langle \phi(\xi_1), \phi(\xi_2) \rangle = \mathbf{E}[\xi_1 \xi_2]$. The rate function in (5.1) can be written also as

$$I(z) = \inf \left\{ 2^{-1} \mathbf{E}[\xi^2]: \xi \in \mathcal{L}, \phi(\xi) = z \right\}. \quad (5.2)$$

The next theorem gives necessary and sufficient conditions for the LDP for Gaussian processes.

Theorem 5.2. *Let $\{U_n(t): t \in T\}$, $n \geq 1$, be a sequence of centered Gaussian processes. Let $\{\varepsilon_n\}$ be a sequence of positive numbers such that $\varepsilon_n \rightarrow 0$. Then the following sets of conditions ((a) and (b)) are equivalent:*

- (a.1) *for each $s, t \in T$, $\varepsilon_n^{-1} \mathbf{E}[U_n(s) U_n(t)]$ converges as $n \rightarrow \infty$;*
- (a.2) *(T, d) is totally bounded, where*

$$d^2(s, t) = \lim_{n \rightarrow \infty} \varepsilon_n^{-1} \mathbf{E}[(U_n(s) - U_n(t))^2];$$

- (a.3) $\sup_{t \in T} |U_n(t)| \xrightarrow{\mathbf{P}} 0$;
- (a.4) $\lim_{\eta \rightarrow 0} \limsup_{n \rightarrow \infty} \sup_{d(s, t) \leq \eta} \varepsilon_n^{-1} \mathbf{E}[(U_n(s) - U_n(t))^2] = 0$;
- (b) *$\{U_n(t): t \in T\}$ satisfies the LDP in $l_\infty(T)$ with speed ε_n^{-1} .*

Moreover, if either (a) or (b) holds, the rate function is defined by (5.2), where

$$R(s, t) = \lim_{n \rightarrow \infty} \varepsilon_n^{-1} \mathbf{E}[U_n(s) U_n(t)].$$

P r o o f. Assume conditions (a). We apply Theorem 3.1. By condition (a.2), condition (a.1) in Theorem 3.1 is satisfied.

Given $t_1, \dots, t_m \in T$ and $\lambda_1, \dots, \lambda_m \in \mathbf{R}$, we have that

$$\varepsilon_n \ln \left(\mathbf{E} \left[\exp \left(\varepsilon_n^{-1} \sum_{j=1}^m \lambda_j U_n(t_j) \right) \right] \right) \rightarrow 2^{-1} \mathbf{E} \left[\left(\sum_{j=1}^m \lambda_j Z(t_j) \right)^2 \right],$$

where $\{Z(t): t \in T\}$ is a centered Gaussian process with covariance $\mathbf{E}[Z(s)Z(t)] = R(s, t)$, $s, t \in T$. The previous limit and the Ellis theorem

(Theorem 4.1) imply condition (a.2) in Theorem 3.1 with the rate function

$$I_{t_1, \dots, t_m}(u_1, \dots, u_m) = \sup_{\lambda_1, \dots, \lambda_m} \left\{ \sum_{j=1}^m \lambda_j u_j - 2^{-1} \sum_{j,k=1}^m \lambda_j \lambda_k \mathbf{E}[Z(t_j) Z(t_k)] \right\}.$$

By Lemma 4.1, this rate function can be expressed as

$$\inf \left\{ 2^{-1} \mathbf{E}[\xi^2] : \xi \in \mathcal{L}, \mathbf{E}[\xi Z(t_j)] = u_j \text{ for each } 1 \leq j \leq m \right\}.$$

By the isoperimetric inequality for Gaussian processes ([30] and [5]),

$$\begin{aligned} & \mathbf{P}^* \left\{ \left| \sup_{d(s,t) \leq \eta} |U_n(s) - U_n(t)| - M_n \right| \geq u \right\} \\ & \leq \exp \left(- \frac{u^2}{2 \sup_{d(s,t) \leq \eta} \mathbf{E}[|U_n(s) - U_n(t)|^2]} \right), \end{aligned}$$

where M_n is the median of $\sup_{d(s,t) \leq \eta} |U_n(s) - U_n(t)|$. This inequality and (a.4) imply (a.3) in Theorem 3.1. Therefore, (b) in Theorem 3.1 holds with the rate function

$$I(z) = \sup \left\{ I_{t_1, \dots, t_m}(z(t_1), \dots, z(t_m)) : t_1, \dots, t_m \in T, m \geq 1 \right\},$$

where $I_{t_1, \dots, t_m}$ was defined above. By Theorem 4.2 this rate function can be expressed as in (5.2).

Assume condition (b). The contraction principle implies that for each $t_1, t_2 \in T$ and each $\lambda_1, \lambda_2 \in \mathbf{R}$, the LDP for $\lambda_1 U_n(t_1) + \lambda_2 U_n(t_2)$ with speed ε_n^{-1} holds. By Theorem 5.1, for each $t_1, t_2 \in T$ and each $\lambda_1, \lambda_2 \in \mathbf{R}$, $\varepsilon_n^{-1} \mathbf{E}[(\lambda_1 U_n(t_1) + \lambda_2 U_n(t_2))^2]$ converges. Therefore, condition (a.1) holds. Besides this implies that the rate function for the finite dimensional distributions is

$$\begin{aligned} & I_{t_1, \dots, t_m}(u_1, \dots, u_m) \\ & = \inf \left\{ 2^{-1} \mathbf{E}[\xi^2] : \xi \in \mathcal{L}, \mathbf{E}[\xi Z(t_j)] = u_j \text{ for each } 1 \leq j \leq m \right\}. \end{aligned}$$

So, $\rho_k(s, t) = \sup\{|u_2 - u_1| : I_{s,t}(u_1, u_2) \leq k\} = d(s, t)(2k)^{1/2}$. So, by condition (a.1) in Theorem 3.1, (a.2) holds.

By condition (a.3) in Theorem 3.1, for each $\tau > 0$,

$$\lim_{\eta \rightarrow 0} \limsup_{n \rightarrow \infty} \left(\mathbf{P}^* \left\{ \sup_{d(s,t) \leq \eta} |U_n(t) - U_n(s)| \geq \tau \right\} \right) = -\infty.$$

This and the fact that $U_n(t) \xrightarrow{\mathbf{P}} 0$, for each $t \in T$, implies (a.3).

Condition (a.3) in Theorem 3.1 also implies that for each $\tau > 0$,

$$\lim_{\eta \rightarrow 0} \sup_{d(s,t) \leq \eta} \limsup_{n \rightarrow \infty} \varepsilon_n \ln \left(\mathbf{P} \left\{ |U_n(t) - U_n(s)| \geq \tau \right\} \right) = -\infty,$$

which implies (a.4). Theorem 5.2 is proved.

Large deviations for Gaussian processes have been considered by several authors. Schilder [28] considered large deviations for the Brownian motion. In our notation, Chevet [8] proved that if $\{\varepsilon_n^{-1/2}U_n(t): t \in T\}$ converges weakly to a Radon Gaussian process $\{Z(t): t \in T\}$, then $\{U_n(t): t \in T\}$ satisfies the LDP with speed ε_n^{-1} . The previous theorem generalizes Theorem 2 in [8]. It is easy to find examples which do not satisfy the conditions in the Chevet theorem. Let $\{g_k\}_{k=1}^\infty$ be a sequence of independent identically distributed r.v.'s with standard normal distribution. Let $\{a_n\}_{n=1}^\infty$ be a sequence of real numbers converging to infinity. Let $U_n(k) = a_n^{-1}(\ln k)^{-1/2}g_k$. It follows from Theorem 5.2 that $\{U_n(k): k \geq 1\}$, $n \geq 1$, satisfies the LDP in $l_\infty(\mathbf{N})$ with speed a_n^2 . However, it is not true that $\{a_n U_n(k): k \geq 1\}$ converges weakly to a Gaussian process in $l_\infty(\mathbf{N})$.

The previous theorem implies that if $\{B(t): t \geq 0\}$ is a Brownian motion and $\{a_n\}$ is a sequence of real numbers such that $n^{-1/2}a_n \rightarrow 0$, then, for each $0 < M < \infty$,

$$\{a_n B(n^{-1}t): 0 \leq t \leq M\}$$

satisfies the LDP in $l_\infty[0, M]$ with speed na_n^{-2} and rate function

$$I(z) = \begin{cases} \int_0^M 2^{-1}|z'(t)|^2 dt, & \text{if } z(0) = 0 \\ & \text{and } z \text{ is absolutely continuous,} \\ \infty, & \text{otherwise} \end{cases} \quad (5.3)$$

(this result is due to [28]). The next theorem considers centered Gaussian processes with stationary increments.

Theorem 5.3. *Let $\{X(t): t \geq 0\}$ be a centered Gaussian process with stationary increments and $X(0) = 0$. Let $0 < M < \infty$. Let $\{a_n\}$ be a sequence of real numbers such that $a_n^2 \mathbf{E}[X^2(n^{-1}M)] \rightarrow 0$. Then the following sets of conditions are equivalent:*

(a.1) *for some $0 < \alpha \leq 1$, for each $0 < t$,*

$$\lim_{n \rightarrow \infty} \frac{\mathbf{E}[X^2(n^{-1}t)]}{\mathbf{E}[X^2(n^{-1}M)]} = t^{2\alpha} M^{-2\alpha};$$

(a.2) $\sup_{0 \leq t \leq M} a_n |X(n^{-1}t)| \xrightarrow{P} 0$;

(b) $\{a_n X(n^{-1}t): 0 \leq t \leq M\}$ *satisfies the large deviation principle with speed $a_n^{-2}(\mathbf{E}[X^2(n^{-1}M)])^{-1}$.*

Moreover, for $\alpha = \frac{1}{2}$, the rate function is given by (5.3); if $0 < \alpha < 1$ and $\alpha \neq \frac{1}{2}$, the rate function is given by

$$I(z) = \inf \left\{ 2^{-1} \tau_\alpha \int_{-\infty}^{\infty} \phi^2(x) dx: \right. \\ \left. \tau_\alpha \int_{-\infty}^{\infty} \phi(x) \left(|x-t|^{(2\alpha-1)/2} - |x|^{(2\alpha-1)/2} \right) dx = z(t), \right. \\ \left. \text{for each } 0 \leq t \leq M \right\}, \quad (5.4)$$

where $z \in l_\infty([0, 1])$ and

$$\tau_\alpha = \left(\int_{-\infty}^{\infty} \left(|x-1|^{(2\alpha-1)/2} - |x|^{(2\alpha-1)/2} \right)^2 dx \right)^{-1};$$

and if $\alpha = 1$, the rate function is

$$I(z) = \begin{cases} 2^{-1}a^2, & \text{if for some } a, z(t) = at \text{ for each } 0 \leq t \leq M, \\ \infty, & \text{otherwise.} \end{cases} \quad (5.5)$$

P r o o f. Without loss of generality, we may assume that $M = 1$. Assume (a), we apply Theorem 5.2. Conditions (a.1)–(a.3) in Theorem 5.2 are assumed. By regular variation,

$$\lim_{\eta \rightarrow 0+} \limsup_{n \rightarrow \infty} \sup_{0 < t \leq \eta} \frac{\mathbf{E}[X^2(n^{-1}t)]}{\mathbf{E}[X^2(n^{-1})]} = 0.$$

This implies condition (a.4) in Theorem 5.2.

Assume condition (b). Theorem 5.2 implies (a.2). By Theorem 5.1, for each $0 \leq s, t \leq 1$,

$$\lim_{n \rightarrow \infty} \frac{\mathbf{E}[X(n^{-1}s)X(n^{-1}t)]}{\mathbf{E}[X^2(n^{-1})]}$$

exists. By Theorem 1.9 in [4], $\mathbf{E}[X^2(n^{-1}t)]$ is regularly varying as $t \rightarrow 0$. Hence there exists an $\alpha \in \mathbf{R}$ such that for each $t > 0$,

$$\lim_{n \rightarrow \infty} \frac{\mathbf{E}[X^2(n^{-1}t)]}{\mathbf{E}[X^2(n^{-1})]} = t^{2\alpha}.$$

Since condition (a.4) in Theorem 5.2 holds, $\alpha > 0$. For $0 \leq s < t$, we have that $\|X(t)\|_2 \leq \|X(s)\|_2 + \|X(t) - X(s)\|_2$. Hence we have that $t^\alpha \leq s^\alpha + |t - s|^\alpha$. Taking $t = 2s$, we get that $\alpha \leq 1$.

If $\alpha \neq \frac{1}{2}$, by the change of variable $x = ty$, we get that

$$\int_{-\infty}^{\infty} \left(|x-t|^{(2\alpha-1)/2} - |x|^{(2\alpha-1)/2} \right)^2 dx = |t|^{2\alpha} \tau_\alpha^{-1}.$$

Hence

$$\begin{aligned} R(s, t) &= 2^{-1}(s^{2\alpha} + t^{2\alpha} - |s-t|^{2\alpha}) \\ &= \tau_\alpha \int_{-\infty}^{\infty} \left(|x-s|^{(2\alpha-1)/2} - |x|^{(2\alpha-1)/2} \right) \\ &\quad \times \left(|x-t|^{(2\alpha-1)/2} - |x|^{(2\alpha-1)/2} \right) dx. \end{aligned}$$

We take the measure μ defined on $\mathbf{R}$ by $d\mu(x) = \tau_\alpha dx$ and $f(x, t) = |x - t|^{(2\alpha-1)/2} - |x|^{(2\alpha-1)/2}$. From Theorem 4.2, we get that the rate function for the LDP when $0 < \alpha < 1$ and $\alpha \neq \frac{1}{2}$, is given by (5.4).

If $\alpha = \frac{1}{2}$, we have that the rate function is given by (5.3).

If $\alpha = 1$, $R(s, t) = st$. We apply Theorem 4.2 with $S = [0, 1]$, $f(x, t) = t$ and μ equal to the Lebesgue measure, we get that the rate function is as in (5.5). Theorem 5.3 is proved.

A centered Gaussian process $\{B_\alpha(t): t \geq 0\}$, is a fractional Brownian motion of order α , $0 < \alpha < 1$, if its covariance is given by

$$\mathbf{E}[B_\alpha(s)B_\alpha(t)] = 2^{-1}(s^{2\alpha} + t^{2\alpha} - |s - t|^{2\alpha}), \quad s, t \geq 0.$$

It is easy to see that Theorem 5.3 applies to the fractional Brownian motion of order α , if $a_n n^{-\alpha} \rightarrow 0$.

Theorems 3.3 and 4.4 allows one to obtain LDP for compositions of Gaussian processes.

Theorem 5.4. *Let $\{B(t): t \in \mathbf{R}\}$ be a Brownian motion. Let $\{a_n\}$ be a sequence of real numbers such that $a_n \rightarrow \infty$ and $n^{-1}a_n^2 \rightarrow 0$. Let $0 < M < \infty$. Then*

$$\left\{ \frac{a_n^{2-2^{-k+1}}}{n^{1-2^{-k+1}}} B^{(k)}(n^{-1}t): 0 \leq t \leq M \right\},$$

where $B^{(k)} = B \circ \dots \circ B$, satisfies the LDP in $l_\infty[0, M]$ with speed na_n^{-2} and rate function

$$I(z) = \begin{cases} \int_0^M \Psi_{(k)}(z'(t)) dt, & \text{if } z(0) = 0 \\ & \text{and } z \text{ is absolutely continuous,} \\ \infty, & \text{otherwise,} \end{cases} \quad (5.6)$$

where

$$\Psi_{(k)}(x) = 2^{-k2^k/(2^k-1)}(2^{k+1} - 2) |x|^{2^k/(2^k-1)}.$$

P r o o f. We only consider the composition of two Brownian motions in detail. The general case is similar. We apply Theorem 4.4 with $\Phi_1(x) = \Phi_2(x) = 2^{-1}x^2$. By Theorem 5.3, for each $0 < M_1 < \infty$,

$$\left\{ n^{-1/2}a_n^{3/2}B(a_n^{-1}t): -M_1 \leq t \leq M_1 \right\}$$

satisfies the LDP with speed na_n^{-2} . We also have that $\{a_n B(n^{-1}t): 0 \leq t \leq M\}$ satisfies the LDP with rate na_n^{-2} . This implies conditions (ii) and (iii) in Theorem 4.4. We need to obtain the LDP for the finite dimensional distributions (condition (i) in this theorem); given $0 \leq t_1 < \dots < t_m \leq M_1$ and $0 \leq s_1 < \dots < s_m \leq M_1$ and $0 \leq r_1 < \dots < r_m \leq M$, consider

$$\left(n^{-1/2}a_n^{3/2}B(-a_n^{-1}t_m), \dots, n^{-1/2}a_n^{3/2}B(-a_n^{-1}t_1), n^{-1/2}a_n^{3/2}B(a_n^{-1}s_1), \dots, \right. \\ \left. n^{-1/2}a_n^{3/2}B(a_n^{-1}s_m), a_n B(n^{-1}r_1), \dots, a_n B(n^{-1}r_m) \right).$$

Given $\lambda_1, \dots, \lambda_m, \tau_1, \dots, \tau_m, \nu_1, \dots, \nu_m \in \mathbf{R}$,

$$\begin{aligned}
& n^{-1} a_n^2 \ln \left(\mathbf{E} \left[\exp \left(n a_n^{-2} \left(\sum_{i=1}^m \lambda_i n^{-1/2} a_n^{3/2} B(-a_n^{-1} t_i) \right. \right. \right. \right. \\
& \quad \left. \left. \left. + \sum_{i=1}^m \tau_i n^{-1/2} a_n^{3/2} B(a_n^{-1} s_i) + \sum_{j=1}^m \nu_j a_n B(n^{-1} r_j) \right) \right) \right] \right) \\
&= n^{-1} a_n^2 \ln \left(\mathbf{E} \left[\exp \left(\sum_{i=1}^m \lambda_i n^{1/2} a_n^{-1/2} B(-a_n^{-1} t_i) \right. \right. \right. \\
& \quad \left. \left. \left. + \sum_{i=1}^m \tau_i n^{1/2} a_n^{-1/2} B(a_n^{-1} s_i) + \sum_{j=1}^m \nu_j n a_n^{-1} B(n^{-1} r_j) \right) \right] \right) \\
&= 2^{-1} n^{-1} a_n^2 \mathbf{E} \left[\left(\sum_{i=1}^m \lambda_i n^{1/2} a_n^{-1/2} B(-a_n^{-1} t_i) \right)^2 \right. \\
& \quad \left. + \left(\sum_{i=1}^m \tau_i n^{1/2} a_n^{-1/2} B(a_n^{-1} s_i) \right)^2 + \left(\sum_{j=1}^m \nu_j n a_n^{-1} B(n^{-1} r_j) \right)^2 \right. \\
& \quad \left. + 2 \sum_{i=1}^m \lambda_i n^{1/2} a_n^{-1/2} B(-a_n^{-1} t_i) \sum_{j=1}^m \tau_j n^{1/2} a_n^{-1/2} B(a_n^{-1} s_j) \right. \\
& \quad \left. + 2 \sum_{i=1}^m \lambda_i n^{1/2} a_n^{-1/2} B(-a_n^{-1} t_i) \sum_{j=1}^m \nu_j n a_n^{-1} B(n^{-1} r_j) \right. \\
& \quad \left. + 2 \sum_{i=1}^m \tau_i n^{1/2} a_n^{-1/2} B(a_n^{-1} s_i) \sum_{j=1}^m \nu_j n a_n^{-1} B(n^{-1} r_j) \right] \\
&\longrightarrow \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m \left(\lambda_i \lambda_j \min(t_i, t_j) + \tau_i \tau_j \min(s_i, s_j) + \nu_i \nu_j \min(r_i, r_j) \right) \\
&= \sum_{j=1}^m \left(\Phi_1 \left(\sum_{i=j}^m \lambda_i \right) (t_j - t_{j-1}) + \Phi_1 \left(\sum_{i=j}^m \tau_i \right) (s_j - s_{j-1}) \right. \\
& \quad \left. + \Phi_2 \left(\sum_{i=j}^m \nu_i \right) (r_j - r_{j-1}) \right).
\end{aligned}$$

We have that $\Psi_{2,1}(x) = \sup_y (xy - \Phi_2(\Phi_1(y))) = 2^{-5/3} \cdot 3|x|^{4/3}$. The rest of the conditions in Theorem 4.4 are trivially satisfied.

In the general case, we prove that

$$\left\{ \frac{a_n^{2-2^{-j+1}}}{n^{1-2^{-j+1}}} B \left(\frac{n^{1-2^{-j+2}}}{a_n^{2-2^{-j+2}}} t \right) : 0 \leq t \leq M \right\}, \quad 1 \leq j \leq k,$$

satisfy the LDP jointly with speed na_n^{-2} and the rate in Theorem 4.4 corresponding to $\Phi(x) = 2^{-1}x^2$. By composition, we get that

$$\left\{ \frac{a_n^{2^{-2^{-k+1}}}}{n^{1-2^{-k+1}}} B^{(k)}(n^{-1}t); 0 \leq t \leq M \right\}$$

satisfies the LDP in $l_\infty[0, M]$ with speed na_n^{-2} and rate as in Theorem 4.4 with

$$\begin{aligned} \Psi_{(k)}(x) &= \sup_y \left(xy - \Phi \circ \dots \circ \Phi(y) \right) = \sup_y (xy - 2^{-2^{k+1}} y^{2^k}) \\ &= \frac{2^{k+1} - 2}{2^{k2^k/(2^k-1)}} |x|^{2^k/(2^k-1)}. \end{aligned}$$

Theorem 5.4 is proved.

Next, we present a law of the iterated logarithm for the iterated Brownian motion.

Theorem 5.5. *Let $\{B(t): t \in \mathbf{R}\}$ be a Brownian motion. Let $0 < M < \infty$. Then, with probability one,*

$$\left\{ \frac{n^{2^{-k}}}{(\ln \ln n)^{1-2^{-k}}} B^{(k)}(n^{-1}t); 0 \leq t \leq M \right\} \quad (5.7)$$

is relatively compact in $l_\infty[0, M]$ and its limit set is

$$\begin{aligned} &\left\{ z: [0, M] \rightarrow \mathbf{R}: z(0) = 0, \text{ } z \text{ is absolutely continuous} \right. \\ &\quad \left. \text{and } \int_0^M \frac{2^{k+1} - 2}{2^{k2^k/(2^k-1)}} |z'(t)|^{2^k/(2^k-1)} dt \leq 1 \right\}. \end{aligned} \quad (5.8)$$

In particular,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{n^{2^{-k}}}{(\ln \ln n)^{1-2^{-k}}} B^{(k)}(n^{-1}M) &= \frac{2^k M^{2^{-k}}}{(2^{k+1} - 2)^{(2^k-1)/2^k}} \quad a.s., \\ \liminf_{n \rightarrow \infty} \frac{n^{2^{-k}}}{(\ln \ln n)^{1-2^{-k}}} B^{(k)}(n^{-1}M) &= \frac{-2^k M^{2^{-k}}}{(2^{k+1} - 2)^{(2^k-1)/2^k}} \quad a.s., \\ \limsup_{n \rightarrow \infty} \sup_{0 \leq t \leq M} \frac{n^{2^{-k}}}{(\ln \ln n)^{1-2^{-k}}} B^{(k)}(n^{-1}t) &= \frac{2^k M^{2^{-k}}}{(2^{k+1} - 2)^{(2^k-1)/2^k}} \quad a.s. \end{aligned}$$

P r o o f. By Theorem 4.2 in [1], for each $0 < M < \infty$,

$$\{(Y_{n,1}(t), \dots, Y_{n,k}(t)): |t| \leq M\}$$

is relatively compact in $l_\infty[-M, M]$ and its limit set is

$$\left\{ (\alpha_1, \dots, \alpha_k): \alpha_j(0) = 0, \text{ for } 1 \leq j \leq k, \text{ and } \sum_{j=1}^k \int_{-M}^M 2^{-1} (\alpha'_j(t))^2 dt \leq 1 \right\},$$

where

$$Y_{n,j}(t) = \left\{ \frac{n^{2^{-j}}}{(\ln \ln n)^{1-2^{-j}}} B^{(j)} \left(\frac{(\ln \ln n)^{1-2^{-j+1}} t}{n^{2^{-j+1}}} \right) : 0 \leq t \leq M \right\}.$$

By composing the stochastic processes, we have that, with probability one, the stochastic process in (5.7) is relatively compact in $l_\infty[-M, M]$ and its limit set is

$$\left\{ \alpha_1 \circ \cdots \circ \alpha_k : \alpha_j(0) = 0, \text{ for } 1 \leq j \leq k, \right. \\ \left. \text{and } \int_{-M}^M 2^{-1} (\alpha'_1(t))^2 dt + \sum_{j=2}^k \int_{-\infty}^{\infty} 2^{-1} (\alpha'_j(t))^2 dt \leq 1 \right\}.$$

The proof of Theorem 4.4 implies that this set is the same one as in (5.8).

The second part of the theorem follows by noticing that

$$\begin{aligned} & \sup \left\{ z(M) : z(0) = 0 \text{ and } \int_0^M \frac{2^{k+1} - 2}{2^{k2^k/(2^k-1)}} |z'(t)|^{2^k/(2^k-1)} dt \leq 1 \right\} \\ &= -\inf \left\{ z(M) : z(0) = 0 \text{ and } \int_0^M \frac{2^{k+1} - 2}{2^{k2^k/(2^k-1)}} |z'(t)|^{2^k/(2^k-1)} dt \leq 1 \right\} \\ &= \sup \left\{ z(t) : 0 \leq t \leq M, z(0) = 0 \text{ and } \right. \\ & \quad \left. \int_0^M \frac{2^{k+1} - 2}{2^{k2^k/(2^k-1)}} |z'(t)|^{2^k/(2^k-1)} dt \leq 1 \right\} = \frac{2^k M^{2^{-k}}}{(2^{k+1} - 2)^{(2^k-1)/2^k}}. \end{aligned}$$

Observe that

$$z(M) \leq \left(\int_0^M |z'(t)|^{2^k/(2^k-1)} dt \right)^{(2^k-1)/2^k} M^{1/2^k} \leq \frac{2^k M^{2^{-k}}}{(2^{k+1} - 2)^{(2^k-1)/2^k}}$$

and we have equality if z' is a constant. Theorem 5.5 is proved.

The law of the iterated logarithm for the composition of two independent Brownian motion was obtained by Deheuvels and Mason [11] and Burdzy [6]. More general versions of the compact law of the iterated logarithm for the composition of two Brownian motions are in [9] and [1].

The next theorem gives the integrability of the iterated Brownian motion.

Theorem 5.6. *Let $\{B(t) : t \in \mathbf{R}\}$ be a Brownian motion. Then*

- (i) $\{n^{-1+2^{-k}} B^{(k)}(t) : 0 \leq t \leq M\}$ satisfies the LDP in $l_\infty[0, M]$ with speed n and the rate function in (5.6);
- (ii) for each $0 < M < \infty$,

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} \lambda^{-2^k/(2^k-1)} \ln \left(\mathbf{P} \left\{ |B^{(k)}(M)| \geq \lambda \right\} \right) &= \frac{-(2^{k+1} - 2)}{2^{k2^k/(2^k-1)} M^{1/(2^k-1)}}, \\ \lim_{\lambda \rightarrow \infty} \lambda^{-2^k/(2^k-1)} \ln \left(\mathbf{P} \left\{ \sup_{0 \leq t \leq M} |B^{(k)}(t)| \geq \lambda \right\} \right) &= \frac{-(2^{k+1} - 2)}{2^{k2^k/(2^k-1)} M^{1/(2^k-1)}}; \end{aligned}$$

(iii) in particular,

$$\begin{aligned} \mathbf{E} \left[\exp \left(\lambda \sup_{0 \leq t \leq M} |B^{(k)}(t)|^{2^k/(2^k-1)} \right) \right] &< \infty \quad \text{if } \lambda < \frac{2^{k+1} - 2}{2^{k2^k/(2^k-1)} M^{1/(2^k-1)}}, \\ \mathbf{E} \left[\exp \left(\lambda \sup_{0 \leq t \leq M} |B^{(k)}(t)|^{2^k/(2^k-1)} \right) \right] &= \infty \quad \text{if } \lambda > \frac{2^{k+1} - 2}{2^{k2^k/(2^k-1)} M^{1/(2^k-1)}}. \end{aligned}$$

P r o o f. By an argument in the proof of Theorem 5.4, we get that

$$\left\{ n^{-2^{-1}+2^{-j}} B(n^{-2^{-(j-1)}} t): 0 \leq t \leq M \right\}, \quad 1 \leq j \leq k,$$

satisfy the LDP jointly with speed n and the rate function $\Phi(t) = 2^{-1}t^2$. Hence

$$\left\{ n^{-2^{-1}+2^{-k}} B \left(n^{-1/2} B \left(n^{1/2} \dots (B(n^{-1/2} B(n^{-1}t))) \dots \right) \right): 0 \leq t \leq M \right\}$$

satisfies the LDP with speed n and the rate function in (5.6). Using that $\{n^{1/2} B(n^{-1}t): 0 \leq t \leq M\}$ has the same distribution as $\{B(t): 0 \leq t \leq M\}$, we get that $\{n^{-1+2^{-k}} B^{(k)}(t): 0 \leq t \leq M\}$ satisfies the LDP with speed n and the rate function in (5.6).

Assertion (ii) follows from the fact that

$$\begin{aligned} \inf \left\{ \int_0^M \frac{2^{k+1} - 2}{2^{k2^k/(2^k-1)}} |z'(t)|^{2^k/(2^k-1)} dt: z(0) = 0, \sup_{0 \leq t \leq M} |z(t)| \geq 1 \right\} \\ = \frac{2^{k+1} - 2}{2^{k2^k/(2^k-1)} M^{1/(2^k-1)}}. \end{aligned}$$

Observe that if $\sup_{0 \leq t \leq M} |z(t)| \geq 1$ and

$$\int_0^M \frac{2^{k+1} - 2}{2^{k2^k/(2^k-1)}} |z'(t)|^{2^k/(2^k-1)} dt < \infty,$$

then there exists a t_0 , $0 < t_0 \leq M$ such that

$$1 \leq |z(t_0)| \leq \left(\int_0^M |z'(t)|^{2^k/(2^k-1)} dt \right)^{(2^k-1)/2^k} t_0^{1/2^k} \leq \frac{2^k (I(z))^{(2^k-1)/2^k} M^{2^{-k}}}{(2^{k+1} - 2)^{(2^k-1)/2^k}},$$

with equality for $t_0 = M$ and z' constant.

(iii) follows immediately from (ii). Theorem 5.6 is proved.

Finally, we consider the iterated fractional Brownian motion.

Theorem 5.7. *Let $\{B_\alpha(t) \in \mathbf{R}\}$ be a fractional Brownian motion of order α with $0 < \alpha < 1$ and $\alpha \neq \frac{1}{2}$. Let $\{a_n\}$ be a sequence of real numbers such that $a_n \rightarrow \infty$ and $n^{-\alpha} a_n \rightarrow 0$. Then*

$$\left\{ n^{(\alpha^k - \alpha)/(1-\alpha)} a_n^{(1-\alpha^k)/(1-\alpha)} B_\alpha^{(k)}(n^{-1}t): 0 \leq t \leq M \right\}$$

satisfies the LDP in $l_\infty[0, M]$ with speed $n^{2\alpha}a_n^{-2}$ and rate function

$$I(\gamma) = \inf \left\{ 2^{-1}\tau_\alpha \sum_{j=1}^k \int_{-\infty}^{\infty} \phi_j^2(x) dx: \gamma(t) = \beta_1 \circ \cdots \circ \beta_k(t) \right. \\ \text{for each } 0 \leq t \leq M, \\ \tau_\alpha \int_{-\infty}^{\infty} \phi_j(x) \left(|x-t|^{(2\alpha-1)/2} - |x|^{(2\alpha-1)/2} \right) dx = \beta_j(t) \\ \left. \text{for each } t \in \mathbf{R} \text{ and each } 1 \leq j \leq k \right\}. \quad (5.9)$$

P r o o f. We only consider the composition of two processes. The general case is similar. We apply Theorem 3.3. By Theorem 5.3, for $0 < M_1 < \infty$,

$$\{n^{-\alpha}a_n^{\alpha+1}B_\alpha(a_n^{-1}t): -M_1 \leq t \leq M_1\}$$

satisfies the LDP with speed $n^{2\alpha}a_n^{-2}$ and the rate function in (5.4). We also have that $\{a_nB_\alpha(n^{-1}t): 0 \leq t \leq M\}$ satisfies the LDP with speed $n^{2\alpha}a_n^{-2}$ and the rate function in (5.4). To prove the joint LPD for the two stochastic processes, we need to prove the LDP for the joint finite dimensional distributions. Given $-M_1 \leq t_1 < \cdots < t_m \leq M_1$ and $0 \leq s_1 < \cdots < s_m \leq M$, we need to prove the LDP for

$$(n^{-\alpha}a_n^{\alpha+1}B_\alpha(a_n^{-1}t_1), \dots, n^{-\alpha}a_n^{\alpha+1}B_\alpha(a_n^{-1}t_p), a_nB_\alpha(n^{-1}s_1), \dots, a_nB_\alpha(n^{-1}s_q)).$$

Given $\lambda_1, \dots, \lambda_p, \tau_1, \dots, \tau_q \in \mathbf{R}$, we have

$$\begin{aligned} & n^{-2\alpha}a_n^2 \ln \left(\mathbf{E} \left[\exp \left(n^{2\alpha}a_n^{-2} \left(\sum_{i=1}^p \lambda_i n^{-\alpha}a_n^{\alpha+1}B_\alpha(a_n^{-1}t_i) \right. \right. \right. \right. \\ & \quad \left. \left. \left. + \sum_{j=1}^q \tau_j a_n B_\alpha(n^{-1}s_j) \right) \right) \right] \right) \\ &= 2^{-1}n^{2\alpha}a_n^{-2} \mathbf{E} \left[\left(\sum_{i=1}^p \lambda_i n^{-\alpha}a_n^{\alpha+1}B_\alpha(a_n^{-1}t_i) \right)^2 + \left(\sum_{j=1}^q \tau_j a_n B_\alpha(n^{-1}s_j) \right)^2 \right. \\ & \quad \left. + 2 \sum_{i=1}^p \sum_{j=1}^q \lambda_i \tau_j n^{-\alpha}a_n^{\alpha+1}B_\alpha(a_n^{-1}t_i) a_n B_\alpha(n^{-1}s_j) \right] \\ &\longrightarrow 2^{-1} \sum_{i=1}^p \sum_{j=1}^p \lambda_i \lambda_j \mathbf{E}[B_\alpha(t_i) B_\alpha(t_j)] + 2^{-1} \sum_{i=1}^q \sum_{j=1}^q \tau_i \tau_j \mathbf{E}[B_\alpha(s_i) B_\alpha(s_j)], \end{aligned}$$

which implies the LDP for the joint finite dimensional distributions. Theorem 5.7 is proved.

The methods used before for the Brownian motion give the following result for the fractional Brownian motion.

Theorem 5.8. Let $\{B_\alpha(t) \in \mathbf{R}\}$ be a fractional Brownian motion of order α , $0 < \alpha < 1$, $\alpha \neq \frac{1}{2}$. Let $0 < M < \infty$. Then, with probability one,

$$\left\{ \frac{n^{\alpha^k}}{(\ln \ln n)^{(1-\alpha^k)/(2(1-\alpha))}} B^{(k)}(n^{-1}t): 0 \leq t \leq M \right\}$$

is relatively compact in $l_\infty[0, M]$ and its limit set is

$$\left\{ \beta_1 \circ \cdots \circ \beta_k: \sum_{j=1}^k 2^{-1} \tau_\alpha \int_{-\infty}^{\infty} \phi_j^2(x) dx \leq 1, \right. \\ \left. \tau_\alpha \int_{-\infty}^{\infty} \phi_j(x) \left(|x-t|^{(2\alpha-1)/2} - |x|^{(2\alpha-1)/2} \right) dx = \beta_j(t) \right. \\ \left. \text{for each } t \in \mathbf{R} \text{ and each } 1 \leq j \leq k \right\}.$$

In particular,

$$\limsup_{n \rightarrow \infty} \frac{n^{\alpha^k}}{(\ln \ln n)^{(1-\alpha^k)/(2(1-\alpha))}} B^{(k)}(n^{-1}M) = C_k(\alpha) M^{\alpha^k} \quad a.s., \\ \liminf_{n \rightarrow \infty} \frac{n^{\alpha^k}}{(\ln \ln n)^{(1-\alpha^k)/(2(1-\alpha))}} B^{(k)}(n^{-1}M) = C_k(\alpha) M^{\alpha^k} \quad a.s., \\ \limsup_{n \rightarrow \infty} \sup_{0 \leq t \leq M} \frac{n^{\alpha^k}}{(\ln \ln n)^{(1-\alpha^k)/(2(1-\alpha))}} B^{(k)}(n^{-1}t) = C_k(\alpha) M^{\alpha^k} \quad a.s.,$$

where

$$C_k(\alpha) = \left(\frac{2(1-\alpha)}{1-\alpha^k} \right)^{(1-\alpha^k)/(1-\alpha)} \alpha^{[(k-1)\alpha^{k+1}-k\alpha^k+\alpha]/(1-\alpha)^2}.$$

P r o o f. The first part follows similarly to that for the Brownian motion. We just need to show that

$$\sup \left\{ \beta_1 \circ \cdots \circ \beta_k(M): \sum_{j=1}^k 2^{-1} \tau_\alpha \int_{-\infty}^{\infty} \phi_j^2(x) dx \leq 1, \right. \\ \left. \tau_\alpha \int_{-\infty}^{\infty} \phi_j(x) \left(|x-t|^{(2\alpha-1)/2} - |x|^{(2\alpha-1)/2} \right) dx = \beta_j(t) \right. \\ \left. \text{for each } t \in \mathbf{R} \text{ and each } 1 \leq j \leq k \right\} = C_k(\alpha).$$

We have that

$$\beta_k(M) \leq \tau_\alpha \|\phi_k\|_2 \left(\int_{-\infty}^{\infty} \left(|x-1|^{(2\alpha-1)/2} - |x|^{(2\alpha-1)/2} \right)^2 dx \right)^{1/2} M^\alpha \\ = \tau_\alpha^{1/2} \|\phi_k\|_2 M^\alpha, \\ \beta_{k-1}(\beta_k(M)) \leq \tau_\alpha^{1/2} \|\phi_{k-1}\|_2 (\tau_\alpha^{1/2} \|\phi_k\|_2)^\alpha M^{\alpha^2},$$

and by induction $\beta_1 \circ \dots \circ \beta_k(M) \leq M^{\alpha^k} \prod_{j=1}^k (\tau_\alpha^{1/2} \|\phi_j\|_2)^{\alpha^{j-1}}$. Moreover, we have equality if $\phi_k(x) = |x - M|^{(2\alpha-1)/2} - |x|^{(2\alpha-1)/2}$ and

$$\phi_j(x) = |x - \beta_{j+1}(M)|^{(2\alpha-1)/2} - |x|^{(2\alpha-1)/2} \quad \text{for } 1 \leq j \leq k-1.$$

Therefore,

$$C_k(\alpha) = \sup \left\{ \prod_{j=1}^k |x_j|^{\alpha^{j-1}} : 2^{-1} \sum_{j=1}^k x_j^2 \leq 1 \right\}.$$

To find the supremum in the previous expression, we use the theorem of the multipliers of Lagrange. We have that $\alpha^{j-1} x_j^{-1} \prod_{i=1}^k |x_i|^{\alpha^{i-1}} = \lambda x_j$, for each $1 \leq j \leq k$, where $\lambda \in \mathbf{R}$. So, $x_j^2 = \alpha^{j-1} c$, where c is a constant. Since

$$1 = 2^{-1} \sum_{j=1}^k x_j^2 = 2^{-1} \sum_{j=1}^k \alpha^{j-1} c = 2^{-1} (1 - \alpha^k) (1 - \alpha)^{-1} c,$$

we have $c = 2(1 - \alpha)(1 - \alpha^k)^{-1}$ and $x_j^2 = \alpha^{j-1} 2(1 - \alpha)(1 - \alpha^k)^{-1}$. Thus,

$$\begin{aligned} C_k(\alpha) &= \sup \left\{ \prod_{j=1}^k (\alpha^{j-1} 2(1 - \alpha)(1 - \alpha^k)^{-1})^{\alpha^{j-1}} \right\} \\ &= \left(\frac{2(1 - \alpha)}{1 - \alpha^k} \right)^{(1 - \alpha^k)/(1 - \alpha)} \alpha^{[(k-1)\alpha^{k+1} - k\alpha^k + \alpha]/(1 - \alpha)^2}. \end{aligned}$$

Theorem 5.8 is proved.

Theorem 5.9. Let $\{B_\alpha(t) \in \mathbf{R}\}$ be a fractional Brownian motion of order α , $0 < \alpha < 1$, $\alpha \neq \frac{1}{2}$. Then

- (i) $\{n^{-\alpha(1-\alpha^k)/(1-\alpha)} B^{(k)}(t) : 0 \leq t \leq M\}$ satisfies the LDP in $l_\infty[0, M]$ with speed $n^{2\alpha}$ and rate function in (5.7);
- (ii) for each $0 < M < \infty$,

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} \lambda^{-2(1-\alpha)/(1-\alpha^k)} \ln \left(\mathbf{P} \{ |B_\alpha^{(k)}(M)| \geq \lambda \} \right) &= -D_k(\alpha) M^{-2\alpha^k(1-\alpha)/(1-\alpha^k)}, \\ \lim_{\lambda \rightarrow \infty} \lambda^{-2(1-\alpha)/(1-\alpha^k)} \ln \left(\mathbf{P} \left\{ \sup_{0 \leq t \leq M} |B_\alpha^{(k)}(t)| \geq \lambda \right\} \right) \\ &= -D_k(\alpha) M^{-2\alpha^k(1-\alpha)/(1-\alpha^k)}, \end{aligned}$$

where

$$D_k(\alpha) = \frac{1 - \alpha^k}{2(1 - \alpha) \alpha^{[(k-1)\alpha^{k+1} - k\alpha^k + \alpha]/[(1-\alpha)(1-\alpha^k)]}}.$$

P r o o f. (i) follows similarly to the case of Brownian motion. As to (ii), by the arguments in the proof of the previous theorem, we have that

$$\begin{aligned} \inf \left\{ I_k(z) : \sup_{0 \leq t \leq M} |z(t)| \geq 1 \right\} &= \inf \left\{ \sum_{j=1}^k 2^{-1} x_j^2 : M^{\alpha^k} \prod_{j=1}^k x_j^{\alpha^{j-1}} \geq 1 \right\} \\ &= \frac{1 - \alpha^k}{2(1 - \alpha) \alpha^{[(k-1)\alpha^{k+1} - k\alpha^k + \alpha]/[(1-\alpha)(1-\alpha^k)]} M^{2\alpha^k(1-\alpha)/(1-\alpha^k)}}, \end{aligned}$$

where the infimum is attained when $x_j^2 = c\alpha^{j-1}$, for some constant c . Theorem 5.9 is proved.

6. The LDP for Poisson processes. In this section, we present several results on the LDP for Poisson processes.

By the Cramér theorem (see for example Theorem 1.26 in [13]) if $\{X_n\}$ is a sequence of independent identically distributed r.v.'s with Poisson distribution and mean n , then $\{n^{-1}X_n\}$ satisfies the LDP with speed n and rate function $h(x) = \sup_{\lambda \in \mathbf{R}} (x\lambda - (e^\lambda - 1))$. It is easy to see that

$$h(x) = \begin{cases} x \ln \left(\frac{x}{e} \right) + 1, & \text{if } x \geq 0; \\ \infty, & \text{if } x < 0. \end{cases} \quad (6.1)$$

We will use the fact that if ξ is a Poisson r.v. with mean λ , then for $a, t > 0$,

$$\mathbf{P}\{\xi \geq a\} \leq e^{-ta} \mathbf{E}[e^{t\xi}] = \exp(-ta + \lambda(e^t - 1)).$$

Taking the supremum over $t > 0$, we get that for $a > \lambda$,

$$\mathbf{P}\{\xi \geq a\} \leq \exp(-\lambda h(\lambda^{-1}a)). \quad (6.2)$$

Instead of dividing by n in $\{n^{-1}X_n\}$, we can divide by a sequence of real numbers of growing faster than n . The LDP in this case is given by the following theorem.

Theorem 6.1. *Let $\{X_n\}$ be a sequence of Poisson random variables with $\mathbf{E}[X_n] = n$ and let $\{a_n\}$ be a sequence of positive numbers such that $n^{-1}a_n \rightarrow \infty$. Then $\{a_n^{-1}X_n\}$ satisfies the LDP in $[0, \infty)$ with speed $a_n \ln(n^{-1}a_n)$ and rate function $I(t) = t$ for $t \geq 0$; $I(t) = \infty$ for $t < 0$.*

P r o o f. By (6.2), given $t > 0$, for n large enough,

$$\mathbf{P}\{a_n^{-1}X_n \geq t\} \leq \exp\left(-a_n t \ln\left(\frac{a_n t}{n}\right) + ta_n - n\right).$$

Hence

$$\begin{aligned} & \limsup_{n \rightarrow \infty} a_n^{-1} (\ln(n^{-1}a_n))^{-1} \ln(\mathbf{P}\{a_n^{-1}X_n \geq t\}) \\ & \leq -\limsup_{n \rightarrow \infty} \left(\frac{t \ln(n^{-1}a_n t)}{\ln(n^{-1}a_n)} + \frac{t}{\ln(n^{-1}a_n)} - \frac{1}{n^{-1}a_n \ln(n^{-1}a_n)} \right) = -t. \end{aligned}$$

This implies that for each closed set $F \subset [0, \infty)$,

$$\limsup_{n \rightarrow \infty} a_n^{-1} (\ln(n^{-1}a_n))^{-1} \ln(\mathbf{P}\{a_n^{-1}X_n \in F\}) \leq -\inf\{t: t \in F\}.$$

Let U be an open set of $[0, \infty)$ and let $t \in U$. Let $k_n = [a_n t]$. For n large enough, by the Stirling formula

$$\begin{aligned} \mathbf{P}\{a_n^{-1}X_n \in U\} & \geq \mathbf{P}\{X_n = k_n\} = e^{-n} \frac{n^{k_n}}{k_n!} \simeq e^{-n} n^{k_n} k_n^{-k_n} e^{k_n} (2\pi k_n)^{-1/2} \\ & \simeq e^{-n} n^{a_n t} (ta_n)^{-ta_n} e^{ta_n} (2\pi ta_n)^{-1/2} = e^{-n} (n^{-1}ta_n)^{-ta_n} e^{ta_n} (2\pi ta_n)^{-1/2}. \end{aligned}$$

So, $\liminf_{n \rightarrow \infty} a_n^{-1} (\ln(n^{-1} a_n))^{-1} \ln(\mathbf{P}\{a_n^{-1} X_n \in U\}) \geq -t$. Therefore, the claim follows. Theorem 6.1 is proved.

Next, we consider the LDP for Poisson processes. The LDP for homogeneous Poisson processes has been considered by Lynch and Sethuraman [24]. We consider nonhomogeneous Poisson processes.

Theorem 6.2. *Let $\{N(t): t \geq 0\}$ be a Poisson process with a mean measure μ such that $\mu[0, \infty) = \infty$. Let $0 < M < \infty$. Then the following conditions are equivalent:*

- (a) *either $\mu[0, x]$ is regularly varying at infinity with index $\alpha > 0$ or $\lim_{n \rightarrow \infty} (\mu[0, Mn])^{-1} \mu[0, Mn] = 0$;*
- (b) *$\{(\mu[0, Mn])^{-1} N(tn): 0 \leq t \leq M\}$ satisfies the LDP in $l_\infty[0, M]$ with speed $\mu[0, Mn]$.*

Moreover, if $\mu[0, x]$ is regularly varying at infinity with index $\alpha > 0$, the rate function is

$$I(z) = \begin{cases} \int_0^M h(\alpha^{-1} t^{1-\alpha} M^\alpha z'(t)) \alpha t^{\alpha-1} M^{-\alpha} dt, & \text{if } z \text{ is absolutely continuous and } z(0) = 0; \\ \infty, & \text{otherwise,} \end{cases} \quad (6.3)$$

where $h(x)$ is as in (6.1). If $\lim_{M \rightarrow \infty} (\mu[0, M])^{-1} \mu[0, M] = 0$, the rate function is

$$I(z) = \begin{cases} h(z(M)) & \text{if } z(t) = 0 \text{ for } 0 \leq t < M, \\ \infty, & \text{otherwise.} \end{cases} \quad (6.4)$$

P r o o f. Let $\varepsilon_n = (\mu[0, nM])^{-1}$. First, we prove that (a) implies (b). Suppose first that $\mu[0, x]$ is regularly varying at infinity with index $\alpha > 0$. We apply Corollary 3.6. Given $0 \leq t_1 < \dots < t_m$ and $\lambda_1, \dots, \lambda_m \in \mathbf{R}$, we have that

$$\begin{aligned} \varepsilon_n \ln \left(\mathbf{E} \left[\exp \left(\sum_{j=1}^m \lambda_j N(t_j n) \right) \right] \right) &= \varepsilon_n \sum_{j=1}^m \left(e^{\sum_{i=j}^m \lambda_i} - 1 \right) \mu(t_{j-1} n, t_j n] \\ &\rightarrow \sum_{j=1}^m \left(\exp \left(\sum_{i=j}^m \lambda_i \right) - 1 \right) (t_j^\alpha - t_{j-1}^\alpha) M^{-\alpha} \\ &= \int_0^M \left(\exp \left(\sum_{j=1}^m \lambda_j \mathbf{1}(0 \leq x \leq t_j) \right) - 1 \right) \alpha x^{\alpha-1} M^{-\alpha} dx. \end{aligned}$$

The previous limit and Theorem II.2 in [17] imply that $(\varepsilon_n N(t_1 n), \dots, \varepsilon_n N(t_m n))$ satisfies the LDP with speed ε_n^{-1} , i.e., condition (ii) in Corollary 3.6 holds.

Given $0 \leq s < t \leq M$,

$$\begin{aligned} \varepsilon_n \ln \left(\mathbf{E} \left[\exp (\lambda_1 N(sn) + \lambda_2 N(tn)) \right] \right) \\ \rightarrow s^\alpha M^{-\alpha} (e^{\lambda_1 + \lambda_2} - 1) + (t^\alpha - s^\alpha) M^{-\alpha} (e^{\lambda_2} - 1). \end{aligned}$$

Hence the rate function for large deviations of $(\varepsilon_n N(sn), \varepsilon_n N(tn))$ is

$$\begin{aligned}
 I_{s,t}(u_1, u_2) &= \sup_{\lambda_1, \lambda_2} \left(\lambda_1 u_1 + \lambda_2 u_2 - s^\alpha M^{-\alpha} (e^{\lambda_1 + \lambda_2} - 1) \right. \\
 &\quad \left. - (t^\alpha - s^\alpha) M^{-\alpha} (e^{\lambda_2} - 1) \right) \\
 &= \sup_{\lambda_1, \lambda_2} \left((\lambda_1 + \lambda_2) u_1 + \lambda_2 (u_2 - u_1) \right. \\
 &\quad \left. - s^\alpha M^{-\alpha} (e^{\lambda_1 + \lambda_2} - 1) - (t^\alpha - s^\alpha) M^{-\alpha} (e^{\lambda_2} - 1) \right) \\
 &= s^\alpha M^{-\alpha} h(s^{-\alpha} M^\alpha u_1) \\
 &\quad + (t^\alpha - s^\alpha) M^{-\alpha} h((t^\alpha - s^\alpha)^{-1} M^\alpha (u_2 - u_1)),
 \end{aligned}$$

if $0 \leq u_1 \leq u_2$; and $I_{s,t}(u_1, u_2) = \infty$, otherwise. Let h_+ denote h restricted to $[1, \infty)$. It is easy to see that h_+ is an increasing one-to-one transformation from $[1, \infty)$ into $[0, \infty)$. So, it has an inverse. We claim that

$$\begin{aligned}
 &\{u_2 - u_1: I_{s,t}(u_1, u_2) \leq k\} \\
 &\subset \left[0, \max \left((t^\alpha - s^\alpha) M^{-\alpha}, (t^\alpha - s^\alpha) M^{-\alpha} h_+^{-1}((t^\alpha - s^\alpha)^{-1} M^\alpha k) \right) \right].
 \end{aligned} \tag{6.5}$$

This holds because if $u_2 - u_1 \geq (t^\alpha - s^\alpha) M^{-\alpha}$ and $I_{s,t}(u_1, u_2) \leq k$, then

$$(t^\alpha - s^\alpha) M^{-\alpha} h_+((t^\alpha - s^\alpha)^{-1} M^\alpha (u_2 - u_1)) \leq I_{s,t}(u_1, u_2) \leq k.$$

We have that $h_+(x)/x$ is increasing and $\lim_{x \rightarrow \infty} h_+(x)/x = \infty$. This implies that $x/h_+^{-1}(x)$ is increasing and $\lim_{x \rightarrow \infty} x/h_+^{-1}(x) = \infty$. Hence $\lim_{x \rightarrow 0} x h_+^{-1}(x^{-1}) = 0$. This limit and (6.5) imply condition (iii) in Corollary 3.6.

Now, suppose that $\lim_{n \rightarrow \infty} (\mu[0, Mn])^{-1} \mu[0, Mn] = 0$. We apply Theorem 3.1. Given $0 \leq t_1 < \dots < t_m = M$ and $\lambda_1, \dots, \lambda_m \in \mathbf{R}$, we have that

$$\begin{aligned}
 &\varepsilon_n \ln \left(\mathbf{E} \left[\exp \left(\sum_{j=1}^m \lambda_j N(t_j n) \right) \right] \right) \\
 &= \varepsilon_n \sum_{j=1}^m \left(e^{\sum_{i=j}^m \lambda_i} - 1 \right) \mu(nt_{j-1}, nt_j] \longrightarrow \exp(\lambda_m) - 1.
 \end{aligned}$$

So, $(\varepsilon_n N(nt_1), \dots, \varepsilon_n N(nt_m))$ satisfies the LDP with speed ε_n^{-1} and rate function

$$I_{t_1, \dots, t_m}(u_1, \dots, u_m) = \begin{cases} h(u_m), & \text{if } u_i = 0 \text{ for } 1 \leq i \leq m-1, \\ \infty, & \text{otherwise.} \end{cases} \tag{6.6}$$

So, for $0 \leq s, t < M$, $\rho_k(s, t) = 0$ and for $0 \leq t < M$, $\rho_k(t, M) = \sup\{u: h(u) \leq k\}$. Hence, for each $k \geq 1$, (T, ρ_k) is totally bounded. For

$$0 < \eta < \rho(0, M),$$

$$\begin{aligned} \mathbf{P}\left\{\sup_{\rho(s,t) \leq \eta} \varepsilon_n |N(tn) - N(sn)| \geq \tau\right\} &= \mathbf{P}\left\{\sup_{0 \leq s, t < M} \varepsilon_n |N(tn) - N(sn)| \geq \tau\right\} \\ &= \mathbf{P}\left\{\varepsilon_n (N(Mn-) - N(0)) \geq \tau\right\}. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} (\mu[0, Mn])^{-1} \mu[0, Mn] = 0$, by Theorem 6.1 for each $\lambda > 0$,

$$\lim_{n \rightarrow \infty} \varepsilon_n \ln \left(\frac{\mu[0, Mn]}{\mu(0, Mn)} \right)^{-1} \ln \left(\mathbf{P}\left\{\varepsilon_n (N(Mn-) - N(0)) \geq \lambda\right\} \right) = -\lambda.$$

Hence, for each $\tau > 0$,

$$\lim_{n \rightarrow \infty} \varepsilon_n \ln \left(\mathbf{P}\left\{\varepsilon_n (N(Mn-) - N(0)) \geq \tau\right\} \right) = -\infty.$$

Hence conditions (a) in Theorem 3.1 hold.

Next, we prove that (b) implies (a). First, we prove that for each $0 \leq t \leq M$, $\{(\mu[0, n])^{-1} \mu[0, tn]\}$ converges. We prove this by contradiction. Suppose that there are $0 \leq c_1 < c_2 < \infty$ and subsequences n'_k and n''_k such that $(\mu[0, n'_k])^{-1} \mu[0, tn'_k] \rightarrow c_1$ and $(\mu[0, n''_k])^{-1} \mu[0, tn''_k] \rightarrow c_2$. This implies that LDP for $\{\varepsilon_n N(tn)\}$ with speed ε_n^{-1} has two rate functions. Therefore, for each $0 \leq t \leq M$, $b(t) = \lim_{n \rightarrow \infty} (\mu[0, Mn])^{-1} \mu[0, tn]$ exists.

Now, we consider two cases according to whether $b(t) > 0$ for each $0 < t < M$ or not. Suppose that there exists a t_0 , $0 < t_0 < M$, such that $b(t_0) = 0$. Since b is nondecreasing for each $0 \leq t \leq t_0$, $b(t) = 0$ for such t . For $0 \leq s, t \leq M$

$$(\mu[0, Mn])^{-1} \mu[0, M^{-1}stn] = (\mu[0, sn])^{-1} \mu[0, M^{-1}stn] (\mu[0, Mn])^{-1} \mu[0, sn].$$

Hence, for $0 \leq s, t \leq M$,

$$b(M^{-1}st) = b(t)b(s). \quad (6.7)$$

Hence, for $t_0 < t < M$, there exists a positive integer k such that $M^{-(k-1)}t^k < t_0$. By (6.7), we have that $(b(t))^k = b(M^{-(k-1)}t^k) = 0$. So, $b(t) = 0$. This implies that for $0 \leq s, t < M$,

$$I_{s,t}(u, v) = \begin{cases} 0, & \text{if } u = v, \\ \infty, & \text{otherwise.} \end{cases} \quad (6.8)$$

Hence, for each $k \geq 1$ and each $0 \leq s, t < M$, $\rho_k(s, t) = 0$. So, the asymptotic equicontinuity condition implies that for each $\tau > 0$,

$$\lim_{n \rightarrow \infty} \varepsilon_n \ln \left(\mathbf{P}\left\{\varepsilon_n (N(Mn-) - N(0)) \geq \tau\right\} \right) = -\infty.$$

This implies that $(\mu[0, n])^{-1} \mu[0, Mn] \rightarrow 0$.

If for each $0 < t \leq M$, $b(t) > 0$, by Theorem 1.9.2 in [4], $\mu[0, x]$ is regularly varying. If it is regularly varying of order $\alpha > 0$, we are done. If $\mu[0, x]$ is slowly varying at infinity, then for each $0 < s < t$, and $\lambda_1, \lambda_2 \in \mathbf{R}$,

$$\varepsilon_n \ln \left(\mathbf{E} \left[\exp(\lambda_1 N(sa_n) + \lambda_2 N(ta_n)) \right] \right) \longrightarrow e^{\lambda_1 + \lambda_2} - 1 + e^{\lambda_2} - 1.$$

Hence the rate function for large deviations of $(N(sa_n), N(ta_n))$ is

$$\begin{aligned} I_{s,t}(u_1, u_2) &= \sup \left\{ \lambda_1 u_1 + \lambda_2 u_2 - (e^{\lambda_1 + \lambda_2} - 1 + e^{\lambda_2} - 1) : \lambda_1, \lambda_2 \in \mathbf{R} \right\} \\ &= h(u_1) + h(u_2 - u_1). \end{aligned}$$

If $\{\varepsilon_n N(tn) : 0 \leq t \leq M\}$ satisfied the LDP with speed ε_n^{-1} , then by Theorem 3.1 $([0, 1], \rho_k)$ would be totally bounded, where

$$\begin{aligned} \rho_k(s, t) &:= \sup \{ |u_2 - u_1| : I_{s,t}(u_1, u_2) \leq k \} \\ &= \sup \{ |u_2 - u_1| : h(u_1) + h(u_2 - u_1) \leq k \}, \end{aligned}$$

in contradiction.

By Theorem 4.2, the rate functions are given by (6.3) and (6.4). Theorem 6.2 is proved.

Next, we consider the case when the normalizing constant is of bigger order than the mean.

Theorem 6.3. *Let $\{N(t) : t \geq 0\}$ be a Poisson process with mean measure μ , let $0 < M < \infty$ and let $\{a_n\}$ be a sequence of positive numbers converging to infinity. Suppose that*

- (i) $\mu[0, x]$ is regularly varying at infinity of order $\alpha \geq 0$;
- (ii) $\mu[0, nM]/a_n \rightarrow 0$.

Then $\{a_n^{-1}N(tn) : 0 \leq t \leq M\}$ does not satisfy the LDP in $l_\infty[0, M]$ with speed $a_n \ln(a_n/\mu[0, nM])$.

P r o o f. We claim that given $0 < s < t < M$, $\{(a_n^{-1}N(sn), a_n^{-1}N(tn))\}$ satisfies the LDP with speed $a_n \ln(a_n/\mu[0, n])$ and rate function $I_{s,t}(u_1, u_2) = u_2$ for $0 \leq u_1 \leq u_2$; $I_{s,t}(u_1, u_2) = \infty$, otherwise. This claim implies the theorem, since by Theorem 3.1, if $\{a_n^{-1}N(tn) : 0 \leq t \leq M\}$ satisfied the LDP in $l_\infty[0, M]$, then for each $k > 0$, $([0, M], \rho_k)$ would be totally bounded, where $\rho_k(s, t) = \sup\{|u_2 - u_1| : I_{s,t}(u_1, u_2) \leq k\} = k$. But this condition does not hold.

By the contraction principle it suffices to show that

$$(U_n, V_n) := (a_n^{-1}N(sn), a_n^{-1}(N(tn) - N(sn)))$$

satisfies the LDP with rate function $I^{(2)}(u_1, u_2) = u_1 + u_2$, if $u_1, u_2 \geq 0$, and $I^{(2)}(u_1, u_2) = \infty$, otherwise. By regular variation,

$$\lim_{n \rightarrow \infty} \frac{\mu[0, ns]}{a_n} = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{a_n \ln(a_n/\mu[0, ns])}{a_n \ln(a_n/\mu[0, n])} = 1.$$

Hence, by Theorem 6.1, U_n satisfies the LDP with rate function $I^{(1)}(t) = t$, if $t \geq 0$, and $I^{(1)}(t) = \infty$, otherwise. Similarly, we get that $\{V_n\}$ satisfies the LDP with rate function $I^{(1)}$. Since U_n and V_n are independent, we have that for each open set O in $\mathbf{R}^2$,

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \left(a_n \ln \left(\frac{a_n}{\mu[0, n]} \right) \right)^{-1} \ln (\mathbf{P}\{(U_n, V_n) \in O\}) \\ & \geq -\{I^{(1)}(u) + I^{(1)}(v) : (u, v) \in O\}. \end{aligned}$$

To check the condition for closed sets, it suffices to prove that for each $t > 0$

$$\limsup_{n \rightarrow \infty} \left(a_n \ln \left(\frac{a_n}{\mu[0, n]} \right) \right)^{-1} \ln (\mathbf{P}\{U_n + V_n \geq t\}) \leq -t.$$

But $U_n + V_n = a_n^{-1}N(tn)$ satisfies the LDP with rate function $I^{(1)}$. Theorem 6.3 is proved.

In the situation of the previous theorem, although, the LDP does not hold in $l_\infty[0, M]$, it does in a set of measures. Let $\mathcal{M}_+([0, M], w)$ be the set of positive measures on $[0, M]$ with the weak topology. As it is well known, this topology is defined as follows: $\mu_n \xrightarrow{w} \mu$ if for each continuous function f on $[0, M]$, $\int_0^M f(x) d\mu_n \rightarrow \int_0^M f(x) d\mu(x)$. Given a Poisson process $\{N(t) : t \geq 0\}$, let $\{T_j\}$ be the jumps of this process. Given $0 < M < \infty$, we have the random measure $\mu_n = a_n^{-1} \sum_{T_j \leq nM} \delta_{n^{-1}T_j}$ on $[0, M]$.

Theorem 6.4. *Let $\{N(t) : t \geq 0\}$ be a Poisson process with mean measure μ and let $\{a_n\}$ be a sequence of positive numbers converging to infinity. Suppose that*

- (i) $\mu[0, x]$ is regularly varying at infinity of order $\alpha > 0$;
- (ii) $\mu[0, nM]/a_n \rightarrow 0$.

Then $\{\mu_n\}$ satisfies the LDP in $\mathcal{M}_+([0, M], w)$ with speed $a_n \ln(a_n/\mu[0, n])$ and rate function $I(\nu) = \nu[0, M]$.

P r o o f. Since $a_n^{-1}N(Mn)$ satisfies the LDP, given a closed set $F \subset \mathcal{M}_+([0, M], w)$,

$$\begin{aligned} & \left(a_n \ln \left(\frac{a_n}{\mu[0, n]} \right) \right)^{-1} \ln (\mathbf{P}\{\mu_n \in F\}) \\ & \leq \left(a_n \ln \left(\frac{a_n}{\mu[0, n]} \right) \right)^{-1} \ln \left(\mathbf{P}\left\{ \mu_n([0, M]) \geq \inf_{\mu \in F} \mu([0, M]) \right\} \right) \\ & = \left(a_n \ln \left(\frac{a_n}{\mu[0, n]} \right) \right)^{-1} \ln \left(\mathbf{P}\left\{ a_n^{-1}N(Mn) \geq \inf_{\mu \in F} \mu([0, M]) \right\} \right) \\ & \longrightarrow -\inf_{\mu \in F} \mu([0, M]). \end{aligned}$$

Given an open set $G \subset \mathcal{M}_+([0, M]^d, w)$ and $\nu_0 \in G$, there are $\delta > 0$ and

$0 \leq t_1 < \dots < t_m \leq 1$ such that

$$\left\{ \nu: |\nu[0, t_1] - \nu_0[0, t_1]| < \delta, \sup_{2 \leq i \leq m} |\nu(t_{i-1}, t_i] - \nu_0(t_{i-1}, t_i]| \leq \delta \right\} \subset G.$$

Hence

$$\mathbf{P}\{\mu_n \in G\} \geq \mathbf{P}\left\{ \sup_{1 \leq i \leq m} \left| a_n^{-1}(N(nt_i) - N(nt_{i-1})) - p_i \right| \leq \delta \right\},$$

where $t_0 = 0$, $p_1 = \nu_0[0, t_1]$ and $p_i = \nu_0(t_{i-1}, t_i]$, for $2 \leq i \leq m$. By an argument in the previous theorem

$$\left\{ a_n^{-1}(N(nt_1) - N(nt_0)), \dots, a_n^{-1}(N(nt_m) - N(nt_{m-1})) \right\}$$

satisfies the LDP with rate function $I(u_1, \dots, u_m) = \sum_{j=1}^m u_j$, $u_j \geq 0$, for each $1 \leq j \leq m$, $I(u_1, \dots, u_m) = \infty$, otherwise. Hence

$$\liminf_{n \rightarrow \infty} \left(a_n \ln \left(\frac{a_n}{\mu[0, n]} \right) \right)^{-1} \ln (\mathbf{P}\{\mu_n \in G\}) \geq - \sum_{i=1}^m (p_i - \delta)^+.$$

Therefore, the claim follows. Theorem 6.4 is proved.

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