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STEPANOV ALMOST PERIODICALLY CORRELATED
AND ALMOST PERIODICALLY UNITARY PROCESSES¹⁾

Данная работа распространяет структуру и свойства почти периодически коррелированных (ППК) и почти периодически унитарных (ППУ) процессов, которые были определены в смысле Бохра, на более широкий класс процессов, для которых почти периодичность понимается в смысле Степанова. Эти процессы, в отличие от боховских ППК и ППУ процессов, не являются с необходимостью непрерывными в среднем квадратическом, но сохраняют смысл почти периодичности. Например, процессы $f(t)X(t)$ или $X(t+f(t))$, полученные из стационарного в широком смысле процесса $X(t)$ в результате модуляций амплитуды или времени с помощью скалярной почти периодической (ПП) функции Степанова $f(t)$, не обязательно непрерывной, являются ППУ и ППК по Степанову. Основные результаты, касающиеся ППК и ППУ процессов, перенесены на новый класс процессов. Характеризация Гладышева ППК корреляционных функций распространена на ППК по Степанову процессы, и показано, что их корреляционные функции полностью представляются рядом Фурье, имеющим счетное число функциональных коэффициентов, которые представляют собой преобразования Фурье комплексных мер. Установлено, что ППУ по Степанову процессы являются также ППК по Степанову и даются формулой $X(t) = U(t)[P(t)]$, где $\{U(t), t \in \mathbf{R}\}$ есть строго непрерывная группа унитарных операторов, а $P(t)$ есть векторнозначная ПП по Степанову функция. Как и в случае ППУ по Бохру процессов, предыдущий факт приводит к представлениям $X(t)$, основанным на спектральной теории для унитарных операторов и для степановских почти периодических функций.

Ключевые слова и фразы: почти периодически коррелированные, почти периодически унитарные случайные процессы, почти периодические функции.

1. Introduction

Although discontinuous (in quadratic-mean) weakly stationary processes are considered pathological, there are simple examples of discontinuous second order nonstationary processes. Consider the process $X(t) = f(t)a(t)$,

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where $f(t) = f(t + T)$ is a periodic discontinuous scalar function and $a(t)$ is a quadratic-mean (q.m.) continuous weakly stationary process. In this case, $X(t)$ is *periodically correlated* (PC) [4] in that $m(t) = \mathbf{E}\{X(t)\} = \mathbf{E}\{X(t + T)\}$ and $R(s, t) = \mathbf{E}\{X(s)\overline{X(t)}\} = R(s + T, t + T)$; as our interest here is in the second order properties, we take $m(t) \equiv 0$. The periodicity of the correlation function $R(s, t)$ of a PC process is in a diagonal sense so that the transformation $s = t + \tau$, $t = t$ rotates the coordinates to put all the periodicity in the coordinate of local time t . That is, $B(t, \tau) = R(t + \tau, t)$ is periodic in the variable t with period T . Thus the Fourier series,

$$B(t, \tau) = \sum_k B_k(\tau) \exp \left[\frac{i2\pi kt}{T} \right], \quad (1.1)$$

is suggested where $B_k(t) = (1/T) \int_0^T B(t, \tau) \exp(-i2\pi kt/T) dt$, and the convergence of (1.1) depends on the properties of $B(\cdot, \tau)$. Gladyshev [4] gave necessary and sufficient conditions in terms of $\{B_k(\tau), k \in \mathbf{Z}\}$ for a continuous function $R(s, t) = R(s + T, t + T)$ to be non-negative definite. A similar result applies to a class of discontinuous correlation functions, including those arising from processes of the sort $X(t) = f(t)a(t)$ mentioned above [5].

In the following examples, $X(t)$ is to be considered as a curve or a mapping from $\mathbf{R}$ to the square integrable random variables, denoted $L_2(\Omega)$, of a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. For two elements X and Y in $L_2(\Omega)$, we denote $\langle X, Y \rangle = \mathbf{E}\{X\overline{Y}\}$.

Every second order PC process $\{X(t), t \in \mathbf{R}\}$, discontinuous or not, has a natural imbedded unitary operator U_X for which $X(t + T) = U_X X(t)$ for every $t \in \mathbf{R}$ [8]. If $X(t)$ is q.m. continuous, it is then *periodically unitary* (PU); that is, there is a strongly continuous group $\{U(t'), t' \in \mathbf{R}\}$ of unitary operators for which

$$\|X(t + t') - U(t')X(t)\|_2 = 0 \quad \text{for every } t \text{ and } n \text{ when } t' = nT, \quad (1.2)$$

where $\|\cdot\|_2$ is the $L_2(\Omega)$ norm. If $R(t, t)$ is Lebesgue integrable on $[0, T]$, then it is $L_2([0, T])$ periodically correlated [8] in that

$$\int_0^T \|X(t + t') - U(t')X(t)\|_2^2 dt = 0 \quad \text{when } t' = nT. \quad (1.3)$$

Properties of the $L_2([0, T])$ PU processes are addressed in [8].

The same simple multiplicative example, $X(t) = f(t)a(t)$ with $f(t)$ uniformly almost periodic (UAP) in the sense of Bohr, is *almost periodically correlated* (APC) in the sense of Gladyshev [4]. That is, $m(t)$ is UAP and for every s, t , $R(s + \alpha, t + \alpha)$ is UAP in the variable α . It follows that for every τ , the function $R(t + \tau, t)$ is a UAP function with respect to the variable t and so the correlation $R(t + \tau, t)$ has the Fourier series $\sum_{\lambda_j \in \Lambda_\tau} a(\lambda_j, \tau) \exp(\lambda_j t)$

where the coefficient functions are given by

$$a(\lambda_j, \tau) = \lim_{A \rightarrow \infty} \frac{1}{2A} \int_{-A}^A R(t + \tau, t) \exp(-i\lambda_j t) dt. \quad (1.4)$$

For fixed τ the set Λ_τ is countable from the theory of UAP functions; for Gladyshev APC processes, it is also true that $\Lambda = \{\lambda: a(\lambda, \tau) \neq 0 \text{ for some } \tau\} = \cup_\tau \Lambda_\tau$ is countable [7]. Gladyshev also gave necessary and sufficient conditions, in terms of the coefficient functions $a(\lambda_j, \tau)$, for $R(s, t)$ to be non-negative definite. The notion of *periodically unitary* processes defined by (1.2) motivates the *almost periodically unitary* (APU) processes [6]. These are the q.m. continuous second order processes $X(t)$ for which there exists a strongly continuous group $\{U(t'), t' \in \mathbf{R}\}$ of unitary operators such that for every $\varepsilon > 0$ the set

$$S(\varepsilon, X, U) = \left\{ t': \sup_{t \in \mathbf{R}} \|X(t + t') - U(t')X(t)\|_2 < \varepsilon \right\} \quad (1.5)$$

is relatively dense (RD), or in more modern terms, has bounded gaps. A set S is called RD if there exists a number $l > 0$ such that every interval of length l contains at least one point of S . Such an l is called an inclusion length for S . The APU processes are precisely those processes given by $X(t) = \sum_j f_j(t) a_j(t)$ where $\{f_j(t), j \in \mathbf{Z}\}$ are UAP functions with $\sum_j |f_j(t)|^2 \leq M$ and the convergence of $\sum_{j=1}^n |f_j(t)|^2$ is uniform in t ; the processes $\{a_j(t), j \in \mathbf{Z}\}$ are jointly stationary and mutually orthogonal at each fixed t , $\langle a_j(t), a_k(t) \rangle = \delta_{jk}$. Equivalently, the APU processes are precisely those given by $X(t) = U(t)[P(t)]$ where $P(t)$ is a $L_2(\Omega)$ -valued almost periodic function. Such a function might also be called an almost periodic process and $\{U(t'), t' \in \mathbf{R}\}$ is a group of unitary operators.

The same multiplicative example $X(t) = f(t)a(t)$ motivates the consideration of the class of processes obtained when $a(t)$ is weakly stationary (and q.m. continuous) but $f(t)$ is in the class S_p of scalar Stepanov [9], [10] almost periodic functions; that is, $f(t)$ is locally L_p : $\int_0^1 |f(t+u)|^p du < \infty$ for every t , and for every $\varepsilon < 0$ the set

$$S(\varepsilon, f) = \left\{ t': \sup_{t \in \mathbf{R}} \int_t^{t+1} |f(u+t') - f(u)|^p du < \varepsilon \right\} \quad (1.6)$$

is RD. We note that S_p contains the periodic extensions of $L_p[0, 1]$. In this paper we introduce the Stepanov APC and APU processes and give some basic structural results. In Section 2 we define the Stepanov APC processes and extend Gladyshev's characterization [4] of correlation functions of (q.m. continuous) APC processes to this case, for which the correlation need not be continuous. In Section 3 we define the Stepanov APU processes, give several characterizations of these processes and show that despite the lack of pointwise continuity of the map $X: \mathbf{R} \rightarrow L_2(\Omega)$, there is still a spectral theory analogous to the one developed in [6]. In particular the set $\Lambda =$

$\{\lambda: a(\lambda, \tau) \neq 0 \text{ for some } \tau\}$ is again countable and the correlation $R(t + \tau, t)$ has the Fourier series $\sum_{\lambda_j \in \Lambda} a(\lambda_j, \tau) \exp(\lambda_j t)$, where the functions $a(\lambda_j, \cdot)$ are Fourier transforms of signed measures of finite total variation.

1.1. Stepanov almost periodic functions. In this section we summarize basic facts about Stepanov almost periodic functions. A primary fact due to Bochner [1] is that $f \in S_p$ if and only if the function $F: \mathbf{R} \rightarrow L_p[0, 1]$ given by $F(t) = f(t + \cdot)$ is UAP. To be more precise, a function $f: \mathbf{R} \rightarrow \mathbf{B}$ (Banach space) is called almost periodic in the sense of Bohr if it is continuous and for every $\varepsilon > 0$ the set

$$S(\varepsilon, f) = \left\{ t': \sup_t \|f(t + t') - f(t)\|_{\mathbf{B}} < \varepsilon \right\}$$

is relatively dense (RD). Every t' in $S(\varepsilon, f)$ is called an ε -almost period of f .

Let $L_p(I, \mathbf{B})$ denote the space of Lebesgue-Bochner measurable L_p -integrable $\mathbf{B}$ -valued functions on $I = [0, 1]$ with the usual norm

$$\|F\|_p = \left(\int_0^1 \|F(u)\|_{\mathbf{B}}^p du \right)^{1/p} < \infty. \quad (1.7)$$

A Lebesgue-Bochner measurable $\mathbf{B}$ -valued function is also called strongly measurable ([12, p. 130]). A strongly measurable L_p -integrable $\mathbf{B}$ -valued function F is Bochner- L_p ([12, p. 133]); we shall say that $F \in L_{\mathbf{B}}^p$. A map $f: \mathbf{R} \rightarrow \mathbf{B}$ will be called locally $L_{\mathbf{B}}^p$ if it is locally strongly measurable and $\int_0^1 \|f(t + u)\|_{\mathbf{B}}^p du < \infty$ for every t . If f is locally $L_{\mathbf{B}}^p$ and we define $F: \mathbf{R} \rightarrow L_p(I, \mathbf{B})$ by $F(t) = f(t + \cdot)$, then the approximations resulting from Bochner- L_p integrability imply that F is continuous at every t .

A locally $L_{\mathbf{B}}^p$ function f is called Stepanov $S_p(\mathbf{B})$ almost periodic or $S_p(\mathbf{B})$ (see [9]) if for every $\varepsilon > 0$ the set

$$S_p(\varepsilon, f) = \left\{ t': \sup_t \int_t^{t+1} \|f(u + t') - f(u)\|_{\mathbf{B}}^p du < \varepsilon \right\}$$

is RD. Since the continuity of F is obtained a-fortiori for every such f , it is easy to see that f is in $S_p(\mathbf{B})$ if and only if the corresponding $L_p(I, \mathbf{B})$ -valued function $F(t) = f(t + \cdot)$ is almost periodic in the sense of Bohr, i.e., it is continuous and for every $\varepsilon > 0$ the set

$$S(\varepsilon, F) = \left\{ t': \sup_t \|F(t + t') - F(t)\|_p < \varepsilon \right\}$$

is RD [1]. The spaces of scalar valued Stepanov AP functions (i.e., $S_p(\mathbf{R})$ or $S_p(\mathbf{C})$) will be denoted by S_p , or $S_p\text{AP}$.

For any S_p function f the mean value

$$M_t\{f\} := \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T+c}^{T+c} f(t) dt$$

exists uniformly with respect to the parameter c . Further, it is clear that

$$\|M_t\{f\}\|_{\mathbf{B}}^p \leq \sup_t \int_0^1 \|f(t+s)\|_{\mathbf{B}}^p ds \leq \sup_t \|f(t)\|_{\mathbf{B}}^p.$$

If f is any S_p almost periodic function, also is $f(t) \exp(-i\lambda t)$ and the Fourier coefficients

$$a(\lambda) = M_t\{f(t) \exp(-i\lambda t)\} \quad (1.8)$$

exist for every λ . The set $\Lambda = \{\lambda: a(\lambda) \neq 0\}$ is countable and the coefficients $a(\lambda)$ (for $\lambda \in \Lambda$) determine f uniquely up to equality almost everywhere, i.e., if for every λ

$$M_t\{f(t) \exp(-i\lambda t)\} = M_t\{g(t) \exp(-i\lambda t)\}$$

then $\|f(t) - g(t)\|_{\mathbf{B}}^p \rightarrow 0$ almost everywhere with respect to the Lebesgue measure on $\mathbf{R}$.

2. Stepanov almost periodically correlated processes

A complex valued second order stochastic process $X = X(t)$, $t \in \mathbf{R}$, with correlation $R(s, t) = \mathbf{E}\{X(s) \overline{X(t)}\}$ is called S_p almost periodically correlated (S_p APC) if for every τ the function

$$B(t, \tau) = R(t + \tau, t)$$

is S_p AP.

For such $R(s, t)$ we define

$$\begin{aligned} a(\lambda, \tau) &= M_t\{B(t, \tau) \exp(-i\lambda t)\} \\ &= \lim_{T \rightarrow \infty} \int_{-T+c}^{T+c} R(t + \tau, t) \exp(-i\lambda t) dt. \end{aligned} \quad (2.1)$$

The limit does not depend on c by (1.8).

Gladyshev [4] gives a necessary and sufficient condition for a continuous function $R(s, t)$ to be a correlation of an APC process. The necessity extends immediately to the S_p APC case; in fact the same proof yields even a slightly more general theorem.

Proposition 2.1. *Let $R(s, t)$, $t, s \in \mathbf{R}$, be the correlation of an S_p APC process. Then for every real $t_1, \dots, t_n$, $\mu_1, \dots, \mu_n$ and complex $z_1, \dots, z_n$ we have*

$$\sum_{k,l} z_k \overline{z_l} a(\mu_l - \mu_k, t_k - t_l) \exp[i\mu_k(t_k - t_l)] \geq 0. \quad (2.2)$$

P r o o f. The proof is identical to Gladyshev's; we include it for completeness.

First for every n and collections $\{t_1, \dots, t_n\}$, $\{w_1, \dots, w_n\}$ we have

$$\sum_{k,l} R(t_k + u, t_l + u) w_k \overline{w_l} \geq 0.$$

Then we note from (2.1) that

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T R(s+u, t+u) \exp(-i\lambda u) du = a(\lambda, s-t) \exp(i\lambda t). \quad (2.3)$$

Substituting $w_k = \exp(i\mu_k(u+t_k))z_k$ and integrating with respect to u , we get

$$\sum_{k,l} z_k \bar{z}_l \frac{1}{2T} \int_{-T}^T R(t_k+u, t_l-u) \exp[-i(\mu_l-\mu_k)u] du \exp[i(\mu_k t_k - \mu_l t_l)] \geq 0.$$

Now using (2.3) and after obvious calculations we conclude (2.2).

We now address the extension of the sufficiency of Gladyshev's theorem to the Stepanov APC case. We first note that the coefficients $a(\lambda, \tau)$ determine $R(t+\tau, t)$ only up to equality almost everywhere, thus a condition like (2.2) will not imply that R is positive definite but at most that it is equal to a positive definite function almost everywhere in a sense to be made precise subsequently. Moreover, Gladyshev's proof requires that not only every Λ_τ but the union $\Lambda = \bigcup_\tau \Lambda_\tau$ be countable. Gladyshev does not address this issue but his proof can be augmented as follows: the uniform joint continuity of $R(t, s)$ implies that for each fixed λ the function $\tau \rightarrow a(\lambda, \tau)$ is continuous; indeed

$$\begin{aligned} |a(\lambda, \tau+h) - a(\lambda, \tau)| &\leq M_t \left\{ |R(t+\tau+h, t) - R(t+\tau, t)| \right\} \\ &\leq \sup_t |R(t+\tau+h, t) - R(t+\tau, t)| < \varepsilon \end{aligned}$$

for $|h|$ sufficiently small. Now continuity of $a(\lambda, \cdot)$ implies that if $a(\lambda, \tau) = 0$ for all rational τ then $a(\lambda, \tau) = 0$ for all $\tau \in \mathbf{R}$. It follows that

$$\Lambda = \bigcup_{\tau \in \mathbf{R}} \Lambda_\tau \subset \bigcup_{\tau \text{ rational}} \Lambda_\tau$$

which implies that Λ is at most countable.

The above reasoning may be used under the weaker assumption that $\tau \rightarrow B(\cdot, \tau)$ is continuous with respect to the S_p -norm; for then,

$$M_t \left\{ |R(t+\tau+h, t) - R(t+\tau, t)| \right\}^p \leq \sup_t \int_t^{t+1} |B(u, \tau+h) - B(u, \tau)|^p du < \varepsilon$$

for $|h|$ sufficiently small, and hence we can conclude that Λ is countable. It also turns out that this continuity assumption is enough to extend Gladyshev's theorem to the S_p case.

Theorem 2.1. Suppose $B(\cdot, \tau) = R(\cdot + \tau, \cdot)$ is S_p almost periodic and $\tau \rightarrow B(\cdot, \tau)$ is a continuous function from $\mathbf{R}$ to S_p . Then if condition (2.2) is satisfied, there exist a positive definite function $\tilde{R}(t, s)$ and a set $M \subset \mathbf{R}$ of Lebesgue measure zero such that $R(t, s) = \tilde{R}(t, s)$ for $(t, s) \notin M \times M$.

P r o o f. The remark before the theorem shows that there exists one countable set $\Lambda = \{\lambda_1, \lambda_2, \dots\}$ such that $a(\lambda, \tau) = 0$ for $\lambda \notin \Lambda$ and $\tau \in \mathbb{R}$ so we can form the Bochner–Fejer approximating polynomials as in Gladyshev's theorem. Namely, let $\beta_1, \beta_2, \dots$ be a rational basis of Λ , i.e., a subset of Λ such that for every $\lambda \in \Lambda$ there exists a sequence of rational numbers $r_1, r_2, \dots, r_n$ such that $\lambda = \sum_{j=1}^n r_j \beta_j$ and $\beta_1, \beta_2, \dots$ are linearly independent over the field of rational numbers. Such a basis can always be selected from elements of Λ ([9, p. 26]). The Bochner–Fejer polynomials are defined as follows ([9, p. 27]):

$$R_m(u, v) = \frac{1}{(m!)^{2m}} \sum_{v_1=0}^{(m!)^2-1} \cdots \sum_{v_m=0}^{(m!)^2-1} \sigma_{v_1 \dots v_m}(u, v)$$

where

$$\begin{aligned} \sigma_{v_1 \dots v_m}(u, v) &= \sum_{k_1=-v_1}^{v_1} \cdots \sum_{k_m=-v_m}^{v_m} \exp \left\{ i \left(\frac{v_1}{m!} \beta_1 + \cdots + \frac{v_m}{m!} \beta_m \right) v \right\} \\ &\times a \left(\frac{v_1}{m!} \beta_1 + \cdots + \frac{v_m}{m!} \beta_m, u - v \right). \end{aligned}$$

Using the formula

$$\sum_{v=0}^m \sum_{k=-v}^v a_k = \sum_{k,j=0}^m a_{k-j}$$

we can rewrite the formula for R_m as

$$\begin{aligned} R_m(u, v) &= \sum_{k_1, \dots, k_m} \sum_{j_1, \dots, j_m} a(\mu_{k_1 \dots k_m} - \mu_{j_1 \dots j_m}, u - v) \\ &\times \exp \left\{ i (\mu_{k_1 \dots k_m} - \mu_{j_1 \dots j_m}) v \right\}, \end{aligned}$$

where we have denoted

$$\mu_{k_1 \dots k_m} = \frac{k_1}{m!} \beta_1 + \cdots + \frac{k_m}{m!} \beta_m \quad (2.3')$$

and the sum is taken over all k_i, j_i such that $0 \leq k_i, j_i \leq (m!)^2 - 1$. Now for any $t_1, \dots, t_n, z_1, \dots, z_n$ after multiplying and dividing by $\exp\{i\mu_{j_1 \dots j_m}(t_k - t_l)\}$ we get

$$\begin{aligned} \sum_{k,l} R_m(t_k, t_l) z_k \bar{z}_l &= \sum_{k_1, \dots, k_m} \sum_{j_1, \dots, j_m} \sum_{k,l} a(\mu_{k_1 \dots k_m} - \mu_{j_1 \dots j_m}, t_k - t_l) \\ &\times \exp \{ i \mu_{j_1 \dots j_m} (t_k - t_l) \} z_{k, j_1 \dots j_m} \bar{z}_{l, k_1 \dots k_m} \end{aligned}$$

where

$$z_{k, j_1 \dots j_m} = \exp \{ -i \mu_{j_1 \dots j_m} t_k \} z_k, \quad z_{l, k_1 \dots k_m} = \exp \{ -i \mu_{k_1 \dots k_m} t_l \} z_l.$$

The above expression is of the form (2.2) with $t_k, \mu_k, a_k, 1 \leq k \leq n$, replaced by multi-index sequences $t_{k,k_1 \dots k_m}, \mu_{k,k_1 \dots k_m}, z_{k,k_1 \dots k_m}$ where $t_{k,k_1 \dots k_m} = t_k$, $\mu_{k,k_1 \dots k_m}$ is given by (2.3') and $z_{k,k_1 \dots k_m}$ is as above. Now condition (2.2) implies

$$\sum_{k,l} R_m(t_k, t_l) z_k \bar{z}_l \geq 0 \quad (2.4)$$

which, in order to apply the approximation theorem, we rewrite as

$$\sum B_m(t_l, t_k - t_l) z_k \bar{z}_l \geq 0.$$

For every τ , $B_m(t, \tau) = R_m(t + \tau, t)$ is the Bochner-Fejer polynomial ([9, p. 27]) corresponding to the AP function $B(\cdot, \tau)$, thus $B_m(\cdot, \tau)$ converges to $B(\cdot, \tau)$ in the Stepanov norm ([9, p. 30]), i.e.,

$$\sup_t \int_t^{t+1} |R_m(u + \tau, u) - B(u + \tau, u)|^p du \rightarrow 0$$

as $m \rightarrow \infty$.

Continuity of $\tau \rightarrow B(\cdot, \tau) \in S_p\text{AP}$ implies that for every N , the set of functions

$$A_N = \{B(\cdot, \tau), \tau \in [-N, N]\}$$

being an image of a compact set $[-N, N]$ is compact in S_p . By Lyusternik's theorem ([9, p. 7]) applied to $X = S_p$ this set of $S_p\text{AP}$ functions satisfies the following conditions:

(i) For every $t_0 \in \mathbf{R}$, the set of $L_p(0, 1)$ -valued functions:

$$E_{t_0, N} = E\{g \in L_p(0, 1); g(u) = B(t_0 + u, \tau), \tau \in [-N, N]\}$$

is compact in $L_p(0, 1)$.

(ii) Functions in A_N are equicontinuous, i.e., for every $\varepsilon > 0$ there exists a $\delta > 0$ depending only on N such that $|t' - t''| < \delta$ implies

$$\|B(t', \tau) - B(t'', \tau)\|_{S_p} < \varepsilon$$

for every τ in $[-N, N]$.

(iii) A_N is equi-almost periodic in the sense that for every $\varepsilon > 0$ there is an $l = l_\varepsilon(N)$, depending on N but not on $\tau \in [-N, N]$ such that every interval $(n, n + l) \subset \mathbf{R}$ contains a common ε -almost period for all $B(\cdot, \tau)$, $\tau \in [-N, N]$.

The proof of the approximation theorem ([9, p. 29]) shows that conditions (i)–(iii) imply that $B_m(\cdot, \tau)$ converges to $B(\cdot, \tau)$ in S_p , uniformly in $\tau \in [-N, N]$; that is, for every $\varepsilon > 0$ there exists $m_0 = m_0(\varepsilon, N)$ such that for $m > m_0$ and $\tau \in [-N, N]$

$$\sup_t \int_t^{t+1} |B_m(u, \tau) - B(u, \tau)|^p du < \varepsilon.$$

From this it follows that for every N , $B_m(\cdot, \tau)$ converges in $L_p(-N, N)$ uniformly in $\tau \in [-N, N]$ and, hence, it converges in measure also uniformly in τ , i.e., for every $\varepsilon_1, \varepsilon_2$ there exists $m > m_0(\varepsilon_1, \varepsilon_2, N)$ such that $n, m > m_0$ implies

$$\text{Leb} \left\{ t: |B_m(t, \tau) - B_n(t, \tau)| > \varepsilon_1 \right\} < \varepsilon_2$$

for all $\tau \in [-N, N]$. Now applying a standard argument on existence of a subsequence converging almost surely (e.g., [6, p. 59]), we can find m_n , depending only on N , and a set $M_N \subset \mathbf{R}$ of Lebesgue measure zero such that for every $t \notin M_N$ and $\tau \in [-N, N]$ we have

$$\lim_{n \rightarrow \infty} B_{m_n}(t, \tau) = B(t, \tau).$$

It follows from the above that there exists a subsequence m_ν such that unless $t \in M = \cup_N M_N$, $B_{m_\nu}(t, \tau)$ converges to $B(t, \tau)$. Now we have for any $t_1, \dots, t_n \notin M$

$$\sum_{k,l} B_{m_\nu}(t_l, t_k - t_l) z_k \bar{z}_l = \sum_{k,l} R_{m_\nu}(t_k, t_l) z_k \bar{z}_l \geq 0 \quad (2.5)$$

converges, as $\nu \rightarrow \infty$ to

$$\sum_{k,l} B(t_l, t_k - t_l) z_k \bar{z}_l = \sum_{k,l} R(t_k, t_l) z_k \bar{z}_l \geq 0. \quad (2.6)$$

Setting $\tilde{R}(t, s) = R(t, s)$ for $(t, s) \in (\mathbf{R} \setminus M) \times (\mathbf{R} \setminus M)$ and $\tilde{R}(t, s) = 0$ elsewhere concludes the proof of the theorem.

R e m a r k. The difficulty in the last part of the proof of Theorem 2.1 is that we are trying to infer something about $R(t, s) = B(s, t - s)$ using convergence of $B_m(t, \tau) = R_m(t + \tau, t)$. If R is jointly uniformly continuous as in Gladyshev's case, then $R_m(t, s) = B_m(s, t - s)$ converges to $B(s, t - s) = R(t, s)$ for every $\tau = t - s$ and every s , hence, for every t and s . In the present situation we only have, for every $\tau = t - s$, almost everywhere convergence with respect to s in a set of Lebesgue measure zero, say M_τ , and unless we can combine $\cup_\tau M_\tau$ into one set of measure zero we can not take a limit in (2.5) to get (2.6).

The same continuity that was instrumental in Theorem 2.1 permits us to conclude the following.

Theorem 2.2. Suppose $B(\cdot, \tau) = R(\cdot + \tau, \cdot)$ is S_p almost periodic and $\tau \rightarrow B(\cdot, \tau)$ is a continuous function from $\mathbf{R}$ to S_p . Then for every $\lambda \in \mathbf{R}$ there exists a signed measure μ_λ of finite total variation for which

$$a(\lambda, \tau) = \int_{-\infty}^{\infty} \exp(i\gamma\tau) \mu_\lambda(d\gamma). \quad (2.7)$$

R e m a r k. Under the same hypotheses, μ_λ will be non-null for only a countable set of λ .

P r o o f. By remarks preceding Theorem 2.1, the function $\tau \rightarrow a(\lambda, \tau)$ is continuous for each λ and since $a(0, \cdot)$ may be shown to be positive definite (see [7]), there exists a finite measure μ_0 such that

$$a(0, \tau) = \int_{-\infty}^{\infty} \exp(i\gamma\tau) \mu_0(d\gamma).$$

Now continuity of $a(\lambda, \cdot)$ and Proposition 3 of [7] show that (2.7) holds for every λ where μ_λ is a signed measure of finite total variation bounded by that of μ_0 , i.e.,

$$\int_{-\infty}^{\infty} |\mu_\lambda(d\gamma)| \leq a(0, 0) = \int_{-\infty}^{\infty} \mu_0(d\gamma).$$

3. Stepanov almost periodically unitary processes

In this section, the Banach space $\mathbf{B}$ of Section 1.1 is specialized to be $L_2(\Omega)$. For a second order stochastic process $X = \{X(t), t \in \mathbf{R}\}$ we denote by $H = H_X$ the $L_2(\Omega)$ -closure of $\text{span}\{X(t), t \in \mathbf{R}\}$. H_X is a Hilbert space with the norm $\|\cdot\|_2 = \|\cdot\|_{L_2(\Omega)}$. We consider processes X that satisfy the following conditions:

Condition A: X is locally strongly measurable,

Condition B: $\int_t^{t+1} \|X(u)\|_2^p du < \infty$ for every t .

Conditions A and B together imply X is locally L_2^p ([12, p. 133]) and $X(t + \cdot)$ as a map $\mathbf{R} \rightarrow L_p(I, L_2(\Omega))$ is continuous.

Definition 3.1. A second order stochastic process $X: \mathbf{R} \rightarrow L_2(\Omega)$ is called S_p almost periodically unitary (S_p APU) if it satisfies conditions A and B and there exists a strongly continuous group $U(t')$, $t' \in \mathbf{R}$, of unitary operators on H_X such that for every $\varepsilon > 0$ the set

$$S_p(\varepsilon, X, U) = \left\{ t': \sup_t \int_t^{t+1} \|X(u+t') - U(t')X(u)\|_2^p du < \varepsilon \right\}$$

is RD.

We now give some examples and describe basic properties of S_p APU processes. In each of the examples we must show that X satisfies conditions A and B and there exists a strongly continuous group of unitary operators satisfying the preceding definition. To show X satisfies condition A (locally strongly measurable) it suffices by a theorem of Pettis ([12, p. 130]) to show

Condition A₁: $X(t)$ is locally weakly measurable; that is, $\langle X(t), Z \rangle$ is a Borel measurable function of t for every $Z \in H_X$;

Condition A₂: $X(t)$ is almost separably valued: there is a Borel measurable set E with $m(E) = 0$ and the range of $X(t)$ for $t \notin E$ is separable.

Example 1. First, an $L_2(\Omega)$ -continuous wide sense stationary pro-

cess is S_p APU for $p > 0$ with $U(t')$ a shift group defined by

$$U(t') \left(\sum_{j=1}^n a_j X(t_j) \right) = \sum_{j=1}^n a_j X(t_j + t')$$

as it follows that $\|X(u + t') - U(t')X(u)\|_2^p = 0$ for every t' and u , so $S_p(\varepsilon, X, U) = \mathbf{R}$. The weak measurability comes from the fact that $\langle X(t), Z \rangle$ is continuous and therefore measurable function of t for every $Z \in H_X$. Further, the continuity of X implies the required separability. Finally, $\int_t^{t+1} \|X(u)\|_2^p du < \infty$ for every t because $\|X(u)\|_2^p = \|X(0)\|_2^p$ from stationarity.

Example 2. Every uniformly APU process is S_p APU with any $p > 0$. The weak measurability and separability come from the continuity of X as in the stationary case. Further, $\int_t^{t+1} \|X(u)\|_2^p du < \infty$ for every t because $\|X(u)\|_2^p$ is continuous and bounded [6]. Finally, the fact that

$$S(\varepsilon, X, U) = \left\{ t' : \sup_t \|X(t + t') - U(t')X(t)\|_2 < \varepsilon \right\}$$

is RD and $\|X(t + t') - U(t')X(t)\|_2$ is a continuous function of t implies $S_p(\varepsilon, X, U)$ is RD. Thus in particular a trigonometric polynomial with jointly stationary coefficients, i.e., a process of the form

$$X(t) = \sum_{j=1}^n a_j(t) \exp(i\lambda_j t)$$

where $\{a_j(t), j = 1, \dots, n\}$ is a set of jointly wide sense stationary processes, is uniformly APU and hence S_p APU.

Example 3. If $\{a_j(t), j = 1, \dots, n\}$ are instantaneously orthogonal and q.m. continuous jointly wide sense stationary, and $\{f_j(t), j = 1, \dots, n\}$ are scalar S_p almost periodic functions, then

$$X(t) = \sum_{j=1}^n f_j(t) a_j(t)$$

is S_p APU. We shall make the argument only for $n = 1$. The weak measurability comes from the fact that $\langle f(t)a(t), Z \rangle = f(t)\langle a(t), Z \rangle$ and the two factors are measurable. The process $f(t)a(t)$ is separably valued because $a(t)$ is separably valued and the scalar multiplier $f(t)$ can only eliminate vectors from the range of $a(t)$. Next, $\int_t^{t+1} \|f(u)a(u)\|_2^p du = \|a(0)\|_2^p \int_t^{t+1} |f(u)|^p du < \infty$ for every t because $f(u)$ is S_p . Choosing the group $\{U(t'), t' \in \mathbf{R}\}$ to be the shift group for the stationary process $a(t)$, then $\|X(u + t') - U(t')X(u)\|_2^p = |f(u + t') - f(u)|^p \|a(u + t')\|_2^p$ implies that $S_p(\varepsilon, X, U)$ is RD. This shows that amplitude scale modulations of wide sense stationary by scalar S_p functions produces S_p APU processes.

Example 4. The next class of examples (time scale modulation) is contained in the following proposition.

Proposition 3.1. *Let X be a mean square continuous wide sense stationary stochastic process and let f be a scalar S_p function. Then the process*

$$Y(t) = X(t + f(t))$$

is S_p APU.

P r o o f. In this example we are required to replace $X(t)$ with a jointly measurable modification $\tilde{X}(t)$ (this means $\tilde{X}(t, \omega) = X(t, \omega)$ almost surely for each t) in order to obtain the weak measurability. For then, since $f(t)$ is measurable,

$$\langle \tilde{X}(t + f(t)), Z \rangle = \int_{\Omega} \tilde{X}(t + f(t), \omega) \overline{Z(\omega)} dP$$

is a measurable function of t for $Z \in H(Y)$. The process $Y(t)$ is separably valued because $\tilde{X}(t)$ is, and $\int_t^{t+1} \|Y(u)\|_2^p du < \infty$ for every t because $\|\tilde{X}(u + f(u))\|_2^p = \|\tilde{X}(0)\|_2^p$ from the stationarity of $\tilde{X}$. To complete the proof we will show that for every $\varepsilon > 0$ there exists $\varepsilon' > 0$ such that if

$$\sup_t \int_t^{t+1} |f(u + t') - f(u)|^p du < \varepsilon'$$

then

$$\sup_t \int_t^{t+1} \|\tilde{X}(f(u + t')) - \tilde{X}(f(u))\|_2^p du < \varepsilon.$$

Then taking $U(t')$ to be the shift group for the process $\tilde{X}$ we find

$$\|Y(t + t') - U(t')Y(t)\|_2 = \|\tilde{X}(f(t + t')) - \tilde{X}(f(t))\|_2,$$

and so for every $\varepsilon > 0$

$$S_p(\varepsilon, \tilde{X}, U) \supset S_p(\varepsilon', f)$$

which will then imply that $S_p(\varepsilon, \tilde{X}, U)$ is separable.

Thus let $\varepsilon > 0$ be given. To prove the existence of ε' we first note that the continuity of the correlation of $\tilde{X}$ implies that there exists $\delta > 0$ such that if $u \in M_\delta$ where

$$M_\delta = \{u \in [t, t + 1]: |f(u + t') - f(u)| < \delta\}$$

then

$$\|\tilde{X}(f(u + t')) - \tilde{X}(f(u))\|_2^p < \frac{\varepsilon}{2}.$$

Now

$$\begin{aligned} & \int_t^{t+1} \|\tilde{X}(f(u + t')) - \tilde{X}(f(u))\|_2^p du \\ &= \left(\int_{M_\delta \cap [t, t+1]} + \int_{[t, t+1] \setminus M_\delta} \right) \|\tilde{X}(f(u + t')) - \tilde{X}(f(u))\|_2^p du \\ &\leq \frac{\varepsilon}{2} m\{M_\delta\} + 2 \sup_u \|\tilde{X}(u)\|_2^p m\{[t, t + 1] \setminus M_\delta\} \end{aligned}$$

where m denotes Lebesgue measure and $\sup_u \|\tilde{X}(u)\|_2 < \infty$ because $\tilde{X}$ is stationary. But $m\{[t, t+1]\} = 1$ and by Chebyshev's inequality

$$m\{[t, t+1] \setminus M_\delta\} \leq \frac{1}{\delta^p} \int_t^{t+1} |f(u+t') - f(u)|^p du.$$

It follows that

$$\sup_u \int_t^{t+1} |f(u+t') - f(u)|^p du < \varepsilon'$$

with $\varepsilon' = (\varepsilon/2)\delta^p(2\sup_u \|\tilde{X}(u)\|_2^p)^{-1}$ implies

$$\int_t^{t+1} \|\tilde{X}(f(u+t')) - \tilde{X}(f(u))\|_2^p du < \varepsilon$$

which proves the claim and hence the proposition.

It is known that $X(t)$ is APU in the Bohr sense if and only if $X(t) = U(t)[P(t)]$ where $U(t)$ is a strongly continuous group of unitary operators on H_X and $P(t)$ is a $L_2(\Omega)$ -valued almost periodic function in the sense of Bohr. To approach a similar result in this case we must see if the conditions A and B are preserved under the action of the unitary group.

Proposition 3.2. *If $\{U(t), t \in \mathbf{R}\}$ is a strongly continuous group of unitary operators on H_X then*

$$Y(t) = U(t)[X(t)] \quad (3.1)$$

satisfies conditions A and B if and only if $X(t)$ satisfies the same conditions.

R e m a r k. The sense of equality in (3.1) is in $L_2(\Omega)$ for almost every t , so that equality in the sense of the Stepanov norm:

$$\|f\|_{S_p} = \sup_t \left(\int_t^{t+1} \|f(u)\|_2^p du \right)^{1/p}$$

is implied.

P r o o f. We shall only give the sufficiency as $U^{-1}(t) = U(-t)$ is also strongly continuous group of unitary operators on H_Y . To show $Y(t)$ satisfies A we show it locally satisfies A_1 and A_2 where $X(t)$ satisfies A_1 and A_2 . Since $X(t)$ is locally almost separably valued, then for any arbitrary t it is separably valued for $s \notin E_t \subseteq (t, t+1)$ where $\text{Leb}(E_t) = 0$. Let $\{\xi_j; j = 1, 2, \dots\}$ be a CONS of $\overline{\text{sp}}\{X(s), s \in (t, t+1), s \notin E_t\}$ so that for such s we can write

$$X(s) = \sum_{j=1}^{\infty} \langle X(s), \xi_j \rangle \xi_j \quad (3.2)$$

where the coefficient functions $f_j(s) = \langle X(s), \xi_j \rangle$ are measurable because X is weakly measurable (A_1) and $\sum_{j=1}^{\infty} |f_j(s)|^2 < \infty$ for $s \notin E_t$. For such s we can write

$$Y(s) = U(s)[X(s)] = \sum_{j=1}^{\infty} \langle X(s), \xi_j \rangle U(s)[\xi_j] \quad (3.3)$$

because (3.2) is strongly convergent. But (3.3) shows that for $s \notin E_t$ the range of Y is separable because the processes $a_j(s) = U(s)[\xi_j]$ are all continuous and thus separable; the scalar coefficients $f_j(s) = \langle X(s), \xi_j \rangle$ in (3.3) can only eliminate vectors. So Y satisfies A_2 .

To show Y satisfies A_1 , we express

$$\langle Y(s), Z \rangle = \sum_{j=1}^{\infty} \langle X(s), \xi_j \rangle \langle U(s)[\xi_j], Z \rangle = \sum_{j=1}^{\infty} f_j(s) g_j(s) \quad (3.3')$$

where the f_j are measurable, the g_j are continuous and hence measurable, and the sum converges absolutely for $s \notin E_t$, and so $\langle Y(s), Z \rangle$ is equal almost everywhere to a measurable function.

Finally, to show $\int_t^{t+1} \|Y(u)\|_2^p du < \infty$ for every t (B), we just note that $\|Y(u)\|_2^p = \|U(u)[X(u)]\|_2^p = \|X(u)\|_2^p$.

Corollary 3.1. *If X and Y are related as in (3.1) by a strongly continuous group $\{U(t), t \in \mathbb{R}\}$ of unitary operators, then Y is locally L_2^p if and only if X is locally L_2^p .*

The following is the basic characterization of S_p APU processes. It is the S_p version of Proposition 1 of [6].

Proposition 3.3. *A second order stochastic process $X = X(t)$, $t \in \mathbb{R}$, is S_p APU with a shift group U if and only if*

$$X(t) = U(t)[P(t)] \quad (3.4)$$

for some $S_p(H_X)$ almost periodic function $P(t)$.

Proof. In view of Proposition 3.2 we only need to show that $S_p(\varepsilon, X, U)$ is RD if and only if $S_p(\varepsilon, P)$ is RD where $P(t) = U(-t)[X(t)]$; the proof is almost identical to the proof of Theorem 1 of [6]. If $X(t)$ is given by (3.4) where $P(t)$ is an $S_p(H_X)$ AP function, then by Corollary 3.1 X is locally L_2^p . Further, in view of

$$\int_t^{t+1} \|X(u+t') - U(t')X(u)\|_2^p du = \int_t^{t+1} \|P(u+t') - P(u)\|_2^p du \quad (3.5)$$

we conclude $S_p(\varepsilon, X, U) = S_p(\varepsilon, P)$ is RD so X is S_p APU. Conversely, if X is S_p APU with the unitary group U then $P(t) = U(-t)X(t)$ is locally L_2^p and p -integrable and (3.5) again implies $S_p(\varepsilon, P) = S_p(\varepsilon, X, U)$ and hence P is $S_p(H_X)$ almost periodic.

Some basic properties of S_p APU processes are summarized in the following proposition.

Theorem 3.1. *Let $X(t)$ be an S_p APU process with the unitary group U . Then*

- (i) X is S_p -bounded, i.e., $\sup_t \int_t^{t+1} \|X(u)\|_2^p du < \infty$;
- (ii) X is S_p -uniformly continuous, i.e., $\lim_{h \rightarrow 0} \sup_t \int_t^{t+1} \|X(u+h) - X(u)\|_2^p du = 0$;

(iii) U is S_p -equicontinuous on X , i.e., $\lim_{h \rightarrow 0} \sup_t \int_t^{t+1} \|U(h)X(u) - X(u)\|_2^p du = 0$;

(iv) The function $Z(v, u) = U(v)P(u) = U(v)U(-u)X(u)$ is jointly S_p -continuous, i.e.,

$$\lim_{h, k \rightarrow 0} \sup_{t, s} \int_t^{t+1} \int_s^{s+1} \|Z(v+k, u+h) - Z(v, u)\|_2^p dv du = 0.$$

P r o o f. From (3.1) we have

$$\sup_t \int_t^{t+1} \|X(u)\|_2^p du = \sup_t \int_t^{t+1} \|P(u)\|_2^p du$$

where P is an $S_p(H_X)$ almost periodic function but for any such function the last expression is finite (see, e.g., [9, p. 34]). This proves (i).

To prove (ii) observe that since X is locally L_2^p integrable, for any t it is continuous in the sense

$$\lim_{h \rightarrow 0} \int_t^{t+1} \|X(u+h) - X(u)\|_2^p du = 0 \quad (3.6)$$

and so it will be uniformly continuous in this sense for t in any bounded interval (a, b) . As in the case of UAP functions, the question of continuity at an arbitrary t can be translated to a single bounded interval. Thus for $\varepsilon > 0$ and arbitrary t write $t = t' + t''$ where $t'' \in [0, l_\varepsilon]$ and $t' \in S(\varepsilon, X)$. Performing obvious changes of variables and adding and subtracting $U(t')X(u+t)$ and $U(t')X(u)$ we get

$$\begin{aligned} & \left(\int_t^{t+1} \|X(u+h) - X(u)\|_2^p du \right)^{1/p} \\ &= \left(\int_{t''}^{t''+1} \|X(u+t'+h) - X(u+t')\|_2^p du \right)^{1/p} \leq \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \end{aligned}$$

where

$$\begin{aligned} \varepsilon_1^p &= \int_{t''}^{t''+1} \|X(u+t'+h) - U(t')X(u+h)\|_2^p du, \\ \varepsilon_2^p &= \int_{t''}^{t''+1} \|U(t')X(u+h) - U(t')X(u)\|_2^p du, \\ \varepsilon_3^p &= \int_{t''}^{t''+1} \|U(t')X(u) - X(u+t')\|_2^p du. \end{aligned}$$

Now since $t' \in S(\varepsilon, X)$, ε_1^p and ε_3^p are bounded by ε . By taking $|h|$ sufficiently small, the remaining term ε_2^p is no bigger than ε from the uniform continuity of X (in the sense of (3.2)) on any bounded open interval containing $[0, l_\varepsilon]$.

Turning to (iii), write

$$\begin{aligned} & \sup_t \left(\int_t^{t+1} \|U(h)X(u) - X(u)\|_2^p du \right)^{1/p} \\ & \leq \sup_t \left(\int_t^{t+1} \|U(h)X(u) - X(u+h)\|_2^p du \right)^{1/p} \\ & \quad + \sup_t \left(\int_t^{t+1} \|X(u+h) - X(u)\|_2^p du \right)^{1/p}. \end{aligned}$$

The second term goes to zero as $h \rightarrow 0$ by (ii). The first one equals

$$\sup_t \left(\int_t^{t+1} \|P(u+h) - P(u)\|_2^p du \right)^{1/p}$$

where P is the $S_p(H_X)$ AP function from Proposition 3.3. It converges to zero by the uniform S_p continuity of S_p almost periodic functions ([9, p. 33]). Finally (iv) follows by noting that

$$\begin{aligned} & \sup_{t,s} \left(\int_s^{s+1} \int_t^{t+1} \|Z(v+k, u+h) - Z(v, u)\|_2^p du dv \right)^{1/p} \\ & \leq \sup_{t,s} \left(\int_s^{s+1} \int_t^{t+1} \|(U(v+k) - U(v))P(u+h)\|_2^p du dv \right)^{1/p} \\ & \quad + \sup_{t,s} \left(\int_s^{s+1} \int_t^{t+1} \|U(v)P(u+h) - U(v)P(u)\|_2^p du dv \right)^{1/p} \end{aligned}$$

and applying (iii) to the first term and using the S_p continuity of p in the second.

We next give representations of S_p APU processes, analogous to those given in [6] for uniformly APU processes, based on $X(t) = U(t)[P(t)]$ where the sense of equality is that of the Stepanov norm. The first is based on Stone's theorem on representation of unitary groups [11] which in our case gives

$$X(t) = U(t)P(t) = \int_{-\infty}^{+\infty} \exp(i\gamma t) dE_\gamma[P(t)].$$

The second is based on a representation of an S_p AP function as a limit in Stepanov norm of a sequence of trigonometric polynomials. The third is based on expression (3.2) in the proof of Proposition 3.2. The proofs of the first two are identical to those in [6] and are therefore omitted.

Proposition 3.4. *A process $X = X(t)$, $t \in \mathbb{R}$, is S_p APU if and only if it has a representation*

$$X(t) = \int_{-\infty}^{+\infty} \exp(i\gamma t) Z(d\gamma, t) \quad (3.7)$$

where for each t , $Z(\cdot, t)$ is an $L_2(\Omega)$ -valued measure orthogonally scattered in that for every t, s and disjoint A, B $\mathbf{E} Z(A, t) Z(B, s) = 0$ and for each A , $Z(A, \cdot)$ is $S_p(H_X)$ almost periodic.

Proposition 3.5. A process $X = X(t)$, $t \in \mathbf{R}$, is S_p APU if and only if it is an S_p -limit of processes of the form $X_n(t) = \sum_{j=1}^{k_n} \eta_{j,n}(t) \exp\{i\gamma_j t\}$ where $\{\eta_{j,n}\}$ is a set of jointly wide sense stationary stochastic processes and $\{\gamma_j\}$ is a countable set of real numbers.

Theorem 3.2. The process $X = X(t)$, $t \in \mathbf{R}$, is S_p APU if and only if

$$X(t) = \sum_{j=1}^{\infty} f_j(t) a_j(t) \quad (3.8)$$

where the sequence $\{a_j(t)$, $j \in \mathbf{Z}\}$ consists of jointly stationary processes with $\mathbf{E}\{|a_j(0)|^2\} = 1 \ \forall j$ that are mutually instantaneously orthogonal, $\mathbf{E}\{a_j(t) \overline{a_k(t)}\} = 0$ for every t and $j \neq k$. The sequence $\{f_j(t)$, $j \in \mathbf{Z}\}$ consists of scalar S_p functions for which $\lim_{N \rightarrow \infty} \int_t^{t+1} \sum_{j > N} |f_j(s)|^2 ds = 0$ uniformly in t .

Proof. *Necessity.* If X is S_p APU it is locally almost separably valued; that is, for every $k \in \mathbf{Z}$, $\overline{\text{sp}}\{X(s), s \in [k, k+1] \setminus E_k\}$ is separable where $\text{Leb}(E_k) = 0$. It follows that $\overline{\text{sp}}\{X(s), s \notin \cup_k E_k\}$ is separable. Let $\{\varepsilon_j; j = 1, 2, \dots\}$ be a CONS of $\overline{\text{sp}}\{X(s), s \notin \cup_k E_k\}$ so that for a.e. s we can write

$$P(s) = \sum_{j=1}^{\infty} \langle P(s), \xi_j \rangle_2 \xi_j \quad (3.9)$$

where the coefficient functions $f_j(s) = \langle P(s), \xi_j \rangle_2$ are measurable because X is weakly measurable, and are locally L^p because $|f_j(s)| = |\langle P(s), \xi_j \rangle_2| \leq \|P(s)\|_2$. Each $f_j(s)$ is a scalar S_p function because

$$|f_j(s+t') - f_j(s)| = |\langle P(s+t') - P(s), \xi_j^t \rangle_2| \leq \|P(s+t') - P(s)\|_2.$$

Using $X(t) = U(t)[P(t)]$ we identify $a_j(t) = U(t)\xi_j$, and note that $\langle a_j(t), a_k(t) \rangle_2 = \langle U(t)\xi_j, U(t)\xi_k \rangle_2 = \langle \xi_j, \xi_k \rangle_2 = \delta_{jk}$.

To show $\int_t^{t+1} \sum_{j > N} |f_j(s)|^2 ds \rightarrow 0$ uniformly in t as $N \rightarrow \infty$, we first observe that

$$\begin{aligned} \int_t^{t+1} \|X(u) - X_N(u)\|_2^2 du &= \int_t^{t+1} \|P(u) - P_N(u)\|_2^2 du \\ &= \int_t^{t+1} \sum_{j \geq N} |f_j(u)|^2 du, \end{aligned} \quad (3.10)$$

where $P_N(t) = \sum_{j \leq N} f_j(t) \xi_j$ and $X_N(t) = U(t)[P_N(t)]$. We next adapt a proof of Levitan ([9, p. 75]) to show that $P_N(t)$ converges in the Stepanov norm uniformly to $P(t)$, and with the help of (3.10), the result will be concluded.

If $P(t)$ is $S(H_X)$ AP, then for every t , the function $\tilde{P}_t$ defined by $\tilde{P}_t(s) = P(t+s)$ is a map from $\mathbf{R}$ to the locally measurable functions $\{X: \int_0^1 \|X(t+u)\|_2^2 du < \infty \text{ for every } t\}$. An inner product may be defined on these functions by $\langle X, Y \rangle = \int_0^1 \langle X(u), Y(u) \rangle_2 du$, and since we already know they form a Banach space, they evidently also are a Hilbert space which we denote as $\mathcal{H}$. Now let $\mathcal{H}(\tilde{P}_t; t \in \mathbf{R})$ be the subspace of $\mathcal{H}$ generated by the uniformly almost periodic function $\tilde{P}_t$; since $\tilde{P}: \mathbf{R} \rightarrow \mathcal{H}$ is continuous, it follows that $\mathcal{H}(\tilde{P}_t; t \in \mathbf{R})$ is separable, and we take $\{e_\mu, \mu \in \mathbf{Z}\}$ to be a CONS for $\mathcal{H}(\tilde{P}_t; t \in \mathbf{R})$. Except for a set of Lebesgue measure zero, every e_μ has a representation

$$e_\mu(s) = \sum_{j=1}^{\infty} \langle e_\mu(s), \xi_j \rangle_2 \xi_j \quad (3.11)$$

where for those s

$$\sum_{j=1}^{\infty} |\langle e_\mu(s), \xi_j \rangle_2|^2 < \infty. \quad (3.12)$$

From the argument subsequent to (3.9) it follows that the complex valued functions $\langle e_\mu(s), \xi_j \rangle_2$ are $L_2[0, 1]$ and

$$\|e_\mu\|_2^2 = \int_0^1 \|e_\mu(s)\|_2^2 ds = \int_0^1 \sum_{j=1}^{\infty} |\langle e_\mu(s), \xi_j \rangle_2|^2 ds = 1.$$

If $\{\varphi_j, j \in \mathbf{N}^+\}$ is a CONS for $L_2[0, 1]$, then we may conclude that another CONS for $\mathcal{H}(\tilde{P}_t; t \in \mathbf{R})$ may be taken from the functions $\{\varphi_m(s)\xi_n, m \in \mathbf{N}^+, n \in \mathbf{N}^+\}$. Denote this new CONS by $\Psi = \{\psi_m, m \in \mathbf{N}^+\}$.

We are now ready to show the uniform convergence of (3.11). Since the function $\tilde{P}_t$ is UAP, the closure of its range $\overline{\mathcal{H}}(\tilde{P}_t; t \in \mathbf{R})$ is a compact subset of the Hilbert space $\mathcal{H}(\tilde{P}_t; t \in \mathbf{R})$ (which is thus a complete metric space). Hence $\overline{\mathcal{H}}(\tilde{P}_t; t \in \mathbf{R})$ is totally bounded, and so for any $\varepsilon > 0$ there is a finite collection of vectors $\{\eta_1, \eta_2, \dots, \eta_n\} \subset \mathcal{H}(\tilde{P}_t; t \in \mathbf{R})$ such that for every fixed t one of these vectors yields $\|\tilde{P}_t - \eta_k\|_2 < \varepsilon/2$; and each η_k has a finite approximant $\hat{\eta}_k = \sum_{j=1}^{N_k} \alpha_{jk} \psi_j$ for which $\|\hat{\eta}_k - \eta_k\|_2 < \varepsilon/2$. This leads to the conclusion that for every t there is an approximant $\hat{\eta}_k$ for which

$$\|\tilde{P}_t - \hat{\eta}_k\|_2 < \varepsilon; \quad (3.13)$$

that is, there exists a finite collection from Ψ , linear combinations of which uniformly approximate $\tilde{P}_t$. Let N be the largest index of the ξ_n appearing in all the approximants $\hat{\eta}_k$ and let S_N be the subspace of $\mathcal{H}(\tilde{P}_t; t \in \mathbf{R})$ formed by functions of the form $\sum_{j \leq N} g_j(t) \xi_j$ where $g_j \in L_2[0, 1]$. Then it is clear that all the approximants $\hat{\eta}_k \in S_N$, and additionally $\tilde{P}_{N,t} = \sum_{j \leq N} \langle P(t+s), \xi_j \rangle_2 \xi_j \in S_N$. But $\tilde{P}_{N,t}$ is just the projection of $\tilde{P}_t$ onto S_N and hence (3.13) implies that $\|\tilde{P}_t - \tilde{P}_{N,t}\|_2 < \varepsilon$ for all t .

Sufficiency. First, the joint stationarity of $\{a_j(t), j \in \mathbf{Z}\}$ implies there is a unitary group $\{U(\tau), \tau \in \mathbf{R}\}$ which, for every j , gives $a_j(t) = U(t)\xi_j$ where $\xi_j = a_j(0)$; the instantaneous orthogonality gives $\langle \xi_j, \xi_k \rangle = \delta_{jk}$. We now wish to show that $P: \mathbf{R} \rightarrow H_X$ given by $P(t) = \sum_j f_j(t)\xi_j$ is $S_p(H_X)$ AP, where $\{f_j(t)\}$ are scalar S_p functions that satisfy $\int_t^{t+1} \sum_{j>N} |f_j(s)|^2 ds \rightarrow 0$ uniformly in t as $N \rightarrow \infty$. But from (3.11) this is enough to conclude that P is locally L_2^2 and that $P_N(t) = \sum_{j \leq N} f_j(t)\xi_j$ converges in the Stepanov norm to $P(t)$ and thus $P(t)$ must be $S_p(H_X)$ AP.

4. Harmonic analysis of the correlation for S_p APU processes

We note that the Bohr APU processes may be shown to be uniformly APC in the sense of Dehay [2], and also equi-APC as well. In the current case, we show only that S_{2p} APU processes are equi- S_p APC.

Proposition 4.1. *If $X = X(t), t \in \mathbf{R}$, is S_{2p} APU, then the family of functions $\{B(\cdot, \tau) = R(\cdot + \tau, \cdot), \tau \in \mathbf{R}\}$ is equi- S_p AP.*

Proof. First we must show that $B(t, \tau)$ is locally L_p : $\int_0^1 |B(t+u, \tau)|^p du < \infty$ for every t . This follows immediately from an application of the Schwarz inequality (on $L_2[0, 1]$) to the function $|B(t+u, \tau)|^p \leq \|X(t+u+\tau)\|^p \|X(t+u)\|^p$. For fixed τ we replace $B(\cdot, \tau)$ by a measurable equivalent, as required; condition B implies that $\|X(t+\cdot)\|^p$ is Lebesgue measurable. To complete the result, we must show that

$$S(B, \varepsilon) = \left\{ t': \sup_{\tau \in \mathbf{R}} \sup_{t \in \mathbf{R}} \int_0^1 |B(t+u+t', \tau) - B(t+u, \tau)|^p du < \varepsilon \right\}$$

is relatively dense. For fixed t , write

$$X(t+t'+\tau) = U(t')X(t+\tau) + \varepsilon(t+\tau, t') \quad (4.1)$$

where $U(\cdot)$ is a shift group for X , and $\{t': \sup_{t \in \mathbf{R}} \sup_{t \in \mathbf{R}} \int_0^1 \|\varepsilon(t+u+\tau, t')\|^{2p} du < \varepsilon\}$ is RD. Then using (4.1) produces

$$\begin{aligned} R(t+u+t'+\tau, t+u+t') - R(t+u+\tau, t+u) \\ = \langle U(t')X(t+u+\tau), \varepsilon(t+u, t') \rangle + \langle U(t')X(t+u), \varepsilon(t+u+\tau, t') \rangle \\ + \langle \varepsilon(t+u+\tau, t'), \varepsilon(t+u, t') \rangle. \end{aligned}$$

Application of the triangle inequality to $\mathbf{C}$ and then the Schwarz inequality (in $L_2(\Omega)$) to each term on the right gives the inequality

$$\begin{aligned} |B(t+u+t', \tau) - B(t+u, \tau)| &\leq \|U(t')X(t+u+\tau)\| \|\varepsilon(t+u, t')\| \\ &+ \|U(t')X(t+u)\| \|\varepsilon(t+u+\tau, t')\| + \|\varepsilon(t+u+\tau, t')\| \|\varepsilon(t+u, t')\|. \end{aligned}$$

We next apply the triangle inequality on $L_p[0, 1]$, the unitarity of $U(\cdot)$ and then the Schwarz inequality on $L_2[0, T]$ to obtain bounding terms of the form

(taking the first on the right of the inequality)

$$\left[\int_0^1 \|X(t+u+\tau)\|^{2p} du \right]^{1/(2p)} \left[\int_0^1 \|\varepsilon(t+u, t')\|^{2p} du \right]^{1/(2p)}.$$

Taking the supremum over t and τ , the first is dominated (say by M_X) and the second is no larger than $[\varepsilon_0]^{1/(2p)}$ provided $t' \in S_p(\varepsilon_0, X, U)$. Combining all the terms produces

$$\begin{aligned} & \sup_{\tau \in \mathbf{R}} \sup_{t \in \mathbf{R}} \int_0^1 |B(t+u+t', \tau) - B(t+u, \tau)|^p du \\ & \leq 2M_X[\varepsilon_0]^{1/(2p)} + [\varepsilon_0]^{1/p} \equiv K(\varepsilon_0) \end{aligned}$$

provided $t' \in S_p(\varepsilon_0, X, U)$. Since $K(\varepsilon_0)$ is monotone for $\varepsilon_0 \geq 0$, we have for any $\varepsilon > 0$,

$$S(B, \varepsilon) \supset S_p(K^{-1}(\varepsilon), X, U).$$

Recall that

$$a(\lambda, \tau) = \lim_{n \rightarrow \infty} \frac{1}{2T} \int_{-T}^T R(t+\tau, t) \exp(-i\lambda t) dt$$

exists for every λ and τ and determines $B(t, \tau)$ uniquely almost everywhere (in the sense described in Section 2). For every τ the set $\Lambda_\tau = \{\lambda: a(\lambda, \tau) \neq 0\}$ is countable; moreover, as we mentioned in the remark just before Theorem 2.1, the continuity in S_p norm implies that there exists one countable set $\Lambda = \{\lambda_1, \lambda_2, \dots\}$ such that $a(\lambda, \tau) = 0$ for $\lambda \notin \Lambda$ and every τ . Thus $R(t+\tau, t)$ is uniquely (up to equality almost everywhere) represented by a countable sequence of Fourier frequencies $\{\lambda_k\}$ and coefficients $a(\lambda_k, \tau)$. We express this as in [9] by writing $R(t+\tau, t) \sim \sum_k a(\lambda_k, \tau) \exp(i\lambda_k t)$. The series on the right may not converge but there exists a sequence of multipliers $\mu_{k,n}$, $k = 1, 2, \dots, n$, $n = 1, 2, \dots$, such that for every τ the sequence

$$R_n(t+\tau, t) = \sum_{k=1}^n \mu_{n,k} a(\lambda_k, \tau) \exp(i\lambda_k t)$$

converges to $R(t+\tau, t)$ in Stepanov norm ([9, p. 35]), i.e.,

$$\lim_{n \rightarrow \infty} \sup_t \int_t^{t+1} |R(u+\tau, u) - R_n(u+\tau, u)|^p du = 0.$$

Due to their construction from the Fejer polynomials [9] the multipliers $\mu_{k,n}$ depend only of the frequencies $\{\lambda_k\}$ and are independent of τ and the coefficients $\{a(\lambda_k, \tau)\}$.

The following shows that the inherent continuity of a S_2 APU process is enough to ensure $B(\cdot, \tau)$ is continuous with respect to τ in the scalar S_2 sense, and by Theorem 2.2 $a(\lambda, \tau)$, for each fixed λ , is a Fourier transform of a signed measure of finite total variation.

Theorem 4.1. Let $X = X(t)$, $t \in \mathbb{R}$, be S_2 APU. Then for every $\lambda \in \mathbb{R}$ the function $a(\lambda, \cdot)$ is a Fourier transform of a signed measure of finite total variation.

P r o o f. To apply Theorem 2.2 we need to show the required continuity. To do this we note

$$\begin{aligned} & \sup_t \left(\int_t^{t+1} |B(u, \tau + h) - B(u, \tau)|^2 du \right) \\ &= \sup_t \left(\int_t^{t+1} \left| \mathbf{E}\{X(u + \tau + h) \overline{X(u)}\} - \mathbf{E}\{X(u + \tau) \overline{X(u)}\} \right|^2 du \right) \\ &\leq \sup_t \left(\int_t^{t+1} \|X(u + \tau + h) - X(u + \tau)\|_2^2 du \right)^{1/2} \\ &\quad \times \sup_t \left(\int_t^{t+1} \|X(u)\|_2^2 du \right)^{1/2}. \end{aligned}$$

By Theorem 3.1 the first term tends to zero as $h \rightarrow 0$ while the second term is bounded, thus proving the required continuity.

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