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HÖLDER EQUALITY FOR CONDITIONAL EXPECTATIONS WITH APPLICATION TO LINEAR MONOTONE OPERATORS

Для стандартного пространства $L_p = L_p(\Omega, \mathfrak{A}, \mathbf{P})$, $1 \leq p < \infty$, и заданных множителя f и σ -алгебры $\mathfrak{B} \subseteq \mathfrak{A}$ выводится некоторый критерий представимости непрерывного линейного оператора $x(X)$, $X \in L_p$, в виде условного математического ожидания: $x(X) = \mathbf{E}(Xf | \mathfrak{B})$. В качестве применения указанное представление (с соответствующим множителем $f \geq 0$) рассматривается для общих линейных монотонных операторов $x(X)$, $X \in K$, заданных на произвольном конусе $K \subseteq L_p^+$, где $L_p^+ = \{X \in L_p : X \geq 0\}$.

Ключевые слова и фразы: условные математические ожидания, неравенство Гёльдера, линейные монотонные операторы, линейные монотонные продолжения.

Let $(\Omega, \mathfrak{A}, \mathbf{P})$ be a probability space with a probability measure $\mathbf{P}(A)$, $A \in \mathfrak{A}$, on a σ -algebra $\mathfrak{A}$ of sets $A \subseteq \Omega$, and let

$$L_p = L_p(\Omega, \mathfrak{A}, \mathbf{P}), \quad 1 \leq p < \infty,$$

be a standard L_p -space of real random variables $X = X(\omega)$, $\omega \in \Omega$, with $\|X\| = (\mathbf{E}|X|^p)^{1/p}$. Here we consider the *conditional expectation*

$$x(X) = \mathbf{E}(Xf | \mathfrak{B}), \quad X \in L_p, \quad (1)$$

associated with an appropriate factor $f = f(\omega)$, $\omega \in \Omega$, and a σ -algebra $\mathfrak{B} \subseteq \mathfrak{A}$. Naturally the factor f should be at least such that $\mathbf{E}|Xf| < \infty$ for all $X \in L_p$.

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Theorem 0.1. *The conditional expectation in (1) represents a well-defined continuous linear operator $x(X)$, $X \in L_p$, taking values in L_p if and only if the factor f belongs to the dual space L_q , $q = p(p-1)^{-1}$ and $\text{ess sup } \mathbf{E}(|f|^q | \mathfrak{B}) < \infty$, according to the following Hölder equality:*

$$\sup_{\|X\| \leq 1} [\mathbf{E}(\mathbf{E}(Xf | \mathfrak{B})|)^{1/p}] = \text{ess sup} [\mathbf{E}(|f|^q | \mathfrak{B})]^{1/q}. \quad (2)$$

For $p = 1$ ($q = \infty$) the right-hand side of the equality means

$$\lim_{q \rightarrow \infty} \text{ess sup} [\mathbf{E}(|f|^q | \mathfrak{B})]^{1/q} = \text{ess sup } |f|. \quad (3)$$

In addition, with respect to (3), it seems interesting to note that there is a certain limit

$$\lim_{q \rightarrow \infty} [\mathbf{E}(|f|^q | \mathfrak{B})]^{1/q} = |f|_{\mathfrak{B}}$$

representing the minimal $\mathfrak{B}$ -measurable majorant $|f|_{\mathfrak{B}}$ for $|f(\omega)|$, $\omega \in \Omega$, with $\text{ess sup } |f|_{\mathfrak{B}} = \text{ess sup } |f|$.

With respect to the equality in (2) we have to recall the Hölder inequality for conditional expectation

$$\mathbf{E}(|Xf| | \mathfrak{B}) \leq [\mathbf{E}(|X|^p | \mathfrak{B})]^{1/p} \cdot [\mathbf{E}(|f|^q | \mathfrak{B})]^{1/q}. \quad (4)$$

It is worth noting that the operators of the (1)-type satisfying (2) have the following property:

$$|x(X)| \leq C [\mathbf{E}(|X|^p | \mathfrak{B})]^{1/p}, \quad X \in L_p, \quad (5)$$

where the minimal constant $C < \infty$ appears as the operator norm

$$\|x\| = \sup_{\|X\| \leq 1} \|\mathbf{E}(Xf | \mathfrak{B})\|.$$

Property (5) is justified by using both the relations (2) and (4). Note that if $\mathfrak{B}$ was the trivial σ -algebra, then the (1)-type operator would be a linear functional on L_p with the corresponding well-known properties (2)–(5). In particular, it would be

$$\|x\| = \sup_{\|X\| \leq 1} |\mathbf{E}(Xf)| = [\mathbf{E}(|f|^q)]^{1/q}.$$

As an application of the above Hölder equality, we consider an arbitrary linear monotone operator with $\mathfrak{B}$ -measurable values: $x(X)$, $X \in K$, initially given on a convex subcone $K \subseteq L_p^+$ in the cone $L_p^+ = \{X \in L_p: X \geq 0\}$ of the separable space L_p as complete vector lattice. We suggest a criterion for the above operator to admit the (1)-type representation (with the corresponding $f \geq 0$) as monotone or strictly monotone extension $x(X)$, $X \in L_p$, so that $x(X_1) \leq x(X_2)$, $X_1 \leq X_2$, or, respectively, $x(X_1) < x(X_2)$, $X_1 < X_2$. Here the strict sign « $<$ » means that the indicated point-wise relation holds on a nonzero measure set.

Theorem 0.2. *The (1)-type monotone continuous extension $x(X)$, $X \in L_p$, does exist if and only if the initial operator $x(X)$, $X \in K$, lays in a sandwich, i.e., the following condition is satisfied:*

$$\mathbf{E}(Yg | \mathfrak{B}) + x(X_1) \leq x(X_2) + \mathbf{E}(Zh | \mathfrak{B}) \quad (6)$$

for all $X_1, X_2 \in K$ and $Y, Z \in L_p^+$ such that $Y + X_1 \leq X_2 + Z$, and for some $g, h \in L_q$ with $g, h \geq 0$ a.s.:

$$\text{ess sup } \mathbf{E}(|g|^q | \mathfrak{B}) < \infty, \quad \text{ess sup } \mathbf{E}(|h|^q | \mathfrak{B}) < \infty.$$

Moreover, the (1)-type strictly monotone continuous extension exists if and only if the above sandwich condition holds with some $g, h > 0$ a.s.

The above introduction made, we now turn to the proofs.

P r o o f o f T h e o r e m 1. At first note that if the (1)-type conditional expectation satisfies (2) and (3), then it is obviously a continuous linear operator mapping $X \in L_p$ into $x(X) = \mathbf{E}(Xf | \mathfrak{B}) \in L_p$.

Vice versa, we consider a continuous linear operator $x(X)$, $X \in L_p$, with values in L_p represented by the conditional expectation in (1). Note that the expectation

$$\mathbf{E}x(X) = \mathbf{E}\mathbf{E}(Xf \mid \mathfrak{B}) = \mathbf{E}Xf, \quad X \in L_p,$$

of this kind of operator represents a certain linear continuous functional with its representative $f \in L_q$ in the dual space L_q , $q = p(p-1)^{-1}$. Actually, further on we proceed with arguments that are quite similar to those usually applied to the linear continuous functionals, cf. [7], for example.

First of all, we study the case $1 < p < \infty$. Considering the Hölder inequality in (4), let us take the variables

$$\xi = [\mathbf{E}(|X|^p \mid \mathfrak{B})]^{1/p}, \quad \varphi = [\mathbf{E}(|f|^q \mid \mathfrak{B})]^{1/q}$$

into account. We can see that $Xf = 0$, $\mathbf{E}(|Xf| \mid \mathfrak{B}) = 0$ on the ω -set $\{\xi\varphi = 0\} \in \mathfrak{B}$. Hence, we are going to consider all the variables involved on the complementary set $\Omega^+ = \{\xi\varphi > 0\}$. Now let us apply the known elementary inequality $\alpha\beta \leq p^{-1}\alpha^p + q^{-1}\beta^q$ ($\alpha, \beta \geq 0$), where the equality sign holds if and only if $\alpha^p = \beta^q$, $q = p(p-1)^{-1}$. For $\alpha = |X|/\xi$, $\beta = |f|/\varphi$ we have

$$\frac{|Xf|}{\xi\varphi} \leq \frac{1}{p} \frac{|X|^p}{\xi^p} + \frac{1}{q} \frac{|f|^q}{\varphi^q},$$

where all the variables involved are *integrable* on the set Ω^+ , since $\xi^{-p}\mathbf{E}(|X|^p \mid \mathfrak{B})$, $\varphi^{-q}\mathbf{E}(|f|^q \mid \mathfrak{B})$ are integrable. Hence,

$$\frac{\mathbf{E}(|Xf| \mid \mathfrak{B})}{\xi\varphi} \leq \frac{1}{p} \frac{\mathbf{E}(|X|^p \mid \mathfrak{B})}{\xi^p} + \frac{1}{q} \frac{\mathbf{E}(|f|^q \mid \mathfrak{B})}{\varphi^q} = 1$$

on the set Ω^+ . This implies the Hölder inequality in (4), where the equality sign holds if and only if

$$\frac{|X|^p}{\xi^p} = \frac{|f|^q}{\varphi^q} \tag{7}$$

for almost all $\omega \in \Omega^+$. Note that the above condition (7) can be equivalently characterized as follows:

$$|X| = a|f|^{1/(p-1)} \quad \text{a.e. or} \quad |f| = b|X|^{1/(q-1)} \quad \text{a.e.} \tag{8}$$

for some $\mathfrak{B}$ -measurable variables a, b on the set Ω^+ . To explain, we just note that condition (8) implies $|X|^p = a^p|f|^q$, $\xi^p = a^p\varphi^q$, from which we conclude that $|X|^p/\xi^p = a^p|f|^q/(a^p\varphi^q) = |f|^q/\varphi^q$ (see (7)). Now we consider

$$X = \begin{cases} f^{q-1}, & f \geq 0, \\ -|f|^{q-1}, & f < 0, \end{cases}$$

and we can see that $|X| = |f|^{1/(p-1)}$ (cf. (8)), and that $|X|^p = |f|^q = |Xf| = Xf$. Thus the equality in (4) holds true as

$$\mathbf{E}(Xf \mid \mathfrak{B}) = [\mathbf{E}(|X|^p \mid \mathfrak{B})]^{1/p} [\mathbf{E}(|f|^q \mid \mathfrak{B})]^{1/q}.$$

Moreover it still remains true by multiplication with any $\mathfrak{B}$ -measurable function (cf. (8)). In particular we can consider $\xi^{-1}1_B$ with $\xi = [\mathbf{E}(|X|^p \mid \mathfrak{B})]^{1/p}$ and an appropriate $B \in \mathfrak{B}$, $B \subseteq \Omega^+$, so that we obtain

$$\mathbf{E}\left(\frac{X}{\xi} 1_B f \mid \mathfrak{B}\right) = \left[\mathbf{E}\left(\left|\frac{X}{\xi}\right|^p 1_B \mid \mathfrak{B}\right)\right]^{1/p} \left[\mathbf{E}(|f|^q \mid \mathfrak{B})\right]^{1/q} = 1_B \left[\mathbf{E}(|f|^q \mid \mathfrak{B})\right]^{1/q}.$$

Then it follows that, for any arbitrary finite C : $C \leq C_0$, with $C_0 := \text{ess sup}[\mathbf{E}(|f|^q \mid \mathfrak{B})]^{1/q}$, and $B = \{[\mathbf{E}(|f|^q \mid \mathfrak{B})]^{1/q} \geq C\}$, we have

$$\mathbf{E}\left|\mathbf{E}\left(\frac{X}{\xi} 1_B f \mid \mathfrak{B}\right)\right|^p \geq C^p \mathbf{E}1_B = C^p \mathbf{E}\left|\frac{X}{\xi} 1_B\right|^p,$$

since $1_B = \mathbf{E}(|X/\xi| 1_B \mid \mathfrak{B})$. The above relation implies $\sup_{\|X\| \leq 1} \|\mathbf{E}(Xf \mid \mathfrak{B})\| \geq C$ and, therefore,

$$\sup_{\|X\| \leq 1} \|\mathbf{E}(Xf \mid \mathfrak{B})\| \geq \text{ess sup}[\mathbf{E}(|f|^q \mid \mathfrak{B})]^{1/q}.$$

On the other hand, thanks to Hölder inequality (4), we have

$$\mathbf{E}|\mathbf{E}(Xf | \mathfrak{B})|^p \leq \mathbf{E}\left[\mathbf{E}(|X|^p | \mathfrak{B}) [\mathbf{E}(|f|^q | \mathfrak{B})]^{p/q}\right] \leq C_0^p \mathbf{E}|X|^p$$

which implies

$$\sup_{\|X\| \leq 1} \|\mathbf{E}(Xf | \mathfrak{B})\| \leq \text{ess sup} [\mathbf{E}(|f|^q | \mathfrak{B})]^{1/q}$$

and hence relation (2).

Now it remains to study the case $p = 1$ ($q = \infty$). First of all we show that limit (3) holds true. Let $C_0 = \text{ess sup } |f|$, then $C_0 \geq |f|$ a.s., $C_0 \geq [\mathbf{E}(|f|^q | \mathfrak{B})]^{1/q}$ a.s. and $C_0 \geq \text{ess sup} [\mathbf{E}(|f|^q | \mathfrak{B})]^{1/q}$, so $C_0 \geq \lim_{q \rightarrow \infty} \text{ess sup} [\mathbf{E}(|f|^q | \mathfrak{B})]^{1/q}$. On the other hand, note that for any $C < C_0$ the set $A = \{|f| \geq C\}$ is of measure $\mathbf{P}(A) > 0$. Let us consider $\alpha = \mathbf{E}(1_A | \mathfrak{B})$, altogether with the sets $B = \{\alpha > 0\} \in \mathfrak{B}$, $B^c = \{\alpha = 0\}$, then we have $\mathbf{P}(B) > 0$, since

$$0 = \mathbf{E}(\alpha 1_{B^c}) = \mathbf{E}\mathbf{E}(1_A 1_{B^c} | \mathfrak{B}) = \mathbf{P}(A \cap B^c)$$

and $\mathbf{P}(B) \geq \mathbf{P}(A \cap B) = \mathbf{P}(A) > 0$. With $\mathbf{P}(B) > 0$ and the point-wise convergence $\lim_{q \rightarrow \infty} \alpha^{1/q}(\omega) = 1_B(\omega)$, $\omega \in \Omega$, we have

$$\lim_{q \rightarrow \infty} \mathbf{P}\{\alpha^{1/q} > (1 - \varepsilon) 1_B\} = \mathbf{P}(B) \quad (\forall \varepsilon > 0),$$

so, for any $\varepsilon > 0$, there is a set $B_\varepsilon \subseteq B$: $\mathbf{P}(B_\varepsilon) > 0$, such that $\alpha^{1/q} \geq (1 - \varepsilon) 1_{B_\varepsilon}$ with $q \geq q_\varepsilon$, for some $q_\varepsilon \geq 1$. Consequently, for $\alpha = \mathbf{E}(1_A | \mathfrak{B})$ and $|f| \geq |f| 1_A \geq C 1_A$, we obtain

$$[\mathbf{E}(|f|^q | \mathfrak{B})]^{1/q} \geq C \alpha^{1/q} \geq C(1 - \varepsilon) 1_{B_\varepsilon}$$

which shows that $\lim_{q \rightarrow \infty} \text{ess sup} [\mathbf{E}(|f|^q | \mathfrak{B})]^{1/q} \geq C$, for any $C \leq C_0 = \text{ess sup } |f|$. This ends the proof of limit (3).

We now conclude the proof by showing that the Hölder equality in (2) stands also for $p = 1$ ($q = \infty$). Obviously, for $C_0 = \text{ess sup } |f|$, we have that

$$\mathbf{E}|\mathbf{E}(Xf | \mathfrak{B})| \leq \mathbf{E}\mathbf{E}(|Xf| | \mathfrak{B}) \leq \mathbf{E}[C_0 \mathbf{E}(|X| | \mathfrak{B})] = C_0 \mathbf{E}|X|,$$

with $\mathbf{E}|X| = \|X\|$ in L_1 -space, so $\sup_{\|X\| \leq 1} \mathbf{E}|\mathbf{E}(Xf | \mathfrak{B})| \leq \text{ess sup } |f|$. On the other hand, for any C : $\mathbf{P}\{|f| > C\} > 0$, let $X = 1_A \text{ sign } f$, where $A = \{|f| \geq C\}$. Then we have

$$\mathbf{E}|\mathbf{E}(Xf | \mathfrak{B})| = \mathbf{E}\mathbf{E}(1_A |f| | \mathfrak{B}) \geq C \mathbf{E}[1_A].$$

And here we obtain $\sup_{\|X\| \leq 1} \mathbf{E}|\mathbf{E}(Xf | \mathfrak{B})| \geq C$, since $\mathbf{E}|X| = \mathbf{E}[1_A]$, and therefore

$$\sup_{\|X\| \leq 1} \mathbf{E}|\mathbf{E}(Xf | \mathfrak{B})| \geq \text{ess sup } |f|.$$

By this we end the proof.

P r o o f o f T h e o r e m 2. Having the representation (1) with properties (2) and (3) for the appropriate factor $f \geq 0$ a.s. (or $f > 0$ a.s. in the strictly monotone case), we can take $g = h := f$, obtaining therefore the sandwich condition (6).

Vice versa, if this condition (6) holds, then the initial operator $x(X)$, $X \in K$, admits a linear monotone (respectively, strictly monotone) continuous extension with $\mathfrak{B}$ -measurable values $x(X)$, $X \in L_p$, which preserves the sandwich in (6) for all $X_1, X_2 \in L_p^+$ and

$$x(\lambda X) = \lambda x(X) \quad (9)$$

for all bounded $\mathfrak{B}$ -measurable multipliers $\lambda = \lambda(\omega)$, $\omega \in \mathfrak{B}$. For this we refer to [3] and [1]. Now we have to show that this sandwich preserving extension admits a representation as in (1) with the properties (2) and (3). We have that the expectation $\mathbf{E}[x(X)] = \mathbf{E}[Xf]$, $X \in L_p$, is linear continuous functional on the L_p -space, with $f \in L_q$, $q = p(p-1)^{-1}$, as its functional representative in the dual space L_q . Hence, according to (9), we have

$$\mathbf{E}[1_B x(X)] = \mathbf{E}[x(1_B X)] = \mathbf{E}[(1_B X) f] = \mathbf{E}[1_B (Xf)], \quad B \in \mathfrak{B},$$

and this justifies that $x(X) = \mathbf{E}(Xf | \mathfrak{B})$, with $f \geq g$ for the sandwich element $g \geq 0$ (or, respectively, $g > 0$ a.e.). By this we end the proof.

With respect to Theorem 2, for the related material, we refer for instance to [1], [2], [4], [5] and [6].

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STILL ON THE SEMIMARTINGALES THAT ARE REPRESENTED AS STABLE INTEGRALS

Приводится результат, дополняющий результаты предыдущей работы автора [7].

Ключевые слова и фразы: семимартингалы без непрерывной мартингальной составляющей, мера скачков, компенсатор, строго α -устойчивые процессы Леви, устойчивые интегралы, формулы представления.

Introduction. In [7] we have characterized the one-dimensional semimartingales X (with vanishing continuous martingale part) which can be represented as stochastic integrals (in the Itô sense) with respect to strictly α -stable Lévy processes $Z^{(\alpha)}$ ($0 < \alpha < 2$), i.e., the semimartingales X for which there exist a predictable process H and a $Z^{(\alpha)}$ (possibly defined on some extension of the original probability space) that verify the following formulas:

$$X_t = \int_{[0,t]} |H_s| dZ_s^{(\alpha)}, \quad t \geq 0, \quad (1)$$

or

$$X_t = \int_{[0,t]} H_s dZ_s^{(\alpha)}, \quad t \geq 0. \quad (2)$$

Such characterization is essentially based on the following hypothesis which fixes the form of the compensator π of the jump-measure μ of X : there exist a predictable process H and a constant α , $0 < \alpha < 2$, such that

$$\pi(ds, dx) = |H_s|^\alpha ds \otimes \nu(dx), \quad (3)$$

where ν is the Lévy measure of an α -stable Lévy motion (this is a synonym of strictly α -stable Lévy process).

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