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THE STRONG LAW OF LARGE NUMBERS
FOR d -DIMENSIONAL ARRAYS
IN VON NEUMANN ALGEBRAS

Пусть $\mathcal{A}$ — алгебра фон Неймана с нормальным точным следовым состоянием τ ; пусть $\tilde{\mathcal{A}}$ $*$ -алгебра измеримых операторов в смысле Сигала. Цель настоящей работы — показать, что если массив $(x(\bar{n}), \bar{n} \in \mathbf{N}^d)$ самосопряженных попарно независимых одинаково распределенных элементов в $\tilde{\mathcal{A}}$ удовлетворяет условию

$$\tau(|x(\bar{1})|(\log^+ |x(\bar{1})|)^{d-1}) < \infty,$$

то

$$\frac{1}{n_1 \cdots n_d} \sum_{k_1=1}^{n_1} \cdots \sum_{k_d=1}^{n_d} x(k)$$

сходится к $\tau(x(\bar{1}))$ билатерально почти равномерно (где $\bar{n} = (n_1, \dots, n_d)$, $\bar{k} = (k_1, \dots, k_d)$, $\bar{1} = (1, \dots, 1) \in \mathbf{N}^d$).

Ключевые слова и фразы: алгебра фон Неймана, усиленный закон больших чисел.

1. Introduction. The strong law of large numbers in von Neumann algebras was considered by some authors (see [1], [3]–[5]). If the state is tracial, this law has been obtained for successively independent sequence of measurable operators. If the state is not tracial some results are proved for orthogonal sequences, for sequences of uncorrelated elements and for uniformly bounded sequences. Note here that the same problem for multidimensional arrays is still not studied.

In this paper, by using the N. Etemadi's method (see [2]), we consider the strong law of large numbers for d -dimensional arrays of *pairwise independent* measurable operators.

*Hanoi University, Vietnam.

Let us begin with some definitions and notations. Throughout this paper, $\mathcal{A}$ will denote a von Neumann algebra with a faithful normal tracial state τ ; $\tilde{\mathcal{A}}$ will denote the $*$ -algebra of measurable operators in Segal's sense (see [7]). Each $x \in \tilde{\mathcal{A}}$ can be written as

$$x = x_1 + ix_2,$$

where x_1 and x_2 are self-adjoint elements in $\tilde{\mathcal{A}}$. Moreover, if $x \in \tilde{\mathcal{A}}$ is self-adjoint, then

$$x = x^+ - x^-,$$

where

$$x^+ = \frac{|x| + x}{2} \geq 0, \quad x^- = \frac{|x| - x}{2} \geq 0, \quad x^+, x^- \in \tilde{\mathcal{A}}.$$

For each self-adjoint element x in $\tilde{\mathcal{A}}$, we denote by $e_\Delta(x)$ the spectral projection of x corresponding to the Borel subset Δ of the real line $\mathbf{R}$.

Let $\mathbf{N}^d = (\bar{n} = (n_1, n_2, \dots, n_d), n_i \in \mathbf{N}; i = 1, \dots, d)$ (where $d \geq 1$ is a fixed integer). $\mathbf{N}^d$ is partially ordered by agreeing that

$$\bar{k} = (k_1, k_2, \dots, k_d) \leq \bar{m} = (m_1, m_2, \dots, m_d) \quad \text{if } k_i \leq m_i, \quad i = 1, \dots, d.$$

If $\bar{k} \in \mathbf{N}^d$, $\bar{m} \in \mathbf{N}^d$, then $(\bar{k} + \bar{m})$ and $\bar{k} \cdot \bar{m}$ denote the usual coordinatewise sums and products. For short, we will denote $|\bar{k}| = k_1 k_2 \dots k_d$ for $\bar{k} = (k_1, k_2, \dots, k_d)$.

Now, let $(x(\bar{n}); \bar{n} \in \mathbf{N}^d)$ be an array of elements in $\tilde{\mathcal{A}}$. The convergences of $(x(\bar{n}); \bar{n} \in \mathbf{N}^d) \in \tilde{\mathcal{A}}$ can be considered as $\bar{n} \rightarrow \infty$ or as $|\bar{n}| \rightarrow \infty$. Namely, we say that an array $(x(\bar{n}); \bar{n} \in \mathbf{N}^d)$ converges to $x \in \tilde{\mathcal{A}}$ as $|\bar{n}| \rightarrow \infty$ almost uniformly (bilaterally almost uniformly) if for each $\varepsilon > 0$ there exists a projection $p \in \mathcal{A}$ such that $\tau(p^\perp) < \varepsilon$, $(x - x(\bar{n}))p \in \mathcal{A}$ ($p(x - x(\bar{n}))p \in \mathcal{A}$) and $\|(x - x(\bar{n}))p\| \rightarrow 0$ ($\|p(x - x(\bar{n}))p\| \rightarrow 0$) as $\max_{1 \leq j \leq d} n_j \rightarrow \infty$ (where $\bar{n} = (n_1, n_2, \dots, n_d) \in \mathbf{N}^d$; $p^\perp = 1 - p$).

If the last condition is replaced by the following weaker one:

$$\|(x - x(\bar{n}))p\| \rightarrow 0 \quad \left(\|p(x - x(\bar{n}))p\| \rightarrow 0 \right) \quad \text{as } \min_{1 \leq j \leq d} n_j \rightarrow \infty$$

we will get the definition of the almost uniform (bilateral almost uniform) convergence as $\bar{n} \rightarrow \infty$.

The almost complete convergence of arrays as $|\bar{n}| \rightarrow \infty$ and as $\bar{n} \rightarrow \infty$ are the same. That is, an array $(x(\bar{n}); \bar{n} \in \mathbf{N}^d) \subset \tilde{\mathcal{A}}$ is said to be convergent almost completely to an element $x \in \tilde{\mathcal{A}}$ if, for each $\varepsilon > 0$, there exists an array $(q(\bar{n}); \bar{n} \in \mathbf{N}^d)$ of projections in $\mathcal{A}$ such that $\sum_{\bar{n} \in \mathbf{N}^d} \tau(q^\perp(\bar{n})) < \infty$, $(x(\bar{n}) - x)q(\bar{n}) \in \mathcal{A}$ and $\|(x(\bar{n}) - x)q(\bar{n})\| < \varepsilon$ for all $\bar{n} \in \mathbf{N}^d$. By the same method as for sequences we can prove that if $x(\bar{n}) \rightarrow x$ almost completely, then $x(\bar{n}) \rightarrow x$ almost uniformly (as $|\bar{n}| \rightarrow \infty$ and as $\bar{n} \rightarrow \infty$).

Various notions of independence for sequences of elements in $\mathcal{A}$ (or $\tilde{\mathcal{A}}$) were introduced in [1]. We shall use the following one only:

Let W_1 and W_2 be two von Neumann subalgebras of $\mathcal{A}$. W_1 is said to be independent of W_2 if for all $x \in W_1$, $y \in W_2$

$$\tau(x \cdot y) = \tau(x) \cdot \tau(y).$$

Two elements x, y in $\tilde{\mathcal{A}}$ are said to be independent if the von Neumann algebras $W(x)$ and $W(y)$ generated by x and y , respectively, are independent. An array $(x(\bar{n}); \bar{n} \in \mathbf{N}^d) \subset \tilde{\mathcal{A}}$ is said to be the array of pairwise independent elements in $\tilde{\mathcal{A}}$ if for all $\bar{m}, \bar{n} \in \mathbf{N}^d$, $\bar{m} \neq \bar{n}$; $x(\bar{m})$ and $x(\bar{n})$ are independent.

For further information we refer to [1], [7] and [8].

2. Lemmas.

Lemma 2.1. Let $(x(\bar{n}); \bar{n} \in \mathbf{N}^d)$ be an array of elements in $\tilde{\mathcal{A}}$; $(C(\bar{n}); \bar{n} \in \mathbf{N}^d)$ be an array of positive numbers, $y(\bar{n}) = x(\bar{n}) e_{[0, C(\bar{n})]}(|x(\bar{n})|)$. If

$$\sum_{\bar{n} \in \mathbf{N}^d} \tau(e_{[C(\bar{n}), \infty)}(|x(\bar{n})|)) < \infty, \quad (2.1)$$

$$\frac{1}{|\bar{n}|} \sum_{\bar{k} \leq \bar{n}} y(\bar{k}) \longrightarrow x \in \tilde{\mathcal{A}} \quad \text{a.u. (almost uniformly)} \quad \text{as } \bar{n} \rightarrow \infty, \quad (2.2)$$

then

$$\frac{1}{|\bar{n}|} \sum_{\bar{k} \leq \bar{n}} x(\bar{k}) \longrightarrow x \in \tilde{\mathcal{A}} \quad \text{a.u. as } \bar{n} \rightarrow \infty. \quad (2.3)$$

P r o o f. Let $\varepsilon > 0$ be given. By (2.1) we can find $\bar{n}_0 \in \mathbf{N}^d$ such that

$$\sum_{\bar{n} \not\leq \bar{n}_0} \tau(e_{[C(\bar{n}), \infty)}(|x(\bar{n})|)) < \frac{\varepsilon}{3}.$$

By the measurableness of elements $x(\bar{k})$ ($\bar{k} \leq \bar{n}_0$), there exists a projection $p_1 \in \mathcal{A}$ such that $\tau(p_1^\perp) < \varepsilon/3$ and $\sum_{\bar{k} \leq \bar{n}_0} x(\bar{k}) p_1 \in \mathcal{A}$.

By (2.2) and the measurableness of x we can find a projection $p_2 \in \mathcal{A}$ and $\bar{n}_1 \in \mathbf{N}^d$ such that $\tau(p_2^\perp) < \varepsilon/3$, $y(\bar{k}) p_2 \in \mathcal{A}$ ($\forall \bar{k} \in \mathbf{N}^d$) and

$$\left\| \left(\frac{1}{|\bar{n}|} \sum_{\bar{k} \leq \bar{n}} y(\bar{k}) - x \right) p_2 \right\| < \frac{\varepsilon}{3} \quad \text{for } \bar{n} \geq \bar{n}_1.$$

Evidently, there exists $\bar{n}_2 \in \mathbf{N}^d$ such that

$$\left\| \frac{1}{|\bar{n}|} \sum_{\bar{k} \leq \bar{n}_0} x(\bar{k}) p_1 \right\| < \frac{\varepsilon}{3}; \quad \left\| \frac{1}{|\bar{n}|} \sum_{\bar{k} \leq \bar{n}_0} y(\bar{k}) p_2 \right\| < \frac{\varepsilon}{3} \quad \text{for } \bar{n} \geq \bar{n}_2.$$

Now, putting

$$p = p_1 \wedge p_2 \wedge \left(\bigwedge_{\bar{n} \not\leq \bar{n}_0} e_{[0, C(\bar{n})]}(|x(\bar{n})|) \right)$$

we have

$$\tau(p^\perp) \leq \tau(p_1^\perp) + \tau(p_2^\perp) + \sum_{\bar{n} \not\leq \bar{n}_0} \tau(e_{[C(\bar{n}), \infty)}(|x(\bar{n})|)) < \varepsilon$$

and

$$\begin{aligned} & \left\| \left(\frac{1}{|\bar{n}|} \sum_{\bar{k} \leq \bar{n}} x(\bar{k}) - x \right) p \right\| \leq \left\| \frac{1}{|\bar{n}|} \sum_{\bar{k} \leq \bar{n}_0} x(\bar{k}) p_1 \right\| \\ & \quad + \left\| \frac{1}{|\bar{n}|} \sum_{\bar{n}_0 \not\leq \bar{k} \leq \bar{n}} (x(\bar{k}) - y(\bar{k})) p \right\| + \left\| \left(\frac{1}{|\bar{n}|} \sum_{\bar{n}_0 \not\leq \bar{k} \leq \bar{n}} y(\bar{k}) - x \right) p \right\| \\ & \leq \left\| \frac{1}{|\bar{n}|} \sum_{\bar{k} \leq \bar{n}_0} x(\bar{k}) p_1 \right\| + \left\| \frac{1}{|\bar{n}|} \sum_{\bar{n}_0 \not\leq \bar{k} \leq \bar{n}} (x(\bar{k}) - y(\bar{k})) e_{[0, C(\bar{k})]}(|x(\bar{k})|) \right\| \\ & \quad + \left\| \left(\frac{1}{|\bar{n}|} \sum_{\bar{k} \leq \bar{n}} y(\bar{k}) - x \right) p \right\| + \left\| \frac{1}{|\bar{n}|} \sum_{\bar{k} \leq \bar{n}_0} y(\bar{k}) p_2 \right\| < \frac{\varepsilon}{3} + 0 + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

for all $\bar{n}$ such that $\bar{n} \geq \bar{n}_0$, $\bar{n} \geq \bar{n}_1$, $\bar{n} \geq \bar{n}_2$; which completes the proof.

Let $x \in \tilde{\mathcal{A}}$, $y \in \tilde{\mathcal{A}}$. If $(x - y)$ is a positive operator then we say that x is larger than y and write $x \geq y$. The following lemma is analogue to the well-known result in classical analysis.

Lemma 2.2. *Let $x(\bar{n})$ ($\bar{n} \in \mathbb{N}^d$) and x be self-adjoint elements in $\mathcal{A}$. The following statements are equivalent:*

$$x(\bar{n}) \rightarrow x \quad \text{in norm} \quad \text{as } \bar{n} \rightarrow \infty, \quad (2.4)$$

for each $\varepsilon > 0$, there exists $\bar{n}_0 \in \mathbb{N}^d$ such that

$$x - \varepsilon \cdot 1 \leq x(\bar{n}) \leq x + \varepsilon \cdot 1 \quad \text{for all } \bar{n} \geq \bar{n}_0 \quad (2.5)$$

(where 1 is an identity operator).

P r o o f. (2.4) $\Rightarrow$ (2.5). Assume that $\mathcal{A}$ acts in a Hilbert space $\mathcal{H}$. We have

$$\|x(\bar{n}) - x\| = \sup \left\{ \left| \left((x(\bar{n}) - x)h, h \right) \right|; \|h\| \leq 1 \right\}.$$

If (2.4) is true then, for a fixed $\varepsilon > 0$, there exists $\bar{n}_0 \in \mathbb{N}^d$ such that $\|h\| \leq 1$ for all $h \in \mathcal{H}$ and for all $\bar{n} \in \mathbb{N}^d$, $\bar{n} \geq \bar{n}_0$

$$\left| \left((x(\bar{n}) - x)h, h \right) \right| < \varepsilon.$$

Thus, for all $h \in \mathcal{H}$, $h \neq 0$, we get

$$\begin{aligned} \left((x - \varepsilon \cdot 1)h, h \right) &= (xh, h) - \varepsilon(h, h) = \left[\left(x \frac{h}{\|h\|}, \frac{h}{\|h\|} \right) - \varepsilon \right] \|h\|^2 \\ &\leq \left(x(\bar{n}) \frac{h}{\|h\|}, \frac{h}{\|h\|} \right) \|h\|^2 \leq \left[\left(x \frac{h}{\|h\|}, \frac{h}{\|h\|} \right) + \varepsilon \right] \|h\|^2 \\ &= (xh, h) + \varepsilon(h, h) = \left((x + \varepsilon \cdot 1)h, h \right), \quad \text{for all } \bar{n} \geq \bar{n}_0, \end{aligned}$$

which yields (2.5).

(2.5) $\Rightarrow$ (2.4) is trivial.

For each $\alpha > 0$ and $\bar{n} = (n_1, n_2, \dots, n_d) \in \mathbb{N}^d$ we put $k_{n_i} = [\alpha^{n_i}]$ and $k(\bar{n}) = (k_{n_1}, k_{n_2}, \dots, k_{n_d})$.

Lemma 2.3. *If $\alpha > 1$, then*

$$\frac{\alpha^d}{|k(\bar{n} + \bar{1})|} \geq \frac{1}{|k(\bar{n}) + \bar{1}|} \quad \text{for all } \bar{n} \in \mathbb{N}^d, \bar{n} \geq (2, 2, \dots, 2), \quad (2.6)$$

$$k(\bar{n}) + \bar{1} \leq k(\bar{n} + \bar{1}) \quad \text{for all } \bar{n} \text{ large enough.} \quad (2.7)$$

P r o o f. It suffices to show that

$$\frac{\alpha}{[\alpha^{n+1}]} \geq \frac{1}{[\alpha^n] + 1} \quad \text{for all } n \in \mathbb{N}, n \geq 2, \quad (2.6')$$

and

$$[\alpha^n] + 1 \leq [\alpha^{n+1}] \quad \text{for all } n \text{ large enough.} \quad (2.7')$$

We have

$$\alpha^p = [\alpha^p] + \beta_p \quad \text{with } 0 \leq \beta_p < 1 \quad \text{for all } p \geq 2,$$

$$\alpha = 1 + \beta_1 \quad \text{with } \beta_1 > 0.$$

Thus,

$$[\alpha^{n+1}] = [\alpha([\alpha^n] + \beta_n)] \leq [\alpha([\alpha^n] + 1)] \leq \alpha([\alpha^n] + 1)$$

which yields (2.6').

On the other hand,

$$[\alpha^{n+1}] = [(1 + \beta_1)([\alpha^n] + \beta_n)] \geq [\alpha^n + \beta_1[\alpha^n]] \geq [\alpha^n] + 1$$

if n large enough. This means that (2.7') holds.

3. The strong law of large numbers for sequences. We are now going to prove an extension of the celebrated Kolmogorov's strong law of large numbers. Note that an analogous result was given by Batty in the case of successively independent sequence (see [1]).

Theorem 3.1. *Let (x_n) be a sequence of self-adjoint pairwise independent identically distributed operators in $\tilde{\mathcal{A}}$. If $\tau(|x_1|) < \infty$ then*

$$n^{-1} \sum_{k=1}^n x_k \rightarrow \tau(x_1) \quad \text{bilaterally almost uniformly (b.a.u.)} \quad \text{as } n \rightarrow \infty. \quad (3.1)$$

P r o o f. Since (x_n^+) and (x_n^-) satisfy the assumption of the theorem, without loss of generality we can assume that $x_n \geq 0$, for all $n = 1, 2, \dots$.

Put

$$S_n = \sum_{i=1}^n x_i, \quad \bar{x}_i = x_i e_{[0,i)}(x_i), \quad \bar{S}_n = \sum_{i=1}^n \bar{x}_i.$$

Let $\alpha > 1$, $k_n = [\alpha^n]$ and $\varepsilon > 0$ be given. Using Chebyshev's inequality [4] we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \tau \left(e_{[\varepsilon, \infty)} \left(\left| \frac{\bar{S}_{k_n} - \tau(\bar{S}_{k_n})}{k_n} \right| \right) \right) &\leq C \sum_{n=1}^{\infty} k_n^{-2} \tau(|\bar{S}_{k_n}|^2) \\ &= C \sum_{n=1}^{\infty} k_n^{-2} \sum_{i=1}^{k_n} \tau(|\bar{x}_i - \tau(\bar{x}_i)|^2) \leq C \sum_{n=1}^{\infty} k_n^{-2} \sum_{i=1}^{k_n} \tau(|\bar{x}_i|^2) \\ &= C \sum_{i=1}^{\infty} i^{-2} \tau(x_1^2 e_{[0,i)}(x_1)) = C \sum_{i=1}^{\infty} i^{-2} \sum_{k=1}^i \tau(x_1^2 e_{[k-1,k)}(x_1)) \\ &= C \sum_{k=1}^{\infty} \tau(k^{-1} x_1^2 e_{[k-1,k)}(x_1)) \leq C \sum_{i=1}^{\infty} \tau(x_1 e_{[0,i)}(x_1)) = C \tau(x_1) < \infty, \end{aligned} \quad (3.2)$$

where C is an unimportant positive constant which is allowed to change.

Put

$$q_n = e_{[0,\varepsilon)} \left(\left| \frac{\bar{S}_{k_n} - \tau(\bar{S}_{k_n})}{k_n} \right| \right). \quad (3.3)$$

It is easy to see that

$$\left\| \frac{\bar{S}_{k_n} - \tau(\bar{S}_{k_n})}{k_n} q_n \right\| < \varepsilon$$

which implies

$$\frac{\bar{S}_{k_n} - \tau(\bar{S}_{k_n})}{k_n} \rightarrow 0 \quad \text{almost completely,}$$

this yields

$$\frac{\bar{S}_{k_n} - \tau(\bar{S}_{k_n})}{k_n} \rightarrow 0 \quad \text{a.u.} \quad \text{as } n \rightarrow \infty.$$

We also have

$$\begin{aligned}\tau(x_1) &= \lim_{n \rightarrow \infty} \tau(x_1 e_{[0,n]}(x_1)) = \lim_{n \rightarrow \infty} \tau(\bar{x}_n) \\ &= \lim_{n \rightarrow \infty} \frac{1}{k_n} \sum_{i=1}^{k_n} \tau(\bar{x}_i) = \lim_{n \rightarrow \infty} \frac{1}{k_n} \tau(\bar{S}_{k_n}),\end{aligned}\quad (3.4)$$

so

$$\frac{\bar{S}_{k_n}}{k_n} \rightarrow \tau(x_1) \quad \text{a.u.} \quad \text{as } n \rightarrow \infty.$$

On the other hand,

$$\begin{aligned}\sum_{n=1}^{\infty} \tau(e_{[n,\infty)}(x_n)) &= \sum_{n=1}^{\infty} \sum_{i=n}^{\infty} \tau(e_{[i,i+1)}(x_n)) = \sum_{i=1}^{\infty} i \tau(e_{[i,i+1)}(x_n)) \\ &\leq \sum_{i=1}^{\infty} \tau(x_1 e_{[i,i+1)}(x_1)) \leq \tau(x_1) < \infty.\end{aligned}\quad (3.5)$$

Thus, by Lemma 2.1, we obtain

$$\frac{S_{k_n}}{k_n} \rightarrow \tau(x_1) \quad \text{a.u.} \quad \text{as } n \rightarrow \infty.$$

It implies

$$\frac{S_{k_n}}{k_n} \rightarrow \tau(x_1) \quad \text{bilaterally almost uniformly (b.a.u.),}$$

also

$$\frac{S_{k_n}}{k_n + 1} \rightarrow \tau(x_1) \quad \text{b.a.u.}$$

Consequently, for fixed $\delta, \eta > 0$, there exists a projection q and an integer $r_0 \geq 1$ such that $\tau(q^\perp) < \delta$ and

$$\begin{aligned}\alpha q \tau(x_1) q + \frac{\eta}{2} \cdot 1 &\geq \alpha q \frac{S_{k(r)}}{k(r)} q, \\ \alpha^{-1} q \tau(x_1) q - \frac{\eta}{2} \cdot 1 &\leq \alpha^{-1} q \frac{S_{k(r)}}{k(r) + 1} q\end{aligned}$$

for all $r \geq r_0$, we put $k(r) = k_r$.

Choose $\alpha > 1$ and $n_0 \in \mathbb{N}$ such that

$$(\alpha - 1) \tau(x_1) < \frac{\eta}{2} \quad (3.6)$$

and

$$k(r_0) + 1 \leq n_0 \leq k(r_0 + 1)$$

(where $k(r) = [\alpha^r]$ with a chosen $\alpha > 1$ satisfying (3.6)).

Now let $n > n_0$, by (2.7) there exists an integer $r \geq r_0$ (r dependent on n) such that

$$k(r) + 1 \leq n \leq k(r + 1).$$

By the monotonicity of S_n and (2.6) we can conclude that

$$\frac{\alpha S_{k(r+1)}}{k(r) + 1} \geq \frac{S_{k(r+1)}}{k(r) + 1} \geq \frac{S_n}{n} \geq \frac{S_{k(r)}}{k(r) + 1} \geq \frac{S_{k(r)}}{\alpha(k(r) + 1)}.$$

It follows that

$$\begin{aligned} q\tau(x_1)q + \eta \cdot 1 &\geq \alpha q\tau(x_1)q + \frac{\eta}{2} \cdot 1 \geq \frac{\alpha q S_{k(r+1)} q}{k(r) + 1}, \\ &\geq \frac{q S_n q}{n} \geq \frac{q S_{k(r)} q}{\alpha(k(r) + 1)} \geq \alpha^{-1} q\tau(x_1)q - \frac{\eta}{2} \cdot 1 \geq q\tau(x_1)q - \eta \cdot 1. \end{aligned}$$

Using Lemma 2.2 we get

$$\frac{q S_n q}{n} \rightarrow q\tau(x_1)q \quad \text{in norm.}$$

This means that S_n/n converges to $\tau(x_1)$ bilaterally almost uniformly; which completes the proof.

4. The strong law of large numbers for d -dimensional arrays. In this section we will use the following result.

Lemma 4.1 [5]. *If for all $x > 0$, $f(x)$ is defined by*

$$f(x) = \text{card} \left(\bar{n} \in \mathbf{N}^d, |\bar{n}| \leq x \right),$$

then

$$f(x) = O\left(x(\log^+ x)^{d-1}\right).$$

Theorem 4.1. *Let $(x(\bar{n}), \bar{n} \in \mathbf{N}^d)$ be an array of self-adjoint pairwise independent identically distributed elements in $\tilde{\mathcal{A}}$. If*

$$\tau\left(|x(\bar{l})| \left(\log^+ |x(\bar{l})|\right)^{d-1}\right) < \infty$$

then

$$\frac{1}{|\bar{n}|} \sum_{\bar{k} \leq \bar{n}} x(\bar{k}) \rightarrow \tau(x(\bar{l})) \quad \text{b.a.u. as } |\bar{n}| \rightarrow \infty. \quad (4.1)$$

P r o o f. At first we will prove that

$$\frac{1}{|\bar{n}|} \sum_{\bar{k} \leq \bar{n}} x(\bar{k}) \rightarrow \tau(x(\bar{l})) \quad \text{b.a.u. as } \bar{n} \rightarrow \infty. \quad (4.2)$$

We will follow the proof of Theorem 3.1. Without loss of generality we assume that $x(\bar{n}) \geq 0$ for all $\bar{n} \in \mathbf{N}^d$. Put

$$S(\bar{n}) = \sum_{\bar{k} \leq \bar{n}} x(\bar{k}), \quad \bar{x}(\bar{n}) = x(\bar{n}) e_{[0, |\bar{n}|)}(x(\bar{n})), \quad \bar{S}(\bar{n}) = \sum_{\bar{k} \leq \bar{n}} \bar{x}(\bar{k}).$$

Moreover, for $\alpha > 1$ and $\bar{n} = (n_1, \dots, n_d) \in \mathbf{N}^d$, put

$$k(n_i) = [\alpha^{n_i}]; \quad k(\bar{n}) = (k(n_1), \dots, k(n_d)).$$

Let $\varepsilon > 0$ be given, then (3.2) adapted to this case becomes

$$\begin{aligned} \sum_{\bar{n} \in \mathbf{N}^d} \tau\left(e_{[\varepsilon, \infty)}\left(\left|\frac{\bar{S}(k(\bar{n})) - \tau(\bar{S}(k(\bar{n})))}{|k(\bar{n})|}\right|\right)\right) &\leq \sum_{\bar{n} \in \mathbf{N}^d} \frac{\tau(|\bar{x}(\bar{n})|^2)}{|\bar{n}|^2} \\ &= C \sum_{i=1}^{\infty} d(i) i^{-2} \sum_{k=1}^i \tau\left(x(\bar{l})^2 e_{[k-1, k)}(x(\bar{l}))\right) \\ &= C \sum_{k=1}^{\infty} \tau\left(x(\bar{l})^2 e_{[k-1, k)}(x(\bar{l}))\right) \cdot \sum i^{-2} d(i), \end{aligned}$$

where $d(i) = \text{card}(\bar{n} \in \mathbf{N}^d, |\bar{n}| = i)$.

Put

$$f(k) = \sum_{i=1}^k d(i) = \text{card}(\bar{n} \in \mathbf{N}^d, |\bar{n}| \leq k).$$

By the Abel's transformation and Lemma 4.1 we get

$$\begin{aligned} \sum_{i=1}^{\infty} i^{-2} d(i) &= -k^{-2} f(k-1) + \sum_{i=k}^{\infty} f(k) (i^{-2} - (i+1)^{-2}) \\ &\leq C \sum_{i=k}^{\infty} i^{-3} f(i) \leq C \sum_{i=k}^{\infty} i^{-3} (i(\log^+ i)^{d-1}) \leq C k^{-1} (\log^+ k)^{d-1}. \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{\bar{n} \in \mathbf{N}^d} \tau \left(e_{[\varepsilon, \infty)} \left(\left| \frac{\bar{S}(k(\bar{n})) - \tau(\bar{S}(k(\bar{n})))}{|k(\bar{n})|} \right| \right) \right) \\ \leq C \sum_{k=1}^{\infty} \tau \left(k^{-1} (\log^+ k)^{d-1} x(\bar{1})^2 e_{[k-1, k)}(x(\bar{1})) \right) \\ \leq C \sum_{k=1}^{\infty} \tau \left(x(\bar{1}) (\log^+ x(\bar{1}))^{d-1} e_{[k-1, k)}(x(\bar{1})) \right) \\ \leq C \tau \left(x(\bar{1}) (\log^+ x(\bar{1}))^{d-1} \right) < \infty. \end{aligned} \quad (4.3)$$

The analogue of (3.5) is

$$\begin{aligned} \sum_{\bar{n} \in \mathbf{N}^d} \tau \left(e_{[|\bar{n}|, \infty)}(x(\bar{n})) \right) &= \sum_{i=1}^{\infty} d(i) \tau \left(e_{[i, \infty)}(x(\bar{1})) \right) \\ &= \sum_{i=1}^{\infty} d(i) \sum_{k=1}^{\infty} \tau \left(e_{[k, k+1)}(x(\bar{1})) \right) = \sum_{k=1}^{\infty} \left(\sum_{i=1}^k d(i) \right) \tau \left(e_{[k, k+1)}(x(\bar{1})) \right) \\ &\leq C \sum_{k=1}^{\infty} \tau \left(k (\log^+ k)^{d-1} e_{[k, k+1)}(x(\bar{1})) \right) \leq C \tau \left(x(\bar{1}) (\log^+ x(\bar{1}))^{d-1} \right) < \infty. \end{aligned} \quad (4.4)$$

We also have

$$\begin{aligned} \left| |\bar{n}|^{-1} \sum_{\bar{k} \leq \bar{n}} \tau \left(\bar{x}(\bar{k}) \right) - \tau \left(x(\bar{1}) \right) \right| &= |\bar{n}|^{-1} \left| \sum_{\bar{k} \leq \bar{n}} \tau \left(\bar{x}(\bar{k}) - x(\bar{1}) \right) \right| \\ &= |\bar{n}|^{-1} \sum_{\bar{k} \leq \bar{n}} \tau \left(x(\bar{1}) e_{[|\bar{k}|, \infty)}(x(\bar{1})) \right) = |\bar{n}|^{-1} \sum_{i \leq |\bar{n}|} d(i) \tau \left(x(\bar{1}) e_{(i, \infty)}(x(\bar{1})) \right) \\ &= |\bar{n}|^{-1} \left(\sum_{i \leq |\bar{n}|} f(i) \tau \left(x(\bar{1}) e_{[i, i+1)}(x(\bar{1})) \right) + f(|\bar{n}|) \sum_{i > |\bar{n}|} f(i) \tau \left(x(\bar{1}) e_{[i, i+1)}(x(\bar{1})) \right) \right) \\ &= S_1 + S_2, \end{aligned}$$

where

$$S_1 = |\bar{n}|^{-1} \left(\sum_{i \leq |\bar{n}|} f(i) \tau \left(x(\bar{1}) e_{[i, i+1)}(x(\bar{1})) \right) \right) \longrightarrow 0 \quad \text{as } |\bar{n}| \rightarrow \infty,$$

because

$$\sum_{i=1}^{\infty} i^{-1} f(i) \tau(x(\bar{i}) e_{[i, i+1)}(x(\bar{i}))) \leq \tau(x(\bar{i}) (\log^+ x(\bar{i}))^{d-1}) < \infty.$$

Moreover

$$\begin{aligned} S_2 &= |\bar{n}|^{-1} f(|\bar{n}|) \sum_{i > |\bar{n}|} f(i) \tau(x(\bar{i}) e_{[i, i+1)}(x(\bar{i}))) \\ &\leq C (\log^+ |\bar{n}|)^{d-1} \sum_{i \geq |\bar{n}|} \tau(x(\bar{i}) e_{[i, i+1)}(x(\bar{i}))) \\ &\leq C \sum_{i \geq |\bar{n}|} \tau(x(\bar{i}) (\log^+ x(\bar{i}))^{d-1} e_{[i, i+1)}(x(\bar{i}))) \rightarrow 0 \end{aligned}$$

as $|\bar{n}| \rightarrow \infty$.

Thus,

$$|\bar{n}|^{-1} \sum_{\bar{k} \leq \bar{n}} \tau(x(\bar{k})) \rightarrow \tau(x(\bar{i})) \quad \text{as } |\bar{n}| \rightarrow \infty \quad (\text{and as } \bar{n} \rightarrow \infty). \quad (4.5)$$

The rest of the proof of (4.2) follows similarly.

We now strengthen (4.2). Let $d = 2$ and let $\varepsilon > 0$, $\delta > 0$ be given. By (4.2) there exist $\bar{n}_0 = (n_1^0, n_2^0)$ and a projection $e \in \mathcal{A}$ such that

$$\tau(e^\perp) < \frac{\varepsilon}{2}; \quad e(|n_1 n_2|^{-1} S_{n_1 n_2} - \tau(x_{1,1})) e \in \mathcal{A}$$

and

$$\|e(|n_1 n_2|^{-1} S_{n_1 n_2} - \tau(x_{1,1})) e\| < \delta$$

for all $(n_1, n_2) > (n_1^0, n_2^0)$.

On the other hand, on each of the lines $x = \ell$ and $y = j$ ($\ell = 1, \dots, n_1^0$, $j = 1, \dots, n_2^0$) the one-dimensional strong law of large numbers holds. Thus, there exist the projections p_ℓ and q_j ($\ell = 1, \dots, n_1^0$, $j = 1, \dots, n_2^0$) such that

$$\tau(p_\ell^\perp) \leq \frac{\varepsilon}{4n_1^0}; \quad \tau(q_j^\perp) \leq \frac{\varepsilon}{4n_2^0}; \quad p_\ell(i^{-1} \sum_{s=1}^i x_{\ell,s} - \tau(x_{1,1})) p_\ell \in \mathcal{A},$$

$$\left\| p_\ell \left(i^{-1} \sum_{s=1}^i x_{\ell,s} - \tau(x_{1,1}) \right) p_\ell \right\| < \delta \quad \text{for all } i > i(\ell)$$

and

$$q_j \left(k^{-1} \sum_{s=1}^k x_{s,j} - \tau(x_{1,1}) \right) q_j \in \mathcal{A};$$

$$\left\| q_j \left(k^{-1} \sum_{s=1}^k x_{s,j} - \tau(x_{1,1}) \right) q_j \right\| < \delta \quad \text{for all } k > k(j).$$

Put

$$p = e \bigwedge \left(\bigwedge_{1 \leq \ell \leq n_1^0} p_\ell \right) \bigwedge \left(\bigwedge_{1 \leq j \leq n_2^0} q_j \right),$$

$$N_1 = \max(n_1^0; k(1), \dots, k(n_2^0)), \quad N_2 = \max(n_2^0; i(1), \dots, i(n_1^0)).$$

Then

$$\tau(p^\perp) \leq \tau(e^\perp) + \sum_{\ell=1}^{n_1^0} \tau(p_\ell^\perp) + \sum_{j=1}^{n_2^0} \tau(q_j^\perp) < \varepsilon$$

and

$$\left\| p(|n_1 n_2|^{-1} S_{n_1 n_2} - \tau(x_{1,1})) p \right\| < \delta$$

for all $(n_1, n_2) \not\leq (N_1, N_2)$. That means (4.1) is true for $d = 2$.

Assume inductively that (4.1) holds for $(d-1)$ dimensions. Then by (4.2) there exist a projection $e \in \mathcal{A}$ and a tupe $\bar{n}_0 = (n_1^0, \dots, n_d^0) \in \mathbb{N}^d$ such that

$$\tau(e^\perp) < \varepsilon/2, \quad e(|\bar{n}|^{-1} S(\bar{n}) - \tau(x(\bar{1}))) e \in \mathcal{A}$$

and

$$\left\| e(|\bar{n}|^{-1} S(\bar{n}) - \tau(x(\bar{1}))) e \right\| < \delta$$

for all $\bar{n} \geq \bar{n}_0$.

On the other hand, by inductive assumption, on each of $(d-1)$ -dimensional hyperplanes defined by $n_i = j_i$ ($i = 1, \dots, d$, $j_i = 1, \dots, n_i^0$) the strong law of large numbers holds. Using the same technique as above we can conclude that there exist a projection $p \in \mathcal{A}$ and a tupe $\bar{N} = (N_1, \dots, N_d)$ such that

$$\tau(p^\perp) < \varepsilon, \quad p(|\bar{n}|^{-1} S(\bar{n}) - \tau(x(\bar{1}))) p \in \mathcal{A}$$

and

$$\left\| p(|\bar{n}|^{-1} S(\bar{n}) - \tau(x(\bar{1}))) p \right\| < \delta$$

for all $\bar{n} \not\leq \bar{N}$.

This means that (4.1) is true for d -dimensional arrays, which completes the proof.

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