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ASYMPTOTIC DISTRIBUTIONS OF MULTIVARIATE INTERMEDIATE ORDER STATISTICS

Пусть $\{X_n = (X_n^{(1)}, \dots, X_n^{(d)}), n \geq 1\}$ есть независимые одинаково распределенные (н.о.р.) случайные векторы с общей невырожденной функцией распределения (ф.р.), и для каждого $n \geq 1$ и каждого $k = 1, \dots, d$ пусть $X_{1;n}^{(k)} \leq \dots \leq X_{n;n}^{(k)}$ есть порядковые статистики для $X_1^{(k)}, \dots, X_n^{(k)}$. Предположим, что ранги $r_n = (r_n^{(1)}, \dots, r_n^{(d)})$ удовлетворяют следующим соотношениям: $r_n^{(k)} \rightarrow \infty$ монотонно, $r_n^{(k)}/n \rightarrow 0$ и $r_n^{(k)}/\sum_{l=1}^d r_n^{(l)} \rightarrow m^{(k)} > 0$ для каждого $k = 1, \dots, d$, и положим $X_{r_n;n} = (X_{r_n^{(1)};n}^{(1)}, \dots, X_{r_n^{(d)};n}^{(d)})$. Данная статья посвящена нахождению класса предельных распределений для надлежащим образом центрированных и нормированных векторов $\{X_{r_n;n}\}$, а также получению необходимых и достаточных условий слабой сходимости.

Ключевые слова и фразы: многомерные промежуточные порядковые статистики, асимптотические распределения.

1. Introduction

Let $\{X_n = (X_n^{(1)}, \dots, X_n^{(d)}), n \geq 1\}$ be a sequence of independent identically distributed (i.i.d.) random vectors with a common nondegenerate distribution function (d.f.) F (a d -dimensional d.f. is said to be nondegenerate if all its 1-dimensional marginal d.f.'s are nondegenerate). For each $n \geq 1$ and $k = 1, \dots, d$, write $X_{1;n}^{(k)} \leq \dots \leq X_{n;n}^{(k)}$ for the order statistics of $X_1^{(k)}, \dots, X_n^{(k)}$. For a sequence $\{r_n = (r_n^{(1)}, \dots, r_n^{(d)}), n \geq 1\}$ of positive integers satisfying $1 \leq r_n^{(k)} \leq n$, denoting $X_{r_n;n}^{(k)} = X_{r_n^{(k)};n}^{(k)}$ for every $k = 1, \dots, d$, we call $\{X_{r_n;n} = (X_{r_n^{(1)};n}^{(1)}, \dots, X_{r_n^{(d)};n}^{(d)}), n \geq 1\}$ the sequence of order statistics with ranks $\{r_n\}$. If the ranks $\{r_n\}$ satisfy

$$\frac{r_n^{(k)}}{n} \rightarrow c^{(k)} \in (0, 1) \quad \text{for } k = 1, \dots, d,$$

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the $\{X_{r_n;n}\}$ will be called *central order statistics*. If $\{r_n\}$ satisfy

$$r_n^{(k)} \rightarrow \infty \quad \text{and} \quad \frac{r_n^{(k)}}{n} \rightarrow 0 \quad \text{for } k = 1, \dots, d, \quad (1.1)$$

the $\{X_{r_n;n}\}$ will be called *intermediate order statistics*. For discussions on asymptotic normality of the central order statistics, see [1] and [3]. In the present paper, we only pay attention to the intermediate order statistics.

In order to discuss the asymptotic behaviour of distributions of multivariate intermediate order statistics, we need some results about the case $d = 1$. Denote V as the set consisting of all nondecreasing and right continuous functions v from $\mathbf{R}$ to $\bar{\mathbf{R}} = [-\infty, \infty]$ satisfying

$$v(\pm\infty) := \lim_{x \rightarrow \pm\infty} v(x) = \pm\infty.$$

A $v \in V$ is said to be nondegenerate if there exists a $x_0 \in \mathbf{R}$ such that $v(x_0) \in \mathbf{R}$. The first result for the case $d = 1$ we need is the following (cf. [5]).

Lemma 1.1. *Suppose that*

$$\min(r_n, n - r_n) \rightarrow \infty \quad \text{as } n \rightarrow \infty \quad (1.2)$$

and denote $\lambda_n = r_n/n$. Then for any $\alpha_n > 0$ and $\beta_n \in \mathbf{R}$, equivalent are:

$$\begin{aligned} \frac{X_{r_n;n} - \beta_n}{\alpha_n} &\xrightarrow{d} G, \quad \text{a d.f.}; \\ \frac{\sqrt{n}[F(\alpha_n x + \beta_n) - \lambda_n]}{\sqrt{\lambda_n(1 - \lambda_n)}} &\xrightarrow{w} v(x), \quad \text{for a } v \in V; \end{aligned} \quad (1.3)$$

the relation between G and v is $G = \Phi(v)$, where Φ denotes the standard normal d.f.

We also need a lemma by Wu [6].

Lemma 1.2. *Under the assumption that $r_n \rightarrow \infty$ nondecreasingly and $\lambda_n \rightarrow 0$, if (1.3) holds for a nondegenerate $v \in V$, there exist $a > 0$, $b \in \mathbf{R}$ and $\gamma \in \mathbf{R}$ such that*

$$v(ax + b) = \log w_\gamma(x) \quad \text{for all } x \in \mathbf{R}, \quad (1.4)$$

where

$$w_\gamma(x) = \begin{cases} (1 + \gamma x)^{1/\gamma} & \text{if } 1 + \gamma x > 0; \\ 0 & \text{if } 1 + \gamma x \leq 0 \quad \text{and } \gamma > 0; \\ \infty & \text{if } 1 + \gamma x \leq 0 \quad \text{and } \gamma < 0. \end{cases}$$

For more detail results on the case $d = 1$, one may refer to [2]. But we will always assume that $d \geq 2$ below. In Section 2, we will present a version of Smirnov's lemma for the multivariate case. In Section 3, we discuss the asymptotic behaviour of distributions of multivariate intermediate order

statistics. We will find all limiting d.f.'s for those intermediate order statistics for which ranks go to infinity at the same rate, i.e.,

$$\begin{aligned} r_n^{(k)} \rightarrow \infty \quad \text{nondecreasingly,} \quad \frac{r_n^{(k)}}{\hat{r}_n} \rightarrow m^{(k)} > 0 \quad \text{for } k = 1, \dots, d, \\ \text{and } \frac{\hat{r}_n}{n} \rightarrow 0, \end{aligned} \quad (1.5)$$

where we denote $\hat{r}_n = \sum_{k=1}^d r_n^{(k)}$. In Section 4, this result is generalized to the case that the components of ranks may be split into several groups, each of which tends to infinity with the same rate. It is interesting that the asymptotic behaviour of distributions of multivariate intermediate order statistics depends only on the 2-dimensional marginal d.f.'s of the original d.f. F .

2. A multivariate analogue of Smirnov's lemma

Consider an array $\{I_{n,i} = (I_{n,i}^{(1)}, \dots, I_{n,i}^{(d)}): i = 1, \dots, n; n \geq 1\}$ of random vectors. Suppose that for each $n \geq 1$, $I_{n,1}, \dots, I_{n,n}$ are i.i.d. with a common d.f. satisfying for each $k = 1, \dots, d$,

$$\mathbf{P}\{I_{n,i}^{(k)} = 1\} = p_n^{(k)}; \quad \mathbf{P}\{I_{n,i}^{(k)} = 0\} = q_n^{(k)} =: 1 - p_n^{(k)}.$$

Denote for $k = 1, \dots, d$, and $i = 1, \dots, n$,

$$\hat{I}_{n,i}^{(k)} = \frac{I_{n,i}^{(k)} - p_n^{(k)}}{\sqrt{np_n^{(k)} q_n^{(k)}}}$$

and $\hat{I}_{n,i} = (\hat{I}_{n,i}^{(1)}, \dots, \hat{I}_{n,i}^{(d)})$. Then for $1 \leq k, l \leq d$,

$$\rho_n^{(k,l)} = \sum_{i=1}^n \mathbf{E} \hat{I}_{n,i}^{(k)} \hat{I}_{n,i}^{(l)} = \begin{cases} \frac{p_n^{(k,l)} - p_n^{(k)} p_n^{(l)}}{\sqrt{p_n^{(k)} q_n^{(k)} p_n^{(l)} q_n^{(l)}}} & \text{if } k \neq l, \\ 1 & \text{if } k = l, \end{cases}$$

where $p_n^{(k,l)} = \mathbf{P}\{I_{n,i}^{(k)} = I_{n,i}^{(l)} = 1\}$. We need the following central limit theorem whose proof is easy and therefore omitted.

Lemma 2.1. *Suppose that*

$$np_n^{(k)} q_n^{(k)} \rightarrow \infty \quad \text{for } k = 1, \dots, d.$$

Then

$$\sum_{i=1}^n \hat{I}_{n,i} \xrightarrow{d} \Psi$$

holds for a d.f. Ψ if and only if

$$\rho_n^{(k,l)} \rightarrow \rho^{(k,l)}$$

holds for all $1 \leq k < l \leq d$; the following relation holds:

$$\Psi = \Phi_\Sigma$$

where Φ_Σ denotes the d -dimensional normal d.f. with mean $(0, \dots, 0)$ and covariance matrix

$$\Sigma = (\rho^{(k,l)})_{d \times d}.$$

Now we are going to extend Smirnov's lemma. For $d \geq 2$, (1.2) will be read as

$$\min(r_n^{(k)}, n - r_n^{(k)}) \rightarrow \infty \quad \text{for each } k = 1, \dots, d. \quad (2.1)$$

Assuming that $\alpha_n = (\alpha_n^{(1)}, \dots, \alpha_n^{(d)})$ is positive, i.e., $\alpha_n^{(k)} > 0$ for all $k = 1, \dots, d$, and $\beta_n = (\beta_n^{(1)}, \dots, \beta_n^{(d)}) \in \mathbf{R}^d$, we consider the limit relation

$$\left(\frac{X_{r_n;n}^{(1)} - \beta_n^{(1)}}{\alpha_n^{(1)}}, \dots, \frac{X_{r_n;n}^{(d)} - \beta_n^{(d)}}{\alpha_n^{(d)}} \right) \xrightarrow{d} G \quad (2.2)$$

where G is a d -dimensional d.f. Moreover, write $F^{(k_1, \dots, k_s)}$ for the distribution function of $(X_n^{(k_1)}, \dots, X_n^{(k_s)})$ ($1 \leq k_1 < \dots < k_s \leq d$) and denote

$$\lambda_n = (\lambda_n^{(1)}, \dots, \lambda_n^{(d)}) =: \left(\frac{r_n^{(1)}}{n}, \dots, \frac{r_n^{(d)}}{n} \right);$$

$$v_n^{(k)}(x) = \frac{\sqrt{n} [F^{(k)}(\alpha_n^{(k)}x + \beta_n^{(k)}) - \lambda_n^{(k)}]}{\sqrt{\lambda_n^{(k)}(1 - \lambda_n^{(k)})}} \quad \text{for } k = 1, \dots, d;$$

$$\rho_n^{(k,l)}(x, y) = \frac{F^{(k,l)}(\alpha_n^{(k)}x + \beta_n^{(k)}, \alpha_n^{(l)}y + \beta_n^{(l)}) - F^{(k)}(\alpha_n^{(k)}x + \beta_n^{(k)})F^{(l)}(\alpha_n^{(l)}y + \beta_n^{(l)})}{\sqrt{\lambda_n^{(k)}(1 - \lambda_n^{(k)})\lambda_n^{(l)}(1 - \lambda_n^{(l)})}}.$$

Then Smirnov's lemma may be extended as follows.

Theorem 2.1. Suppose that (2.1) holds. For any $\beta_n \in \mathbf{R}^d$ and positive α_n , (2.2) holds for a nondegenerate d.f. G if and only if there exists a vector $v = (v^{(1)}, \dots, v^{(d)})$ of functions on $\mathbf{R}$, in which $v^{(k)} \in V$ is nondegenerate for each $k = 1, \dots, d$, such that

$$v_n^{(k)} \xrightarrow{w} v^{(k)} \quad \text{for } k = 1, \dots, d \quad (2.3)$$

and there exists a $d \times d$ symmetric and non-negative matrix $\Sigma = (\rho^{(k,l)})$, in which $\rho^{(k,k)} = 1$ for all $k = 1, \dots, d$, such that for $1 \leq k < l \leq d$,

$$\rho_n^{(k,l)}(x, y) \rightarrow \rho^{(k,l)} \quad (2.4)$$

holds for one or all $(x, y) \in C(v^{(k)}) \times C(v^{(l)})$, where $C(f)$ consists of all points at which the function f is continuous and finite; at the same time, the following relation holds:

$$G = \Phi_{\Sigma}(v). \quad (2.5)$$

P r o o f. First note that for any $k = 1, \dots, d$ and $x \in C(v^{(k)})$, either

$$v_n^{(k)}(x) \rightarrow v^{(k)}(x) \quad \text{as } n \rightarrow \infty$$

or

$$\hat{v}_n^{(k)}(x) := \frac{\sqrt{n}[F^{(k)}(\alpha_n^{(k)}x + \beta_n^{(k)}) - \lambda_n^{(k)}]}{\sqrt{F^{(k)}(\alpha_n^{(k)}x + \beta_n^{(k)})[1 - F^{(k)}(\alpha_n^{(k)}x + \beta_n^{(k)})]}} \rightarrow v^{(k)}(x)$$

as $n \rightarrow \infty$ implies

$$\frac{F^{(k)}(\alpha_n^{(k)}x + \beta_n^{(k)})}{\lambda_n^{(k)}} \rightarrow 1 \quad \text{as } n \rightarrow \infty. \quad (2.6)$$

We split the proof into three parts.

(1) If (2.3) holds and (2.4) holds for all $(x, y) \in C(v^{(k)}) \times C(v^{(l)})$, then by (2.6) and Lemma 2.1, (2.2) holds for the G defined by (2.5).

(2) For the converse we show that if (2.2) holds for a nondegenerate d.f. G , then (2.4) holds for some $(x_0, y_0) \in C(v^{(k)}) \times C(v^{(l)})$ and (2.3) holds for a function v with nondegenerate components $v^{(k)} \in V$ ($k = 1, \dots, d$).

It is easily seen that (2.2) implies

$$\frac{X_{r_n; n}^{(k)} - \beta_n^{(k)}}{\alpha_n^{(k)}} \xrightarrow{d} G^{(k)} \quad \text{for } k = 1, \dots, d; \quad (2.7)$$

$$\left(\frac{X_{r_n; n}^{(k)} - \beta_n^{(k)}}{\alpha_n^{(k)}}, \frac{X_{r_n; n}^{(l)} - \beta_n^{(l)}}{\alpha_n^{(l)}} \right) \xrightarrow{d} G^{(k, l)} \quad \text{for } 1 \leq k < l \leq d. \quad (2.8)$$

By Lemma 1.1, (2.7) implies (2.3) with a v whose components are all nondegenerate. Hence, for any $1 \leq k < l \leq d$, there exists a $x_0 \in C(v^{(k)})$ and a $y_0 \in C(v^{(l)})$ such that

$$np_n^{(k)} q_n^{(k)} \sim n\lambda_n^{(k)}(1 - \lambda_n^{(k)}) \rightarrow \infty \quad \text{and} \quad np_n^{(l)} q_n^{(l)} \sim n\lambda_n^{(l)}(1 - \lambda_n^{(l)}) \rightarrow \infty$$

where

$$\begin{aligned} p_n^{(k)} &= F^{(k)}(\alpha_n^{(k)}x_0 + \beta_n^{(k)}) \quad \text{and} \quad q_n^{(k)} = 1 - p_n^{(k)}; \\ p_n^{(l)} &= F^{(l)}(\alpha_n^{(l)}y_0 + \beta_n^{(l)}) \quad \text{and} \quad q_n^{(l)} = 1 - p_n^{(l)}. \end{aligned}$$

For $n \geq 1, i = 1, \dots, n$, write $I_{n,i}^{(k)}$ and $I_{n,i}^{(l)}$ for the indicators of the sets $\{X_i^{(k)} \leq \alpha_n^{(k)} x_0 + \beta_n^{(k)}\}$ and $\{X_i^{(l)} \leq \alpha_n^{(l)} y_0 + \beta_n^{(l)}\}$ respectively. We have

$$\begin{aligned} \mathbf{P}\{I_{n,i}^{(k)} = 1\} &= p_n^{(k)}, \quad \mathbf{P}\{I_{n,i}^{(k)} = 0\} = q_n^{(k)}, \\ \mathbf{P}\{I_{n,i}^{(l)} = 1\} &= p_n^{(l)}, \quad \mathbf{P}\{I_{n,i}^{(l)} = 0\} = q_n^{(l)}, \\ \widehat{v}_n^{(k)}(x_0) &\sim v_n^{(k)}(x_0) \rightarrow v^{(k)}(x_0), \end{aligned} \quad (2.9)$$

$$\widehat{v}_n^{(l)}(y_0) \sim v_n^{(l)}(y_0) \rightarrow v^{(l)}(y_0), \quad (2.10)$$

and

$$\widehat{\rho}_n^{(k,l)} := \frac{p_n^{(k,l)} - p_n^{(k)} p_n^{(l)}}{\sqrt{p_n^{(k)} q_n^{(k)} p_n^{(l)} q_n^{(l)}}} \sim \rho_n^{(k,l)}(x_0, y_0),$$

where $p_n^{(k,l)} = F^{(k,l)}(\alpha_n^{(k)} x_0 + \beta_n^{(k)}, \alpha_n^{(l)} y_0 + \beta_n^{(l)})$. Now we will show that for some $0 \leq \rho^{(k,l)} \leq 1$,

$$\rho_n^{(k,l)}(x_0, y_0) \rightarrow \rho^{(k,l)},$$

i.e., if there exist two subsequences such that

$$\rho_{n'}^{(k,l)}(x_0, y_0) \rightarrow \rho' \quad \text{and} \quad \rho_{n''}^{(k,l)}(x_0, y_0) \rightarrow \rho'', \quad (2.11)$$

then $\rho' = \rho''$. Note the above two limits of subsequences are respectively equivalent to

$$\widehat{\rho}_{n'}^{(k,l)} \rightarrow \rho' \quad \text{and} \quad \widehat{\rho}_{n''}^{(k,l)} \rightarrow \rho''.$$

It follows from (2.9), (2.10), (2.11) and Lemma 2.1 that

$$\begin{aligned} &\mathbf{P} \left\{ \frac{X_{r_{n'}^{(k)}; n'}^{(k)} - \beta_{n'}^{(k)}}{\alpha_{n'}^{(k)}} \leq x_0, \frac{X_{r_{n'}^{(l)}; n'}^{(l)} - \beta_{n'}^{(l)}}{\alpha_{n'}^{(l)}} \leq y_0 \right\} \\ &= \mathbf{P} \left\{ \sum_{i=1}^{n'} I_{n',i}^{(k)} \geq r_{n'}^{(k)}, \sum_{i=1}^{n'} I_{n',i}^{(l)} \geq r_{n'}^{(l)} \right\} \\ &= \mathbf{P} \left\{ \sum_{i=1}^{n'} \widehat{I}_{n',i}^{(k)} \geq -\widehat{v}_{n'}^{(k)}(x_0), \sum_{i=1}^{n'} \widehat{I}_{n',i}^{(l)} \geq -\widehat{v}_{n'}^{(l)}(y_0) \right\} \\ &\rightarrow \Phi_{\rho'}(v^{(k)}(x_0), v^{(l)}(y_0)) \end{aligned}$$

where Φ_ρ denotes the two dimensional normal d.f. with mean $(0,0)$, variance $(1,1)$ and correlation coefficient ρ , and by the same way, (2.9), (2.10) and (2.11) give

$$\mathbf{P} \left\{ \frac{X_{r_{n''}^{(k)}; n''}^{(k)} - \beta_{n''}^{(k)}}{\alpha_{n''}^{(k)}} \leq x_0, \frac{X_{r_{n''}^{(l)}; n''}^{(l)} - \beta_{n''}^{(l)}}{\alpha_{n''}^{(l)}} \leq y_0 \right\} \rightarrow \Phi_{\rho''}(v^{(k)}(x_0), v^{(l)}(y_0)).$$

Comparing these limits with (2.8), we obtain

$$\Phi_{\rho'}(v^{(k)}(x_0), v^{(l)}(y_0)) = G^{(k,l)}(x_0, y_0) = \Phi_{\rho''}(v^{(k)}(x_0), v^{(l)}(y_0)).$$

Because for a fixed $(x, y) \in \mathbf{R}^2$, the function $\Phi_{\rho}(x, y)$ of ρ is strictly increasing, the above equation gives $\rho' = \rho''$ as expected.

(3) It remains to prove that under (2.1), if (2.2) holds for a $(x_0, y_0) \in C(v^{(k)}) \times C(v^{(l)})$, then it holds for all $(x, y) \in C(v^{(k)}) \times C(v^{(l)})$.

In fact, for any two dimensional d.f. H with marginal d.f.'s $H^{(1)}$ and $H^{(2)}$, we have

$$|H(x, y) - H(x_0, y_0)| \leq |H^{(1)}(x) - H^{(1)}(x_0)| + |H^{(2)}(y) - H^{(2)}(y_0)|.$$

Hence,

$$\begin{aligned} & |\rho_n^{(k,l)}(x, y) - \rho_n^{(k,l)}(x_0, y_0)| \\ & \leq \frac{2|F^{(k)}(\alpha_n^{(k)}x + \beta_n^{(k)}) - F^{(k)}(\alpha_n^{(k)}x_0 + \beta_n^{(k)})|}{\sqrt{\lambda_n^{(k)}(1 - \lambda_n^{(k)})\lambda_n^{(l)}(1 - \lambda_n^{(l)})}} \\ & \quad + \frac{2|F^{(l)}(\alpha_n^{(l)}y + \beta_n^{(l)}) - F^{(l)}(\alpha_n^{(l)}y_0 + \beta_n^{(l)})|}{\sqrt{\lambda_n^{(k)}(1 - \lambda_n^{(k)})\lambda_n^{(l)}(1 - \lambda_n^{(l)})}} \\ & = \frac{2|v_n^{(k)}(x) - v_n^{(k)}(x_0)|}{\sqrt{n\lambda_n^{(l)}(1 - \lambda_n^{(l)})}} + \frac{2|v_n^{(l)}(y) - v_n^{(l)}(y_0)|}{\sqrt{n\lambda_n^{(k)}(1 - \lambda_n^{(k)})}} \rightarrow 0. \end{aligned}$$

The proof of the theorem is completed.

3. The limiting distributions of intermediate order statistics

We start from a corollary to Theorem 2.1.

Lemma 3.1. Under (1.1), for any $\beta_n \in \mathbf{R}^d$ and positive α_n , (2.2) holds for a nondegenerate d -dimensional d.f. G if and only if there exists a vector $v = (v^{(1)}, \dots, v^{(d)})$ with nondegenerate $v^{(k)} \in V$ such that

$$\bar{v}_n^{(k)}(x) =: \sqrt{r_n^{(k)}} \left[\frac{F^{(k)}(\alpha_n^{(k)}x + \beta_n^{(k)})}{\lambda_n^{(k)}} - 1 \right] \xrightarrow{w} v^{(k)}(x) \quad (3.1)$$

for all $k = 1, \dots, d$ and a $d \times d$ non-negative symmetric matrix $\Sigma = (\rho^{(k,l)})$ with $\rho^{(k,k)} = 1$ for $k = 1, \dots, d$ such that

$$\bar{\rho}_n^{(k,l)}(x, y) =: \frac{F^{(k,l)}(\alpha_n^{(k)}x + \beta_n^{(k)}, \alpha_n^{(l)}y + \beta_n^{(l)})}{\sqrt{\lambda_n^{(k)}\lambda_n^{(l)}}} \rightarrow \rho^{(k,l)} \quad (3.2)$$

holds for one or all $(x, y) \in C(v^{(k)}) \times C(v^{(l)})$.

P r o o f. It is easily seen that under (1.1)

$$v_n^{(k)}(x) \sim \frac{nF^{(k)}(\alpha_n^{(k)}x + \beta_n^{(k)}) - r_n^{(k)}}{\sqrt{r_n^{(k)}}} = \bar{v}_n^{(k)}(x),$$

so that (3.1) and (2.3) are equivalent. Moreover, (3.1) implies (2.6) and therefore

$$\frac{F^{(k)}(\alpha_n^{(k)}x + \beta_n^{(k)})}{\sqrt{\lambda_n^{(k)}}} \rightarrow 0, \quad k = 1, \dots, d.$$

Hence under (2.1), for every $1 \leq k < l \leq d$ and every $(x, y) \in C(v^{(k)}) \times C(v^{(l)})$, $\{\bar{\rho}_n^{(k,l)}(x, y)\}$ and $\{\rho^{(k,l)}(x, y)\}$ have the same limit if one of them exists. This shows that (3.2) and (2.4) are also equivalent. Hence the assertion of the lemma follows from Theorem 2.1.

Now we need to introduce some notation. Denote by M the set consisting of all vectors $m = (m^{(1)}, \dots, m^{(d)})$ whose components satisfy

$$m^{(k)} > 0 \quad \text{for } k = 1, \dots, d \quad \text{and} \quad \sum_{k=1}^d m^{(k)} = 1.$$

A symmetric matrix $\Sigma = (\rho^{(k,l)})_{d \times d}$ with $\rho^{(k,k)} = 1$ for $k = 1, \dots, d$ is said to be *matchable* to a $m \in M$ if there exists a measure μ on

$$Z = \left\{ x = (x^{(1)}, \dots, x^{(d)}): x^{(k)} = 0 \text{ or } 1 \text{ for } k = 1, \dots, d; \sum_{k=1}^d x^{(k)} > 0 \right\}$$

such that

$$\mu(\{x \in Z: x^{(k)} = 1\}) = m^{(k)} \quad \text{for } k = 1, \dots, d; \quad (3.3)$$

$$\mu(\{x \in Z: x^{(k)} = x^{(l)} = 1\}) = \rho^{(k,l)} \sqrt{m^{(k)} m^{(l)}} \quad \text{for } 1 \leq k < l \leq d. \quad (3.4)$$

For a given $m \in M$, we will write $S(m)$ for the set of all the matrices Σ matchable to the vector m . Moreover, let V_0 denote the set of all functions in V for which there exist constants $a > 0$ and $b \in \mathbf{R}$ such that (1.4) holds for some $\gamma \in \mathbf{R}$; let D denote the set of all d -dimensional d.f.'s G for which there exist a sequence $\{X_n\}$ of i.i.d. random vectors, a positive α_n and a $\beta_n \in \mathbf{R}^d$ such that (2.2) holds. Then we may state our results as follows.

Theorem 3.1. *If (1.5) holds, then*

$$D = \{\Phi_\Sigma(v^{(1)}, \dots, v^{(d)}): v^{(k)} \in V_0, k = 1, \dots, d; \Sigma \in S(m)\};$$

moreover, (2.2) holds for a $G \in D$ if and only if for every $k = 1, \dots, d$, (3.1) holds and for every $1 \leq k < l \leq d$, (3.2) holds for one or all $(x, y) \in C(v^{(k)}) \times C(v^{(l)})$.

P r o o f. The second statement of the theorem follows immediately from Lemma 3.1. We need only to show the first one.

Assume that $G \in D$. It follows from Lemma 3.1 that $G = \Phi_{\Sigma}(v^{(1)}, \dots, v^{(d)})$, where the $v^{(k)}$ is determined by (3.1) and the elements $\rho^{(k,l)}$ of the Σ are determined by (3.2). By Lemma 1.2, for every $k = 1, \dots, d$, we have $v^{(k)} \in V_0$, i.e., there exist $a^{(k)} > 0$, $b^{(k)} \in \mathbf{R}$ and $\gamma^{(k)} \in \mathbf{R}$ such that

$$v^{(k)}(a^{(k)}x + b^{(k)}) = \log w_{\gamma^{(k)}}(x) \quad \text{for all } x \in \mathbf{R}.$$

Without loss of generality, we may assume that for all $k = 1, \dots, d$, $a^{(k)} = 1$ and $b^{(k)} = 0$. Then (3.1), which implies (2.6), gives

$$\frac{nF^{(k)}(\alpha_n^{(k)}x + \beta_n^{(k)})}{\hat{r}_n} \rightarrow m^{(k)}$$

for all x satisfying $1 + \gamma^{(k)}x > 0$, and (3.2) implies

$$\frac{nF^{(k,l)}(\alpha_n^{(k)}x + \beta_n^{(k)}, \alpha_n^{(l)}y + \beta_n^{(l)})}{\hat{r}_n} \rightarrow \rho^{(k,l)} \sqrt{m^{(k)}m^{(l)}}$$

for all x and y satisfying $1 + \gamma^{(k)}x > 0$ and $1 + \gamma^{(l)}y > 0$ respectively. Letting $x = y = 0$ in the above limits and writing $I_n^{(k)}$ for the indicator of set $\{X_1^{(k)} \leq \beta_n^{(k)}\}$, we then obtain

$$\begin{aligned} \frac{n\mathbf{P}\{I_n^{(k)} = 1\}}{\hat{r}_n} &= \frac{nF^{(k)}(\beta_n^{(k)})}{\hat{r}_n} \rightarrow m^{(k)} \quad \text{for } k = 1, \dots, d; \\ \frac{n\mathbf{P}\{I_n^{(k)} = I_n^{(l)} = 1\}}{\hat{r}_n} &= \frac{nF^{(k,l)}(\beta_n^{(k)}, \beta_n^{(l)})}{\hat{r}_n} \rightarrow \rho^{(k,l)} \sqrt{m^{(k)}m^{(l)}} \\ &\quad \text{for } 1 \leq k < l \leq d. \end{aligned}$$

Define a measure on Z by

$$\mu_n(A) = \frac{n\mathbf{P}\{(I_n^{(1)}, \dots, I_n^{(d)}) \in A\}}{\hat{r}_n} \quad \text{for every } A \subset Z$$

and choose a subsequence $\{\mu_{n'}\}$ such that for every $A \subset Z$, the limit of the $\{\mu_{n'}(A)\}$ exists. Obviously, $\mu := \lim_{n' \rightarrow \infty} \mu_{n'}$ determines a measure on Z satisfying (3.3) and (3.4). This gives $\Sigma \in S(m)$, so that

$$D \subset \{\Phi_{\Sigma}(v^{(1)}, \dots, v^{(d)}): v^{(k)} \in V_0, k = 1, \dots, d; \Sigma \in S(m)\}.$$

Conversely, for a given $m \in M$ and a $\Sigma \in S(m)$, let μ be a measure on $S(m)$ such that (3.3) and (3.4) hold and denote

$$\hat{m}^{(i_1, \dots, i_s)} = \mu(\{x \in Z: x^{(k)} = 1, k = i_1, \dots, i_s, \text{ and } x^{(k)} = 0 \text{ otherwise}\})$$

for $s \geq 1$ and $1 \leq i_1 < \dots < i_s \leq d$. Since $\hat{m}^{(i_1, \dots, i_s)} \geq 0$, we may construct i.i.d. random vectors $\{(Y_n^{(1)}, \dots, Y_n^{(d)}), n \geq 1\}$ such that for $y^{(1)}, \dots, y^{(d)} > 0$,

$$\begin{aligned} & \mathbf{P}\{Y_n^{(k)} \geq y^{(k)}, k = 1, \dots, d\} \\ &= \exp \left\{ - \sum_{s=1}^d \sum_{1 \leq i_1 < \dots < i_s \leq d} \hat{m}^{(i_1, \dots, i_s)} \max(y^{(i_1)}, \dots, y^{(i_s)}) \right\} \quad (3.5) \end{aligned}$$

(see [7] or [4]). For $s \geq 1$ and $1 \leq i_1 < \dots < i_s \leq d$, let

$$m^{(i_1, \dots, i_s)} = \sum_{t=s}^d \sum_{\{j_1, \dots, j_t\} \supset \{i_1, \dots, i_s\}} \hat{m}^{(j_1, \dots, j_t)}.$$

Then we have

$$\begin{aligned} m^{(i_1, \dots, i_s)} &= \mu(\{x \in Z: x^{(k)} = 1, k = i_1, \dots, i_s\}); \\ \hat{m}^{(i_1, \dots, i_s)} &= \sum_{t=s}^d (-1)^{t-s} \sum_{\{j_1, \dots, j_t\} \supset \{i_1, \dots, i_s\}} m^{(j_1, \dots, j_t)}. \end{aligned}$$

Noting that for any real numbers $u_1, \dots, u_d$,

$$\min(u_1, \dots, u_d) = \sum_{s=1}^d (-1)^{s+1} \sum_{1 \leq i_1 < \dots < i_s \leq d} \max(u_{i_1}, \dots, u_{i_s}),$$

we may rewrite (3.5) as

$$\begin{aligned} & \mathbf{P}\{Y_n^{(k)} \geq y^{(k)}, k = 1, \dots, d\} \\ &= \exp \left\{ \sum_{s=1}^d (-1)^s \sum_{1 \leq i_1 < \dots < i_s \leq d} m^{(i_1, \dots, i_s)} \min(y^{(i_1)}, \dots, y^{(i_s)}) \right\}. \quad (3.6) \end{aligned}$$

Moreover, for a given $v = (v^{(1)}, \dots, v^{(d)}) \in V_0$, let $F^{(k)}$ be the d.f. for which there exist $\alpha_n^{(k)} > 0$ and $\beta_n^{(k)}$ such that (3.1) holds, write $(F^{(k)})^-$ for its left continuous inverse and define

$$X_n^{(k)} = (F^{(k)})^- (1 - \exp\{-m^{(k)} Y_n^{(k)}\}) \quad \text{for all } n \geq 1 \text{ and } k = 1, \dots, d.$$

We will show that by choosing $\{\alpha_n = (\alpha_n^{(1)}, \dots, \alpha_n^{(d)})\}$ and $\{\beta_n = (\beta_n^{(1)}, \dots, \beta_n^{(d)})\}$, (2.2) holds for the $\{X_n = (X_n^{(1)}, \dots, X_n^{(d)})\}$ defined as above and $\{r_n = (r_n^{(1)}, \dots, r_n^{(d)})\}$ satisfying (1.5). In fact, at this moment (3.6) gives

$$\begin{aligned} \mathbf{P}\{X_n^{(k)} \leq x\} &= \mathbf{P}\{1 - \exp\{-m^{(k)} Y_n^{(k)}\} \leq F^{(k)}(x)\} \\ &= \mathbf{P}\left\{Y_n^{(k)} \leq -\frac{\log[1 - F^{(k)}(x)]}{m^{(k)}}\right\} = 1 - \mathbf{P}\left\{Y_n^{(k)} > -\frac{\log[1 - F^{(k)}(x)]}{m^{(k)}}\right\} \\ &= 1 - [1 - F^{(k)}(x)] = F^{(k)}(x), \end{aligned}$$

i.e., X_n has $\{F^{(k)}, k = 1, \dots, d\}$ as its one dimensional d.f.'s, so that (3.1) is satisfied. On the other hand, the two dimensional d.f.'s of X_n are

$$\begin{aligned} F^{(k,l)}(x, y) &= \mathbf{P}\{X_n^{(k)} \leq x, X_n^{(l)} \leq y\} \\ &= 1 - \mathbf{P}\{X_n^{(k)} > x\} - \mathbf{P}\{X_n^{(l)} > y\} + \mathbf{P}\{X_n^{(k)} > x, X_n^{(l)} > y\} \\ &= F^{(k)}(x) F^{(l)}(y) + [1 - F^{(k)}(x)] [1 - F^{(l)}(y)] \\ &\quad \times \left\{ \exp \left\{ -m^{(k,l)} \max \left(\frac{\log[1 - F^{(k)}(x)]}{m^{(k)}}, \frac{\log[1 - F^{(l)}(y)]}{m^{(l)}} \right) \right\} - 1 \right\}. \end{aligned}$$

Letting $F_n^{(k)}(x) = F^{(k)}(\alpha_n^{(k)}x + \beta_n^{(k)})$ and noting that $m^{(k,l)} = \rho^{(k,l)}\sqrt{m^{(k)}m^{(l)}}$, we get

$$\begin{aligned} &\lim_{n \rightarrow \infty} \bar{\rho}_n^{(k,l)}(x, y) \\ &= \lim_{n \rightarrow \infty} \frac{F^{(k,l)}(\alpha_n^{(k)}x + \beta_n^{(k)}, \alpha_n^{(l)}y + \beta_n^{(l)})}{\sqrt{\lambda_n^{(k)}\lambda_n^{(l)}}} \\ &= \lim_{n \rightarrow \infty} \frac{1}{\sqrt{\lambda_n^{(k)}\lambda_n^{(l)}}} \left\{ \exp \left\{ -m^{(k,l)} \max \left(\frac{\log[1 - F_n^{(k)}(x)]}{m^{(k)}}, \frac{\log[1 - F_n^{(l)}(y)]}{m^{(l)}} \right) \right\} - 1 \right\} \\ &= - \lim_{n \rightarrow \infty} \frac{m^{(k,l)}}{\sqrt{\lambda_n^{(k)}\lambda_n^{(l)}}} \max \left(\frac{\log[1 - F_n^{(k)}(x)]}{m^{(k)}}, \frac{\log[1 - F_n^{(l)}(y)]}{m^{(l)}} \right) \\ &= \lim_{n \rightarrow \infty} \frac{m^{(k,l)}}{\sqrt{\lambda_n^{(k)}\lambda_n^{(l)}}} \min \left(\frac{F^{(k)}(\alpha_n^{(k)}x + \beta_n^{(k)})}{m^{(k)}}, \frac{F^{(l)}(\alpha_n^{(l)}y + \beta_n^{(l)})}{m^{(l)}} \right) \\ &= \lim_{n \rightarrow \infty} \rho^{(k,l)} \min \left(\sqrt{\frac{\lambda_n^{(k)}m^{(l)}}{m^{(k)}\lambda_n^{(l)}}}, \sqrt{\frac{\lambda_n^{(l)}m^{(k)}}{m^{(l)}\lambda_n^{(k)}}} \right) = \rho^{(k,l)}, \end{aligned}$$

so that (3.2) holds. Hence by Lemma 3.1, (2.2) is obtained. This proves

$$D \supset \{\Phi_\Sigma(v^{(1)}, \dots, v^{(d)}): v^{(k)} \in V_0, k = 1, \dots, d; \Sigma \in S(m)\}.$$

The proof of the theorem is now complete.

4. Generalization

Consider the case that components $r_n^{(1)}, \dots, r_n^{(d)}$ of the ranks r_n can be split into several groups and in each group, its components tend to infinity with the same rate. For simplification, we may assume that the groups are

in a natural order. That is, letting $\widehat{d}_s = \widehat{d}_{s-1} + d_s$ for $s = 1, \dots, u$ with $\widehat{d}_0 = 0$ and $\widehat{d}_u = d$, the ranks may be written as

$$r_n = (r_n^{(1)}, \dots, r_n^{(d_1)}; \dots; r_n^{(\widehat{d}_{u-1}+1)}, \dots, r_n^{(d)}).$$

First, we give the following.

Lemma 4.1. *Under (1.1), suppose that there exist a positive α_n and a $\beta_n \in \mathbf{R}^d$ such that (2.2) holds. If*

$$\frac{r_n^{(k)}}{r_n^{(l)}} \rightarrow 0$$

for some $1 \leq k < l \leq d$, then the (k, l) th element of matrix Σ

$$\rho^{(k, l)} = 0.$$

P r o o f. Under conditions of the lemma, (2.6) holds so that

$$\begin{aligned} 0 \leq \bar{\rho}_n^{(k, l)}(x, y) &= \frac{F^{(k, l)}(\alpha_n^{(k)}x + \beta_n^{(k)}, \alpha_n^{(l)}y + \beta_n^{(l)})}{\sqrt{\lambda_n^{(k)}\lambda_n^{(l)}}} \\ &\leq \frac{F^{(k)}(\alpha_n^{(k)}x + \beta_n^{(k)})}{\sqrt{\lambda_n^{(k)}\lambda_n^{(l)}}} \sim \sqrt{\frac{\lambda_n^{(k)}}{\lambda_n^{(l)}}} = \sqrt{\frac{r_n^{(k)}}{r_n^{(l)}}} \rightarrow 0. \end{aligned}$$

This completes the proof of the lemma.

For $s = 1, \dots, u$, denote

$$\begin{aligned} M_s &= \left\{ m_s = (m_s^{(1)}, \dots, m_s^{(d_s)}): m_s^{(k)} > 0 \text{ for } k = 1, \dots, d_s \right. \\ &\quad \left. \text{and } \sum_{k=1}^{d_s} m_s^{(k)} = 1 \right\}; \\ \widehat{r}_n^{(s)} &= \sum_{k=1}^{d_s} r_n^{(\widehat{d}_{s-1}+k)}; \quad \widehat{v}_s = (v^{(\widehat{d}_{s-1}+1)}, \dots, v^{(\widehat{d}_s)}). \end{aligned}$$

Then we get a generalization of Theorem 3.1 as follows.

Theorem 4.1. *If*

$$r_n^{(k)} \rightarrow \infty \text{ nondecreasingly and } \lambda_n^{(k)} \rightarrow 0 \quad (4.1)$$

hold for all $k = 1, \dots, d$,

$$\frac{r_n^{(\widehat{d}_{s-1}+k)}}{\widehat{r}_n^{(s)}} \rightarrow m_s^{(k)} > 0, \quad k = 1, \dots, d_s, \quad (4.2)$$

hold for all $s = 1, \dots, u$ and

$$\frac{\hat{r}_n^{(s)}}{\hat{r}_n^{(t)}} \longrightarrow 0 \text{ or } \infty \quad (4.3)$$

hold for all $1 \leq s < t \leq u$, then

$$D = \{ \Phi_{\Sigma_1}(\hat{v}_1) \cdots \Phi_{\Sigma_u}(\hat{v}_u) : \hat{v}_s \in V_0^{d_s}, \Sigma_s \in S(m_s), s = 1, \dots, u \}$$

where $m_s \in M_s$ and $\Sigma_s = (\rho_s^{(k,l)})_{d_s \times d_s}$. Moreover, (2.2) holds for a $G \in D$ if and only if for each $k = 1, \dots, d$, (3.1) holds and for every $s = 1, \dots, u$ and $1 \leq k < l \leq d_s$,

$$\rho_n^{(\hat{d}_{s-1}+k, \hat{d}_{s-1}+l)}(x, y) \longrightarrow \rho_s^{(k,l)}$$

holds for one or all $(x, y) \in C(v^{(\hat{d}_{s-1}+k)}) \times C(v^{(\hat{d}_{s-1}+l)})$.

P r o o f. It follows from Lemma 4.1 that under the conditions of the theorem,

$$\left\{ \left(\frac{X_{r_n;n}^{(\hat{d}_{s-1}+1)} - \beta_n^{(\hat{d}_{s-1}+1)}}{\alpha_n^{(\hat{d}_{s-1}+1)}}, \dots, \frac{X_{r_n;n}^{(\hat{d}_s)} - \beta_n^{(\hat{d}_s)}}{\alpha_n^{(\hat{d}_s)}} \right), s = 1, \dots, u \right\}$$

are asymptotically independent. Moreover, by Theorem 3.1, for every $s = 1, \dots, u$, we have

$$\left(\frac{X_{r_n;n}^{(\hat{d}_{s-1}+1)} - \beta_n^{(\hat{d}_{s-1}+1)}}{\alpha_n^{(\hat{d}_{s-1}+1)}}, \dots, \frac{X_{r_n;n}^{(\hat{d}_s)} - \beta_n^{(\hat{d}_s)}}{\alpha_n^{(\hat{d}_s)}} \right) \xrightarrow{d} \Phi_{\Sigma_s}(\hat{v}_s).$$

Hence the assertion of the theorem is true.

Now we will give an important corollary to Theorem 4.1 on asymptotic normality of multivariate intermediate order statistics.

Corollary 4.1. Suppose that (4.1)–(4.3) hold and that for each $k = 1, \dots, d$, the one dimensional marginal d.f. $F^{(k)}$ has the positive derivative. If there exist positive α_n and β_n such that (2.2) holds, then the limiting d.f. has to be

$$G(x^{(1)}, \dots, x^{(d)}) = \Phi_{\Sigma_1}(x^{(1)}, \dots, x^{(d_1)}) \times \cdots \times \Phi_{\Sigma_u}(x^{(\hat{d}_{u-1}+1)}, \dots, x^{(d)})$$

where $\Sigma_s \in S(m_s)$; moreover, we may choose

$$\beta_n^{(k)} = (F^{(k)})^{-1}(\lambda_n^{(k)}), \quad \alpha_n^{(k)} = \sqrt{\frac{\lambda_n^{(k)}}{n}} [(F^{(k)})^{-1}]'(\beta_n^{(k)}) \quad (4.4)$$

and then (2.2) holds if and only if for every $s = 1, \dots, u$ and $1 \leq k < l \leq d_s$,

$$\rho_n^{(\hat{d}_{s-1}+k, \hat{d}_{s-1}+l)}(x, y) \longrightarrow \rho_s^{(k,l)}$$

holds for one or all $(x, y) \in \mathbf{R}^2$.

P r o o f. It is easily seen that (3.1) is equivalent to

$$\frac{1}{\alpha_n^{(k)}} \left\{ (F^{(k)})^- \left(\sqrt{\frac{\lambda_n^{(k)}}{n}} y + \lambda_n^{(k)} \right) - \beta_n^{(k)} \right\} \xrightarrow{w} (v^{(k)})^-(y).$$

If $F^{(k)}$ has the positive derivative, then one may easily obtain that for the $\alpha_n^{(k)}$ and $\beta_n^{(k)}$ determined in (4.4),

$$(v^{(k)})^-(y) = y \quad \text{for all } y \in \mathbf{R},$$

so that the conclusions of the corollary follow from Theorem 4.1.

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