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ASYMPTOTIC INFERENCE
FOR NEARLY UNSTABLE
MULTIDIMENSIONAL AR PROCESSES

В статье изучаются близкие к неустойчивым многомерные модели авторегрессии в случае, когда матрица коэффициентов имеет специальный вид. Доказана слабая сходимость последовательности подходящим образом нормированных оценок наименьших квадратов для матрицы коэффициентов. Показана естественная связь между моделью с дискретным временем и соответствующей ей моделью с непрерывным временем.

Ключевые слова и фразы: процессы авторегрессии с дискретным и непрерывным временем; модели, близкие к неустойчивым.

1. **Introduction.** Consider the d -dimensional autoregressive model

$$\begin{cases} X_k = QX_{k-1} + \varepsilon_k, & k = 1, 2, \dots, \\ X_0 = 0, \end{cases} \quad (1)$$

where the d -dimensional random column vector ε_k contains the (unobservable) random innovations (disturbances, noises) at time k , and the $(d \times d)$ -matrix Q is the unknown parameter of the model. The least-squares estimator (LSE) of Q based on the observations $X_1, \dots, X_n$ is given by

$$\hat{Q}_n = \left(\sum_{k=1}^n X_k X_{k-1}' \right) \left(\sum_{k=1}^n X_{k-1} X_{k-1}' \right)^{-1}. \quad (2)$$

Let $\rho(Q)$ denote the spectral radius of the matrix Q , i.e., the maximum of the absolute value of the eigenvalues of the matrix Q .

When $\rho(Q) < 1$, the model is said to be *asymptotically stationary*. Under the assumption that the ε_k 's are i.i.d. with $E\varepsilon_k = 0$, $E\varepsilon_k \varepsilon_k' = \Sigma$, the LSE of Q is asymptotically normal:

$$(\hat{Q}_n - Q) \left(\sum_{k=1}^n X_{k-1} X_{k-1}' \right)^{1/2} \xrightarrow{\mathcal{D}} \mathcal{N}_{d \times d}(0, I) \quad \text{as } n \rightarrow \infty, \quad (3)$$

where $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution and I is the unit matrix (see [16], [1]). By another normalization:

$$\sqrt{n}(\hat{Q}_n - Q) \xrightarrow{\mathcal{D}} \mathcal{N}_{d \times d}(0, F^{-1} \otimes \Sigma) \quad \text{as } n \rightarrow \infty,$$

where $F = \sum_{k=0}^{\infty} Q^k \Sigma (Q')^k$ is the covariance matrix of the stationary distribution (being the unique solution of the equation $\Sigma + QFQ' = F$).

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When $\rho(Q) = 1$, the model (1) is said to be *unstable*. It was shown by White [23] that in the case of the one-dimensional unstable AR(1) model $X_k = \beta X_{k-1} + \varepsilon_k$, $k \geq 1$, with $\beta = 1$, the variables $n(\hat{\beta}_n - \beta)$ converge in law to a random variable:

$$n(\hat{\beta}_n - \beta) \xrightarrow{\mathcal{D}} \frac{\int_0^1 W(t) dW(t)}{\int_0^1 W^2(t) dt}, \quad (4)$$

where $W(t)$, $t \geq 0$, is a standard Wiener process. In case of the one-dimensional unstable AR(p) model Chan and Wei [6] proved that with suitable normalizing matrices δ_n the sequence $\delta_n^{-1}(\hat{\beta}_n - \beta)$ converges in law and gave the representation of the limit distribution. This representation involves multiple stochastic integrals with respect to Wiener processes and has a very complicated form.

Fountis and Dickey [8] considered multidimensional unstable models with a matrix Q having one eigenvalue equal to 1 and the rest less than 1 in magnitude and obtained the limit distribution of the appropriately normalized LSE of the largest eigenvalue. The general multidimensional unstable models are studied in [20] and in [22]. Arató [3] has drawn the attention to the connection between discrete and continuous time multidimensional unstable models. This relationship was pointed out by Kormos [10], [11], and by Kormos and Piterbarg [13] in connection with hypothesis testing of nonstationarity for a (real-valued) Gaussian one-dimensional autoregressive time series.

The result (4) led to the study of the following so-called nearly nonstationary one-dimensional AR(1) model (better to call it nearly unstable):

$$\begin{cases} X_{n,k} = \beta_n X_{n,k-1} + \varepsilon_{n,k}, & k = 1, 2, \dots, n, \\ X_{n,0} = 0, \end{cases} \quad (5)$$

where $\beta_n = 1 + h/n$. It was shown by Chan and Wei [4], [5] that

$$\left(\sum_{k=1}^n X_{n,k-1}^2 \right)^{1/2} (\hat{\beta}_n - \beta_n) \xrightarrow{\mathcal{D}} \frac{\int_0^1 Y(t) dW(t)}{(\int_0^1 Y^2(t) dt)^{1/2}}, \quad (6)$$

where $Y(t)$, $t \in [0, 1]$, is an Ornstein-Uhlenbeck process defined as the solution of the stochastic differential equation

$$dY(t) = hY(t) dt + dW(t), \quad Y(0) = 0.$$

By another normalization

$$n(\hat{\beta}_n - \beta_n) \xrightarrow{\mathcal{D}} \frac{\int_0^1 Y(t) dW(t)}{\int_0^1 Y^2(t) dt}, \quad (7)$$

see, for example, [18], [9], [7]. (The above model is called also near integrated and is applied often in economic theory; see [18].)

Recently, Jeganathan [9] has considered nearly nonstationary one-dimensional AR(p) models, i.e., AR(p) models near to an unstable model. He proved that the appropriately normalized LSE of the coefficients converges in law and gave a very complicated representation for the limiting distribution in terms of multiple stochastic integrals with respect to Wiener processes. In [17] a simpler form and explanation is given for the asymptotic behaviour of the least-squares estimators in the nearly nonstationary AR(p) model and the relation between discrete and continuous time models is clarified.

Nearly unstable multidimensional AR processes are generated according to the scheme

$$\begin{cases} X_{n,k} = Q_n X_{n,k-1} + \varepsilon_{n,k}, & k = 1, 2, \dots, n, \\ X_{n,0} = 0, \end{cases} \quad (8)$$

where $\{\varepsilon_{n,k}\}$ is an array of d -dimensional random vectors and Q_n , $n \geq 1$, is a sequence of $(d \times d)$ -matrices such that $Q_n \rightarrow Q$, where Q is a matrix with $\rho(Q) = 1$. Phillips [18] treated the case, where $Q_n = e^{A/n}$, $n \geq 1$, where A is a fixed $(d \times d)$ -matrix. Kormos and Pap [12] investigated the case, where $Q_n = e^{(\gamma I + A)/n}$, $n \geq 1$, where $\gamma \in \mathbb{R}$ and A is a skew-symmetric matrix and obtained the weak convergence of the log-likelihood ratio under the assumption that ε_k 's are i.i.d. normal. The limit distribution turned out to be the Radon-Nikodym derivative of some continuous time multidimensional Ornstein-Uhlenbeck process. This indicates a natural connection between discrete and continuous time models. This indicates a natural connection between discrete and continuous time models. Stockmarr and Jacobsen [21] studied essentially the case, where $Q_n = I + n^{-1}A$. We remark that these authors considered in fact an additional matrix in their model. However, this situation will also be covered in our model below, since one can easily include this extra matrix in the noise process.

The aim of the present paper is to investigate nearly unstable models (8), where the coefficient matrices have some special form which includes the situations considered by the authors mentioned above. First we give a necessary and sufficient condition on the innovations $\{\varepsilon_{n,k}\}$ for convergence (in the Skorokhod space $D([0, 1] \rightarrow \mathbb{R}^d)$) of the rotated sequence

$$\frac{1}{\sqrt{n}} e^{-[nt]B} X_{n,[nt]}, \quad t \in [0, 1], \quad n = 1, 2, \dots,$$

to a d -dimensional continuous time AR(1) process, where B is a suitable $(d \times d)$ -matrix. Then we shall prove weak convergence of the sequence of appropriately normalized LSE's and finally we give a natural connection between the discrete and the related continuous time models. It should be remarked that our results essentially include the earlier work of the above mentioned authors and our method is completely different.

2. Convergence results. For every $n = 1, 2, \dots$ consider the d -dimensional AR(1) model

$$\begin{cases} X_{n,k} = Q_n e^B X_{n,k-1} + \varepsilon_{n,k}, & k = 1, 2, \dots, n, \\ X_{n,0} = 0, \end{cases} \quad (9)$$

where $\{\varepsilon_{n,k}\}$, is an array of d -dimensional random vectors and $Q_n = e^{A_n/n}$, $n \geq 1$, are $(d \times d)$ -matrices such that $A_n \rightarrow A$, B is a known skew-symmetric $(d \times d)$ -matrix, and $A_n B = B A_n$, $n \geq 1$. (We remark that for each orthogonal matrix C there exists a unique matrix B such that $C = e^B$ and B is a skew-symmetric matrix, i.e., $B' = -B$.) The model (9) is nearly unstable, since $Q_n e^B \rightarrow e^B$ and e^B is an orthogonal matrix.

Since the matrices e^{A_n} and e^B also commute, the rotated observations

$$Z_{n,k} = e^{-kB} X_{n,k}, \quad k = 0, 1, \dots, n; \quad n = 1, 2, \dots,$$

form again a nearly unstable d -dimensional model

$$\begin{cases} Z_{n,k} = e^{A_n/n} Z_{n,k-1} + \zeta_{n,k}, & k = 1, 2, \dots, n, \\ Z_{n,0} = 0, \end{cases} \quad (10)$$

where $\{\zeta_{n,k}\} = \{e^{-kB} \varepsilon_{n,k}\}$ is the rotated array of the random disturbances.

The random step functions

$$\begin{aligned} Y_n(t) &= \frac{1}{\sqrt{n}} e^{-[nt]B} X_{n,[nt]}, & t \in [0, 1], \\ M_n(t) &= \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} e^{-kB} \varepsilon_{n,k}, & t \in [0, 1], \end{aligned}$$

can be considered as random elements in the Skorokhod space $D([0, 1] \rightarrow \mathbf{R}^d)$. We denote by $C([0, 1] \rightarrow \mathbf{R}^d)$ the space of $\mathbf{R}^d$ -valued continuous functions endowed with the supremum norm. The supremum norm and the Skorokhod metric on the space $D([0, 1] \rightarrow \mathbf{R}^d)$ will be denoted by $\|\cdot\|_\infty$ and ρ , respectively. We shall need the following simple lemma, which is based on the continuous mapping theorem and the Skorokhod-construction.

Lemma 1. Let $\Phi, \Phi_n: D([0, 1] \rightarrow \mathbf{R}^k) \rightarrow D([0, 1] \rightarrow \mathbf{R}^l)$, $n = 1, 2, \dots$, be measurable mappings such that $\|\Phi_n(x_n) - \Phi(x)\|_\infty \rightarrow 0$ for all $x, x_n \in D([0, 1] \rightarrow \mathbf{R}^k)$ with $\|x_n - x\|_\infty \rightarrow 0$. Let Z, Z_n , $n = 1, 2, \dots$, be stochastic processes with values in $D([0, 1] \rightarrow \mathbf{R}^k)$ such that $Z_n \xrightarrow{\mathcal{D}} Z$ in $D([0, 1] \rightarrow \mathbf{R}^k)$ and almost all trajectories of Z are continuous. Then $\Phi_n(Z_n) \xrightarrow{\mathcal{D}} \Phi(Z)$ in $D([0, 1] \rightarrow \mathbf{R}^l)$.

P r o o f. Due to the Skorokhod-construction we can find processes $\tilde{Z}_n$ and a process $\tilde{Z}$, such that $\tilde{Z}_n \stackrel{\mathcal{D}}{=} Z_n$, $\tilde{Z} \stackrel{\mathcal{D}}{=} Z$ and $\rho(\tilde{Z}_n, \tilde{Z}) \rightarrow 0$ a.s. Using the fact that $\tilde{Z}$ has continuous trajectories a.s., we conclude that $\|\tilde{Z}_n - \tilde{Z}\|_\infty \rightarrow 0$ a.s. Thus we have $\|\Phi_n(\tilde{Z}_n) - \Phi(\tilde{Z})\|_\infty \rightarrow 0$ a.s. and hence

$$\Phi_n(\tilde{Z}_n) \xrightarrow{\mathcal{D}} \Phi(\tilde{Z}) \text{ in } D([0, 1] \rightarrow \mathbf{R}^l).$$

The last relation implies the desired result. Lemma 1 is proved.

Let $M(t)$, $t \in [0, 1]$, be a continuous semimartingale with values in $\mathbf{R}^d$. Consider the continuous time autoregressive process $Y(t)$, $t \in [0, 1]$, defined as the solution of the stochastic differential equation

$$dY(t) = AY(t)dt + dM(t), \quad Y(0) = 0. \quad (11)$$

Our first theorem gives a necessary and sufficient condition for the weak convergence $Y_n \xrightarrow{\mathcal{D}} Y$ in $D([0, 1] \rightarrow \mathbf{R}^d)$.

Theorem 1. The following statements are equivalent:

- (i) $M_n \xrightarrow{\mathcal{D}} M$ in $D([0, 1] \rightarrow \mathbf{R}^d)$;
- (ii) $Y_n \xrightarrow{\mathcal{D}} Y$ in $D([0, 1] \rightarrow \mathbf{R}^d)$;
- (iii) $(M_n, Y_n) \xrightarrow{\mathcal{D}} (M, Y)$ in $D([0, 1] \rightarrow \mathbf{R}^{2d})$.

R e m a r k. The statement (i) is in other words: the functional central limit theorem holds for the rotated triangular array $\{e^{-kB}\varepsilon_{n,k}\}_{k=1, \dots, n; n \geq 1}$.

P r o o f. (i) $\implies$ (iii). Using Itô's formula we obtain

$$Y(t) = M(t) + A \int_0^t e^{(t-s)A} M(s) ds, \quad t \in [0, 1].$$

A similar formula holds for the random step functions $Y_n(t)$, $t \in [0, 1]$:

$$Y_n(t) = M_n(t) + A_n \int_0^{[nt]/n} e^{\gamma_n([nt]/n - s)} M_n(s) ds, \quad (12)$$

since (10) implies

$$Y_n(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} e^{([nt]-k)A_n/n} \zeta_{n,k},$$

and

$$\begin{aligned} A_n \int_0^{[nt]/n} e^{-sA_n} M_n(s) ds &= A_n \sum_{j=1}^{[nt]} \int_{(j-1)/n}^{j/n} e^{-sA_n} ds M_n\left(\frac{j-1}{n}\right) \\ &= \sum_{j=1}^{[nt]} \left(e^{-(j-1)A_n/n} - e^{-jA_n/n}\right) M_n\left(\frac{j-1}{n}\right) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\sqrt{n}} \sum_{j=1}^{[nt]} \sum_{k=1}^{j-1} \left(e^{-(j-1)A_n/n} - e^{-jA_n/n} \right) \zeta_{n,k} \\
&= \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]-1} \sum_{j=k+1}^{[nt]} \left(e^{-(j-1)A_n/n} - e^{-jA_n/n} \right) \zeta_{n,k} \\
&= \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]-1} \left(e^{-kA_n/n} - e^{-[nt]A_n/n} \right) \zeta_{n,k} \\
&= \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} \left(e^{-kA_n/n} - e^{-[nt]A_n/n} \right) \zeta_{n,k} = e^{-[nt]A_n/n} \left(Y_n(t) - M_n(t) \right).
\end{aligned}$$

Hence the processes (M, Y) and (M_n, Y_n) can be expressed as $(M, Y) = \Phi(M)$, $(M_n, Y_n) = \Phi_n(M_n)$, $n = 1, 2, \dots$, where the measurable mappings $\Phi, \Phi_n: D([0, 1] \rightarrow \mathbf{R}^d) \rightarrow D([0, 1] \rightarrow \mathbf{R}^{2d})$, $n = 1, 2, \dots$, are defined as

$$\begin{aligned}
\Phi(x)(t) &= \left(x(t), x(t) + A \int_0^t e^{(t-s)A} x(s) ds \right), \\
\Phi_n(x)(t) &= \left(x(t), x(t) + A_n \int_0^{[nt]/n} e^{([nt]/n-s)A_n} x(s) ds \right).
\end{aligned}$$

Applying Lemma 1 we obtain $(M_n, Y_n) \xrightarrow{\mathcal{D}} (M, Y)$ in $D([0, 1] \rightarrow \mathbf{R}^{2d})$.

(iii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (ii) are trivial.

(ii) $\Rightarrow$ (iii). Since the process $Y(t)$, $t \in [0, 1]$, is a solution of the stochastic differential equation (11) we have $Y(t) = A \int_0^t Y(s) ds + M(t)$, thus $M(t) = Y(t) - A \int_0^t Y(s) ds$. A similar formula holds for the partial sum process $M_n(t)$, $t \in [0, 1]$:

$$\begin{aligned}
M_n(t) &= \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} (Z_{n,k} - e^{A_n/n} Z_{n,k-1}) = \frac{1}{\sqrt{n}} Z_{n,[nt]} - \frac{1}{\sqrt{n}} (e^{A_n/n} - I) \sum_{k=1}^{[nt]-1} Z_{n,k} \\
&= Y_n(t) - n(e^{A_n/n} - I) \int_0^{[nt]/n} Y_n(s) ds.
\end{aligned}$$

Hence the processes (M, Y) and (M_n, Y_n) can be expressed as $(M, Y) = \Psi(Y)$, $(M_n, Y_n) = \Psi_n(Y_n)$, $n = 1, 2, \dots$, where the measurable mappings $\Psi, \Psi_n: D([0, 1] \rightarrow \mathbf{R}^d) \rightarrow D([0, 1] \rightarrow \mathbf{R}^{2d})$, $n = 1, 2, \dots$, are defined as follows

$$\begin{aligned}
\Psi(x)(t) &= \left(x(t) - A \int_0^t x(s) ds, x(t) \right), \\
\Psi_n(x)(t) &= \left(x(t) - n(e^{A_n/n} - I) \int_0^{[nt]/n} x(s) ds, x(t) \right).
\end{aligned}$$

Applying Lemma 1 we obtain $(M_n, Y_n) \xrightarrow{\mathcal{D}} (M, Y)$ in $D([0, 1] \rightarrow \mathbf{R}^{2d})$. Theorem 1 is proved.

3. Least-squares estimators. The least-squares estimator of the matrix Q_n in the model (9) can be obtained by minimizing the sum of squares

$$\sum_{k=1}^n \|X_{n,k} - Q_n e^B X_{n,k-1}\|^2.$$

The orthogonality of e^B implies that the above sum is equal to $\sum_{k=1}^n \|Z_{n,k} - Q_n Z_{n,k-1}\|^2$, consequently, the LSE is

$$\hat{Q}_n = \left(\sum_{k=1}^n Z_{n,k} Z'_{n,k-1} \right) \left(\sum_{k=1}^n Z_{n,k-1} Z'_{n,k-1} \right)^{-1}. \quad (13)$$

From now on we shall put the following condition on the random disturbances $\{\varepsilon_{n,k}\}$.

(C) $\varepsilon_{n,k}$, $k = 1, \dots, n$, $n \geq 1$, is a triangular array of d -dimensional square integrable martingale differences with respect to the filtrations $(\mathcal{F}_{n,k})_{k=0,1,\dots,n;n \geq 1}$ such that for all $t \in [0, 1]$

$$\frac{1}{n} \sum_{k=1}^{[nt]} \mathbb{E}(\varepsilon_{n,k} \varepsilon'_{n,k} \mid \mathcal{F}_{n,k-1}) \xrightarrow{\mathbf{P}} I \quad \text{as } n \rightarrow \infty$$

and

$$\forall \alpha > 0 \quad \frac{1}{n} \sum_{k=1}^{[nt]} \mathbb{E}(\|\varepsilon_{n,k}\|^2 \chi_{\{\|\varepsilon_{n,k}\| > \alpha \sqrt{n}\}} \mid \mathcal{F}_{n,k-1}) \xrightarrow{\mathbf{P}} 0 \quad \text{as } n \rightarrow \infty.$$

Theorem 2. Suppose that the array $\varepsilon_{n,k}$, $k = 1, \dots, n$, $n \geq 1$, satisfies the condition (C). Then

$$n(\hat{Q}_n - Q_n) \xrightarrow{\mathcal{D}} \int_0^1 (dW(t)) Y'(t) \left(\int_0^1 Y(t) Y'(t) dt \right)^{-1},$$

where the process $Y(t)$, $t \in [0, 1]$, is given by $dY(t) = AY(t) dt + dW(t)$, $Y(0) = 0$, and $W(t)$, $t \in [0, 1]$, is a standard d -dimensional Wiener process.

By another normalization

$$(\hat{Q}_n - Q_n) \left(\sum_{k=1}^n X_{k-1} X'_{k-1} \right)^{1/2} \xrightarrow{\mathcal{D}} \int_0^1 (dW(t)) Y'(t) \left(\int_0^1 Y(t) Y'(t) dt \right)^{-1/2}.$$

P r o o f. Orthogonality of the matrix e^B implies that the rotated array $\zeta_{n,k}$, $k = 1, \dots, n$, $n \geq 1$, also satisfies the condition (C), so using a version of the functional central limit theorem on the space $D([0, 1] \rightarrow \mathbf{R}^d)$ (Theorem 7.11 in [15]) we obtain $M_n \xrightarrow{\mathcal{D}} W$ in $D([0, 1] \rightarrow \mathbf{R}^d)$. Applying Theorem 1 we can conclude $(M_n, Y_n) \xrightarrow{\mathcal{D}} (W, Y)$ in $D([0, 1] \rightarrow \mathbf{R}^{2d})$. This implies weak convergence of the stochastic integrals

$$\left(\int_0^1 Y_n(t) Y'_n(t) dt, \int_0^1 (dM_n(t)) Y'_n(t) \right) \xrightarrow{\mathcal{D}} \left(\int_0^1 Y(t) Y'(t) dt, \int_0^1 (dW(t)) Y'(t) \right)$$

(see, for example, Proposition 6 in [9]). Moreover,

$$\begin{aligned} \hat{Q}_n &= \left(\sum_{k=1}^n (Q_n Z_{n,k-1} + \zeta_{n,k}) Z'_{n,k-1} \right) \left(\sum_{k=1}^n Z_{n,k-1} Z'_{n,k-1} \right)^{-1} \\ &= Q_n + \frac{1}{n} \left(\int_0^1 (dM_n(t)) Y'_n(t) \right) \left(\int_0^1 Y_n(t) Y'_n(t) dt \right)^{-1}. \end{aligned}$$

Since the matrix $\int_0^1 Y(t) Y'(t) dt$ is invertible with probability 1 we can apply the continuous mapping theorem. Theorem 2 is proved.

4. Connection with continuous time AR processes. Let $W(t)$, $t \in [0, 1]$, be a standard d -dimensional Wiener process. Consider the continuous time autoregressive process given by

$$dY(t) = AY(t) dt + dW(t), \quad Y(0) = 0. \quad (14)$$

Consider the measures P_W and P_Y on $C([0, 1] \rightarrow \mathbf{R}^d)$ generated by the processes W and Y , respectively. It is known that P_Y is absolutely continuous with respect to the measure P_W and the Radon-Nikodym derivative has the form

$$\frac{dP_Y}{dP_W}(Y) = \exp \left\{ -\frac{1}{2} \int_0^1 \|AY(t)\|^2 dt + \int_0^1 \langle AY(t), dY(t) \rangle \right\}. \quad (15)$$

Consequently the maximum likelihood estimator (MLE) of the matrix A based on the observations $Y(t)$, $t \in [0, 1]$, in the model (14) is given by

$$\hat{A} = \int_0^1 (dY(t)) Y'(t) \left(\int_0^1 Y(t) Y'(t) dt \right)^{-1}.$$

(See, for example, [14], [2].)

Corollary 1. Suppose that the array $\varepsilon_{n,k}$, $k = 1, \dots, n$, $n \geq 1$, satisfies the condition (C). Then $n(\hat{Q}_n - I) \xrightarrow{D} \hat{A}$, where $\hat{A}$ is the maximum likelihood estimator of the matrix A based on the observations $Y(t)$, $t \in [0, 1]$, in the model (14).

Proof. Using Itô's formula we obtain

$$\hat{A} = A + \int_0^1 (dW(t)) Y'(t) \left(\int_0^1 Y(t) Y'(t) dt \right)^{-1}.$$

Theorem 2 implies

$$n(\hat{Q}_n - I) = n(\hat{Q}_n - Q_n) + n(e^{A_n/n} - I) \xrightarrow{D} \hat{A}.$$

Corollary 1 is proved. * *

Consider now the quadratic form

$$L_n(X_{n,1}, \dots, X_{n,n}) = -\frac{1}{2} \left(\sum_{k=1}^n \|X_{n,k} - Q_n e^B X_{n,k-1}\|^2 - \sum_{k=1}^n \|X_{n,k} - e^B X_{n,k-1}\|^2 \right),$$

connected with the LSE $\hat{Q}_n$. We remark that if the innovations $\{\varepsilon_{n,n}\}$ are normal then $L_n(X_{n,1}, \dots, X_{n,n})$ is just the log-likelihood ratio. Denote the log-likelihood ratio of the process $Y(t)$, $t \in [0, 1]$, by

$$L(Y) = \log \frac{dP_Y}{dP_W}(Y).$$

Corollary 2. Suppose that the array $\varepsilon_{n,k}$, $k = 1, \dots, n$, $n \geq 1$, satisfies the condition (C). Then

$$L_n(X_{n,1}, \dots, X_{n,n}) \xrightarrow{D} L(Y) \quad \text{in } \mathbf{R}.$$

Proof. Using the same arguments as in the proof of Theorem 1 we obtain

$$\begin{aligned} -2L_n(X_{n,1}, \dots, X_{n,n}) &= \sum_{k=1}^n \left\| (Q_n - I) Z_{n,k-1} \right\|^2 \\ &\quad - 2 \sum_{k=1}^n \left\langle Z_{n,k} - Z_{n,k-1}, (Q_n - I) Z_{n,k-1} \right\rangle \\ &= \int_0^1 \|n(Q_n - I)Y_n(t)\|^2 dt - 2 \int_0^1 \langle dY_n(t), n(Q_n - I)Y_n(t) \rangle \\ &\xrightarrow{D} \int_0^1 \|AY(t)\|^2 dt - 2 \int_0^1 \langle dY(t), AY(t) \rangle. \end{aligned}$$

The assertion is proved.

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