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CUTPOINTS AND EXCHANGEABLE EVENTS FOR RANDOM WALKS¹⁾

Для марковской цепи $\{S_n\}$ назовем S_k *точкой разреза* (cutpoint) и k — *моментом разреза* (cut-epoch), если не существует возможных переходов из S_i в S_j ни при каких $i < k < j$. Покажем, что невозвратное случайное блуждание с ограниченным размером шага на целочисленной решетке имеет бесконечно много точек разреза почти наверное. Для простого случайного блуждания на $\mathbb{Z}^d$, $d \geq 4$, этот факт установлен Лолером (Lawler). Далее, пусть G — конечно порожденная группа с по крайней мере полиномиальным порядком роста степени 5, тогда для любого симметричного случайного блуждания на G такого, что шаги имеют ограниченный носитель, который порождает G , моменты разреза имеют положительную плотность.

Также будет показано, что для любой марковской цепи, имеющей бесконечно много точек разреза почти наверное, предельные числа размещения (eventual occupation numbers) порождают перестановочную σ -алгебру. Объединяя эти результаты, можно получить ответ на вопрос, поставленный Каймановичем (Kaimanovich), и частично разрешить предположение Диакониса (Diaconis) и Фридмана (Freedman).

Ключевые слова и фразы: точка разреза, перестановочный, цепь Маркова, пуассоновская граница, случайные блуждания на группах.

1. Introduction

Let $S_0, S_1, \dots$ be a homogenous Markov chain on a countable state space X . Considering an event to be a set of trajectories, we say an event A is *exchangeable* if it is invariant under finite permutations, i.e., if $(s_0, s_1, \dots) \in A$ then so is $(s_{\pi(0)}, \dots, s_{\pi(n)}, s_{n+1}, \dots)$ for any n and any permutation π of $\{0, \dots, n\}$. A transient process visits each state only finitely often, and so for each state x there is a random variable $V(x)$ which counts the number of visits, $V(x) = \#\{n \geq 0 \mid S_n = x\}$. We call the collection $\{V(x)\}_{x \in X}$ the *occupation numbers* of the process. If an event is defined in terms of these occupation numbers then it is invariant under all permutations,

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of these occupation numbers then it is invariant under all permutations, even infinite ones, and thus is exchangeable. A natural question, posed by Kaimanovich [10], is under what conditions does the converse hold:

For which transient Markov chains is the indicator of every exchangeable event equal a.e. to a function of the occupation numbers?

Kaimanovich [10] showed that this is the case for random walks on $\mathbf{Z}$ with non-zero mean, and explicitly asked about transient mean-zero random walks in higher dimensions.

We now state our main result in greater generality, for random walks on finitely-generated discrete groups. Let G be such a group and let p be a probability measure on G whose support U is finite and generates G as a semigroup. The transition probabilities are then defined by $p(x, y) = p(x^{-1}y)$ for x, y in G . More detailed definitions are given in the next section.

Theorem 1.1. *Let $\{S_n\}$ be a transient random walk with bounded increments on the lattice $\mathbf{Z}^d$, or a symmetric random walk, with bounded generating support, on a group G that has growth at least polynomial of degree 5. Then any exchangeable event is given by the occupation numbers, up to a set of measure zero.*

The key idea for the proof of this theorem is to apply a sample-path property of transient random walks, the existence of *cutpoints*. This method also proves a conjecture of Diaconis and Freedman [4], specialized to such random walks; see Corollary 1.2 below. For general transient Markov chains this conjecture, as well as Kaimanovich's question, are still unresolved. A closely related question posed by Kaimanovich and Vershik [11], is still open even for simple random walk on $\mathbf{Z}^3$ — see Remark 2 at the end of the paper.

Let $S_0, S_1, \dots$ be a Markov chain on a countable state space, with transition probabilities $p(x, y) = \mathbf{P}\{S_{n+1} = y \mid S_n = x\}$. The initial distribution of S_0 is arbitrary. We say there is a *permitted transition* from x to y if $p(x, y) > 0$.

Definition 1. If there is no permitted transition from S_i to S_j for all i and j with $0 \leq i < n < j$, then n is a *cut-epoch* for S and S_n is a *cutpoint*.

If S_n is a cutpoint then deleting S_n from the trajectory cuts the trajectory into two disjoint components, $S[0, n)$ and $S(n, \infty)$, with no possible transition from the first to the second.

In Section 2 we establish the existence of cutpoints. We only consider step distributions p on groups G , where the support U of p is finite and generates G as a semigroup.

Theorem 1.2. (a) *Any transient random walk with bounded increments on the lattice $\mathbf{Z}^d$ has infinitely many cutpoints with probability 1.*

(b) *If the growth of G is at least polynomial of degree 5, and p is symmetric, then S_n is a cutpoint for infinitely many n with probability 1.*

A random walk on $\mathbf{Z}^d$ can be transient only in two different ways: either the increment distribution p has non-zero mean, or p has mean zero and

$d \geq 3$. In the first case the theorem is easy to prove because it reduces to the case of a one-dimensional walk. For simple random walk on $\mathbf{Z}^d$ with $d \geq 4$, the existence of cutpoints is essentially proved by Lawler [13] (he considered a related notion of «loop-free» points), but the case $d = 3$ is new even for simple random walk. Part (b) and its proof are based on a result of Lawler ([13, Sections 3.2 and 7.7]).

In Section 3 we return to the problem of describing the exchangeable σ -field. Theorem 1.1 is an immediate consequence of Theorem 1.2 and the following general statement.

Proposition 1.1. *Let $\{S_n\}$ be a Markov chain. If $\{S_n\}$ almost surely has infinitely many cutpoints, then the exchangeable σ -field is generated by the occupation numbers.*

In 1955 Hewitt and Savage [7] proved that the exchangeable σ -field is trivial if S_n are i.i.d. Blackwell and Freedman [2] determined the exchangeable σ -field for recurrent Markov chains; it is atomic, and depends only on the distribution of S_0 . In the case that $\{S_n\}$ is a sequence of independent random variables with a common state space X , Aldous and Pitman [1] gave a necessary condition for the exchangeable σ -field to be trivial, a condition which is sufficient if X is finite. They also gave other sufficient conditions for infinite X . Palacios [16] extended their results to some recurrent non-homogeneous Markov chains.

For transient Markov chains, the exchangeable σ -field has an alternative representation, described in [9] and [10]. Let $V_n(x) = \#\{k \mid 0 \leq k \leq n \text{ and } S_k = x\}$ denote the number of visits to x up to time n . Then $\{(S_n, V_{n-1})\}$ is again a Markov chain, whose states are pairs (x, v) where $x \in X$, and $v: X \rightarrow \mathbf{Z}$ vanishes at all but finitely many points. The exchangeable σ -field of $\{S_n\}$ is exactly the (completed) tail σ -field of the extended chain $\{(S_n, V_{n-1})\}$. We now specialize to the case where X is a group G . There is a natural multiplication rule for the states of the extended chain, which makes them into a group. First let $(T_x v)(y) = v(x^{-1}y)$ for $x, y \in G$ and then the multiplication is $(x, v)(y, w) = (xy, v + T_x w)$. This group is the semi-direct product by G of the direct sum $\bigoplus_G \mathbf{Z}$; it is called the *wreath product* of $\mathbf{Z}$ by G , and denoted by $\mathbf{Z} \wr G$. Let $\delta_e: G \rightarrow \mathbf{Z}$ be 1 at the identity e and 0 elsewhere. Given a probability measure p on G we can define a new measure μ_p on $\mathbf{Z} \wr G$ by $\mu_p(x, \delta_e) = p(x)$ for $x \in G$ and μ_p vanishes elsewhere. The process (S_n, V_{n-1}) is now a random walk on $\mathbf{Z} \wr G$ starting at $(S_0, 0)$, with transition probabilities given by μ_p . Applying the general boundary theory in [11] and [10] yields the following consequence of Theorem 1.2 and Proposition 1.1.

Corollary 1.1. *If the group G and the step distribution p satisfy the conditions in part (a) or (b) of Theorem 1.2, then the distribution of the occupation field V can be identified with the Poisson boundary for the random walk on $\mathbf{Z} \wr G$ with step distribution μ_p .*

While studying extensions of De Finetti's theorem to Markov chains,

Diaconis and Freedman [4] stated a general conjecture for transient Markov chains $\{S_n\}$. Let $M_n(x, y)$ be the number of transitions made from x to y up to time n , so that $M_n(x, y)$ increases to a finite limit $M(x, y)$ as $n \rightarrow \infty$. They conjectured that the intersection of σ -fields

$$\bigcap_n \sigma(M_n, S_n, S_{n+1}, S_{n+2}, \dots)$$

is always generated by M .

Corollary 1.2. *The intersection $\bigcap_n \sigma(M_n, S_n, S_{n+1}, \dots)$ is generated by M whenever the Markov chain $\{S_n\}$ has infinitely many cutpoints almost surely.*

In particular, this applies to the random walks in Theorem 1.2.

P r o o f. The transition counts M_n are the visit counts for the 2-block chain $\{S_n, S_{n+1}\}$. Therefore the intersection in the corollary can be identified as the exchangeable σ -field for the 2-block chain. If k is a cut-epoch for the original chain $\{S_n\}$, then it is clearly also a cut-epoch for the 2-block chain. Hence, since M is the occupation field for the 2-block chain, the assertion of the corollary follows from Proposition 1.1.

2. Cutpoints

In this section, G will always be a finitely-generated, infinite discrete group with identity e , and p a symmetric probability measure on G , whose support U is finite and generates G as a semigroup. The measure p defines transition probabilities by $p(x, y) = p(x^{-1}y)$ and these are bounded away from zero by $p_{\min} = \min_{u \in U} p(u)$. A point y is a *neighbor* of x if y is in xU . Let $U^n = \{u_1 u_2 \dots u_n \mid u_i \in U\}$ for $n \geq 1$ and $U^0 = \{e\}$. The union of these sets is G . We write $\rho(x) = \min\{n \mid x \in U^n\}$ for $x \in G$.

The ball U_n of radius n about the identity in the Cayley graph is $U^0 \cup U^1 \cup \dots \cup U^n$. The group G is of *polynomial growth* if there exists $d > 0$ such that $\#U_n = O(n^d)$. In this case the infimum of these d is called the *degree of growth* of G . This does not depend on the choice of generators U , but is a property of the group G . It is known from work of Gromov [5] that the degree is always an integer.

Let p_n be the n th convolution power of p . The *Green function* θ , evaluated at a point x , is the expected number of visits to x of a random walk starting at e : $\theta(x) = \sum_{n=0}^{\infty} p_n(x)$.

Recall that the walk is assumed transient, hence $\theta(x)$ is finite for every x . We will need a discrete version of the *gradient* of the Green function, $\nabla\theta(x) = \max_{y \in xU} |\theta(y) - \theta(x)|$.

We will be interested in the following three hypotheses in the case $d \geq 3$:

$$(A_0 \rho(x))^{2-d} \leq \theta(x) \leq \left(\frac{\rho(x)}{A_1} \right)^{2-d} \quad (1)$$

for all $x \in G$ for some A_0, A_1 ;

$$\nabla\theta(x) \leq \left(\frac{\rho(x)}{A_2}\right)^{1-d} \quad \text{for all } x \in G \quad \text{for some } A_2; \quad (2)$$

$$(A_3\rho(x))^{1-d} \leq \nabla\theta(x) \quad (3)$$

for all but finitely many $x \in G$ for some A_3 .

In the proposition below we shall exhibit a particular kind of cutpoint. If the random walk $\{S_n\}$ starts inside some sphere and visits that sphere only once, at time n say, then n is a cut-epoch for S , because deleting S_n disconnects the past of the trajectory, which is inside the sphere, from the future, which is outside. This approach works nicely to prove existence of approximate cut-spheres for Brownian motion (see Proposition 1 in [3]) but in the generality of Theorem 1.2 more work is required to obtain the necessary estimates. The obvious choice for such a sphere is the set of points at some fixed distance from the identity, but we find it easier to work with sets on which the Green function is almost constant. For the rest of this section, given d , A_0 and A_2 we take

$$0 < \eta < d - 2 \quad \text{and} \quad \delta = \eta(A_0 A_2)^{1-d}.$$

For $k \geq 1$ we define

$$B_k = \left\{ x \in G \mid \theta(x) \geq \left(\frac{\delta}{k}\right)^{d-2} \right\} \quad \text{and} \quad (4)$$

$$\partial B_k = \{x \in B_k \mid xU \text{ contains some } y \notin B_k\}.$$

The difference set $B_k \setminus \partial B_k$ is called the *interior* of B_k . If x is in the interior of B_k and $y \notin B_k$, then clearly every path from x to y intersects ∂B_k .

Let cut_k be the event that $\{S_n\}$ starts in the interior of B_k and visits ∂B_k exactly once:

$$\text{cut}_k = \bigcup_{n=1}^{\infty} \{S_i \in B_k \setminus \partial B_k \text{ for } i < n, S_n \in \partial B_k, \text{ and} \\ S_j \notin B_k \text{ for } j > n\}. \quad (5)$$

Proposition 2.1. *If G is of polynomial growth of degree $d \geq 3$ and (1)–(3) hold then with probability 1, the events cut_k happen for infinitely many k .*

In Theorem 5.1 of [6], Hebisch and Saloff-Coste show that if p is symmetric then (1), (2) hold without any further conditions on G .

Corollary 2.1. *If the measure p is symmetric, the group G is of polynomial growth of degree $d \geq 3$, and (3) holds, then with probability 1, the events cut_k happen for infinitely many k .*

The proof of Proposition 2.1 is based on the second moment method and the following three lemmas. The first lemma separates the spheres ∂B_k for distinct k , the second compares hitting probabilities of two spheres for

the walk started in the shell between them, and the third lemma bounds the maximal distance from a point $x \in \partial B_k$ to ∂B_{k+1} .

Lemma 2.1. *Let $r_k = (\delta/k)^{d-2}$ and $R_k = (1 + \eta/k)(\delta/k)^{d-2}$. If (1), (2) hold then for all $k \geq 1$ and $x \in \partial B_k$, we have $r_k \leq \theta(x) \leq R_k$.*

Proof. Let $x \in \partial B_k$ and let $y \notin B_k$ be a neighbor of x , i.e. $y \in xU$. Then (1) and (4) yield $(A_0\rho(y))^{2-d} \leq \theta(y) < (k/\delta)^{2-d}$ hence $\rho(y) > k/(\delta A_0)$, and by (2) we get $\nabla\theta(y) \leq (\delta A_0 A_2/k)^{d-1}$. An upper bound for $\theta(x)$ is

$$\theta(x) \leq \theta(y) + \nabla\theta(y) \leq \left(1 + \frac{\delta(A_0 A_2)^{(d-1)}}{k}\right) \left(\frac{\delta}{k}\right)^{d-2} = R_k.$$

We note the following facts about these bounds:

$$r_k - R_{k+1} \sim \delta^{d-2} \frac{d-2-\eta}{k^{d-1}} \quad \text{as } k \rightarrow \infty; \quad (6)$$

$$R_k - r_{k+1} \sim \delta^{d-2} \frac{d-2+\eta}{k^{d-1}} \quad \text{as } k \rightarrow \infty; \quad (7)$$

$$0 < r_k - r_m \leq \delta^{d-2} (d-2) \frac{m-k}{k^{d-2}m} \quad \text{for } 1 \leq k < m; \quad (8)$$

$$R_k - R_m \geq \delta^{d-2} \frac{m-k}{k^{d-2}m} > 0 \quad \text{for } 1 \leq k < m. \quad (9)$$

Here the symbol $\sim$ means that the ratio of the two sides tends to 1. The last two inequalities follow from the expansion

$$\left(\frac{1}{k}\right)^{d-2} - \left(\frac{1}{m}\right)^{d-2} = \left(\frac{1}{k} - \frac{1}{m}\right) \sum_{j=0}^{d-3} \left(\frac{1}{k}\right)^j \left(\frac{1}{m}\right)^{d-3-j}.$$

The asymptotic relation (6) implies that for k sufficiently large, $r_k - R_{k+1} > 0$, and hence the spheres ∂B_k and ∂B_{k+1} are disjoint.

Lemma 2.2. *If (1), (2) hold then there exist a real C and an integer $K \geq 1$ such that for all $K \leq k < m$ and all starting points $x \in \partial B_{k+1}$,*

$$\frac{m}{Ck(m-k)} \leq \mathbf{P}_x\{\{S_n\} \text{ visits } \partial B_m \text{ before } \partial B_k\} \leq \frac{Cm}{k(m-k)}, \quad (10)$$

and consequently for all $x \in \partial B_{k+1}$,

$$\frac{1}{Ck} \leq \mathbf{P}_x\{\{S_n\} \text{ never visits } \partial B_k\} \leq \frac{C}{k}. \quad (11)$$

Proof. Let $h(x) = \mathbf{P}_x\{\{S_n\} \text{ visits } \partial B_m \text{ before } \partial B_k\}$. The function h is p -harmonic in the shell $B_m \setminus (\partial B_m \cup B_k)$. It has boundary values 0 on ∂B_k and 1 on ∂B_m . The Green function θ is p -harmonic at all points except the identity. Let

$$u(x) = \frac{r_k - \theta(x)}{r_k - r_m} \quad \text{and} \quad U(x) = \frac{R_k - \theta(x)}{R_k - R_m}.$$

It follows from Lemma 2.1 that $u(x) \leq h(x) \leq U(x)$ for $x \in \partial B_k \cup \partial B_m$. The maximum principle for p -harmonic functions then forces these inequalities to extend to $x \in B_m \setminus B_k$. For $x \in \partial B_{k+1}$ we infer that

$$u(x) \geq \frac{r_k - R_{k+1}}{r_k - r_m} \quad \text{and} \quad U(x) \leq \frac{R_k - r_{k+1}}{R_k - R_m}.$$

The assertion (10) is now implied by (6)–(9), and the inequalities (11) follow by letting $m \rightarrow \infty$.

The following lemma says that for each point $x \in \partial B_k$ there is a path starting at x and leading to ∂B_{k+1} which does not re-enter ∂B_k , and the length of the path can be bounded uniformly in x and k .

Lemma 2.3. *If (1)–(3) hold then there are integers J and k_0 such that for any $k \geq k_0$ and any $x \in \partial B_k$ there is a finite path $x_0 = x, x_1, \dots, x_j$ in G of length $j \leq J$ that terminates in $x_j \in \partial B_{k+1}$, such that $x_i \in x_{i-1}U$ and $x_i \notin \partial B_k$ for $i = 1, \dots, j$.*

P r o o f. Choose k_0 so large that the inequality in (3) holds for x outside B_{k_0} . Let $k \geq k_0$ and $x \in \partial B_k$. The definition of ∂B_k implies that x has a neighbor $y \in xU$ which is outside B_k . Let $x_1 = y$. We now follow the line of steepest descent of the Green function. Having defined x_i choose $x_{i+1} \in x_iU$ such that $\theta(x_{i+1}) = \min_{z \in x_iU} \theta(z)$. Since θ is p -harmonic and $\theta(x_0) > \theta(x_1)$, the sequence $\{\theta(x_i)\}_{i \geq 0}$ is strictly decreasing. The group G is discrete and θ vanishes at ∞ so $\theta(x_i) \downarrow 0$. Choose j such that $\theta(x_j) \geq r_{k+1} > \theta(x_{j+1})$.

We now give an upper bound for j . Obviously

$$(j-1) \min_{i < j} (\theta(x_i) - \theta(x_{i+1})) \leq \theta(x_1) - \theta(x_j). \quad (12)$$

Each x_i has a neighbor $w \in x_iU$ such that $\theta(w)$ differs from $\theta(x_i)$ by $\nabla\theta(x_i)$. If $\theta(w)$ is greater than $\theta(x_i)$ then the (weighted) average of θ over the other neighbors of x_i must be correspondingly less, and it follows from the definition of x_{i+1} that $\theta(x_i) - \theta(x_{i+1}) \geq p_{\min} \nabla\theta(x_i)$. Since $x_i \in B_{k+1}$ for $i \leq j$, we have

$$\left(\frac{\delta}{k+1}\right)^{d-2} \leq \theta(x_i) \leq \left(\frac{A_1}{\rho(x_i)}\right)^{d-2}$$

where the right-hand inequality is from (1). Therefore $\rho(x_i) \leq A_1(k+1)/\delta$, and $\nabla\theta(x_i) \geq (A_1 A_3(k+1)/\delta)^{1-d}$ by (3). On the other hand

$$\theta(x_1) - \theta(x_j) \leq r_k - r_{k+1} \leq \delta^{d-2}(d-2)k^{1-d}$$

by (8). Substituting this into (12) gives

$$(j-1) p_{\min} \left(\frac{A_1 A_3(k+1)}{\delta}\right)^{1-d} \leq \delta^{d-2}(d-2)k^{1-d}.$$

Recalling that $\delta = \eta(A_0 A_2)^{1-d}$, we conclude that

$$j \leq J = 1 + 2^{d-1} (A_0 A_1 A_2 A_3)^{d-1} \frac{d-2}{\eta p_{\min}} \quad \text{for all } k \geq k_0.$$

Proof of Proposition 2.1. As in the preceding Lemma, choose k_0 so that the inequality (3) holds for x outside B_{k_0} . Let $k \geq k_0$. One way in which a trajectory (starting from the interior of B_k) can be in cut_k is that it moves randomly until it reaches ∂B_k , then marches out to ∂B_{k+1} , and never visits ∂B_k again. Now the probability of reaching ∂B_k is 1. According to Lemma 2.3, from each $x \in \partial B_k$ there is a path of length at most J , which leads to ∂B_{k+1} without re-entering ∂B_k . Thus the probability of going from $x \in \partial B_k$ to ∂B_{k+1} without revisiting ∂B_k is at least $p_{\min}^J$. Therefore, using the lower bound in (11), we obtain $P(\text{cut}_k) \geq p_{\min}^J / (Ck)$. On the other hand, every trajectory in cut_k must visit ∂B_{k+1} and never return to ∂B_k , hence by the upper bound in (11), $P(\text{cut}_k) \leq C/k$.

Let $k_0 \leq k < m$ and consider a trajectory in the event $\text{cut}_k \cap \text{cut}_m$. It visits ∂B_k , then some point x in ∂B_{k+1} and from x it reaches ∂B_m without revisiting ∂B_k . After reaching ∂B_{m+1} it never re-enters B_m . Thus by Lemma 2.2,

$$\begin{aligned} P\{\text{cut}_k \cap \text{cut}_m\} &\leq \max_{x \in \partial B_{k+1}} P_x\{\{S_n\} \text{ visits } \partial B_m \text{ before } \partial B_k\} \\ &\quad \times \max_{y \in \partial B_{m+1}} P_y\{(\forall n) S_n \notin B_m\} \\ &\leq \frac{Cm}{k(m-k)} \frac{C}{m} = \frac{C^2}{k(m-k)}. \end{aligned}$$

It follows that

$$\sum_{k=k_0}^K P(\text{cut}_k) \geq \sum_{k=k_0}^K \frac{p_{\min}^J}{Ck} \sim \frac{p_{\min}^J}{C} \log K$$

and

$$\sum_{k,m=k_0}^K P(\text{cut}_k \cap \text{cut}_m) \leq 2 \sum_{k_0 \leq k < m \leq K} \frac{C^2}{k(m-k)} + \sum_{k=k_0}^K \frac{C}{k} \sim 2C^2 \log^2 K.$$

By the Kochen-Stone Lemma [12],

$$P(\text{cut}_k \text{ i.o.}) \geq \limsup_{K \rightarrow \infty} \frac{(\sum_{k=1}^K P(\text{cut}_k))^2}{\sum_{k,m=1}^K P(\text{cut}_k \cap \text{cut}_m)} \geq \frac{p_{\min}^{2J}}{2C^4}.$$

Now suppose the first n steps of the walk are specified. We may choose k_0 large enough so that $e, x_1, \dots, x_n$ all lie in the interior of B_{k_0} and repeat the reasoning above to obtain

$$P\{\text{cut}_k \text{ i.o.} \mid S_0 = e, \dots, S_n = x_n\} \geq \frac{p_{\min}^{2J}}{2C^4}.$$

In other words the martingale $E(1_{\{\text{cut}_k \text{ i.o.}\}} \mid S_0, \dots, S_n)$, which converges a.s. to $1_{\{\text{cut}_k \text{ i.o.}\}}$, is bounded away from zero. Therefore $P(\text{cut}_k \text{ i.o.}) = 1$.

P r o o f o f T h e o r e m 1.2. (a) First we consider the case that p has non-zero mean. Suppose there is a homomorphism of a group G onto another group G' which maps p to p' , and S_n to S'_n . If S'_n is a cutpoint then so is S_n . The mean of p has a component which is non-zero and we may take our homomorphism to be the projection onto that component. This gives a random walk S'_n on $\mathbf{Z}$ of bounded stepsize with non-zero drift. Using the ergodic theorem, it is easy to see that this walk has infinitely many cutpoints a.s. See part (b), where it is shown that a decay assumption (14) on the convolution powers of p implies the existence of cutpoints; this assumption is certainly satisfied for walks with non-zero mean.

Now suppose p has mean zero. Then d must be at least 3 for the walk to be transient. By Proposition 2.1, it suffices to show that the Green function satisfies (1)–(3). This will follow from a bound on the error in the local central limit theorem for random walks on $\mathbf{Z}^d$ due to Lawler and Polaski [14]. Let Σ be the covariance matrix of p , i.e., $\Sigma = (\mathbf{E}[Y^{(i)}Y^{(j)}])$ where $Y = (Y^{(1)}, \dots, Y^{(d)})$ has distribution p . For $x \in \mathbf{Z}^d$ we write $\|x\| = (x \cdot \Sigma^{-1}x)^{1/2}$. Let $b = (1/2)\Gamma(d/2 - 1)\pi^{-d/2}(\det \Sigma)^{-1/2}$. In our case p has finite moments of all orders, but the lemma below holds whenever p has a finite $(4d - 6)$ th moment, and this moment assumption could be weakened with more work.

Lemma 2.4. *We have*

$$\theta(x) = b\|x\|^{2-d} + o(\|x\|^{1-d}).$$

Section 3 (p. 796–799) of [14] states this with an error term $O(\|x\|^{1-d})$ under the weaker assumption that p has a finite $(4d - 8)$ th moment, but their work contains a range of bounds on the error depending on how many moments of p are assumed to be finite. We explain briefly how the above lemma follows from their estimates.

We may assume p is strongly aperiodic. (See [17, p. 310] for details of this reduction.) Recall that p_n denotes the n th convolution power of p . We approximate $p_n(x)$ by

$$\tilde{p}_n(x) = (2\pi n)^{-d/2}(\det \Sigma)^{-1/2} \exp \left\{ -\frac{\|x\|^2}{2n} \right\}$$

for $n \geq 1$, and let $\tilde{p}_0$ be the point mass at the origin. Now

$$|\theta(x) - b\|x\|^{2-d}| \leq \sum_{n=0}^{\infty} |p_n(x) - \tilde{p}_n(x)| + \left| \sum_{n=0}^{\infty} \tilde{p}_n(x) - b\|x\|^{2-d} \right|.$$

The last term is $O(\|x\|^{-d})$. The first term can be split into three parts:

$$\left(\sum_{0 \leq n < \|x\|^a} + \sum_{\|x\|^a \leq n < \|x\|^2} + \sum_{\|x\|^2 \leq n} \right) |p_n(x) - \tilde{p}_n(x)|, \quad (13)$$

where $a < 2$ is an arbitrary constant, to be chosen below. Lawler and Polaski show the last sum in (13) is $O(\|x\|^{-d})$ and the second sum is $O(\|x\|^{-\gamma})$, where $\gamma = [4 + a(d - 2)]/2$. We will choose a such that $2(d - 3)/(d - 2) < a < 2$, hence $\gamma > d - 1$. In the first sum in (13), $\sum_{0 \leq n < \|x\|^a} \tilde{p}_n(x)$ is negligible because $a < 2$. Lemma 3.2 of [14], which follows from Theorems 2.1 and 2.3 of Ney and Spitzer [15], show that if p has finite $(2m)$ th moment (m an integer) then

$$\sum_{0 \leq n < \|x\|^a} p_n(x) = O(\|x\|^{-\beta}), \quad \text{where } \beta = m\left(1 - \frac{a}{2}\right) + \frac{(d-2)a}{2}.$$

Choosing a close to $2(d - 3)/(d - 2)$ yields $\beta > d - 1$, since $m > 2(d - 2)$. Thus the error term is $o(\|x\|^{1-d})$, and this completes the proof the lemma.

Proof of Theorem 1.2, continued. If $u \in \mathbf{Z}^d$ is fixed, then

$$\|x + u\|^{2-d} = \|x\|^{2-d} + O(\|x\|^{1-d}),$$

hence

$$\nabla \theta(x) = \max_{u \in U} |\theta(x + u) - \theta(x)| = O(\|x\|^{1-d}).$$

Because U , the support of p , generates $\mathbf{Z}^d$ as a semigroup, there is a constant $c > 0$ such that for all $x \in \mathbf{Z}^d$ we can choose $u_x \in U$ such that $u_x \cdot \Sigma^{-1}x \geq c\|x\|$. Now

$$\begin{aligned} \|x + u_x\|^{2-d} &= (\|x\|^2 + 2u_x \cdot \Sigma^{-1}x + \|u_x\|^2)^{1-d/2} \\ &\leq \|x\|^{2-d} (1 + 2c\|x\|^{-1})^{1-d/2} \\ &= \|x\|^{2-d} (1 - (d-2)c\|x\|^{-1} + o(\|x\|^{-1})). \end{aligned}$$

Thus

$$\theta(x) - \theta(x + u_x) \geq b(d-2)c\|x\|^{1-d} + o(\|x\|^{1-d}).$$

Now $\rho(\cdot)$ and $\|\cdot\|$ are equivalent, each being bounded by a multiple of the other, hence (1)–(3) hold.

(b) For symmetric p with bounded generating support, Varopoulos [18], [19] has shown that the growth of G is at least polynomial of degree 5 if and only if the convolution powers p_n satisfy

$$p_n(e) = O(n^{-5/2}). \quad (14)$$

We will need only (14) to show the existence of cutpoints, and (14) holds for many asymmetric walks as well. Of course $p_n(x)$ decays at the same rate for any fixed x .

We may assume that $S_0 = e$. We extend S_n to be doubly infinite. Let $X_n = S_{n-1}^{-1}S_n$, $n \geq 1$, be the increments of the walk. These are i.i.d. with common distribution p . Choose X_n , $n \leq 0$, to be i.i.d. with distribution p and independent of X_n , $n \geq 1$. For $n \geq 1$ we define $S_{-n} = X_0^{-1} \dots X_{-n+1}^{-1}$.

Recall n is a cutpoint for S if $S[0, n)U \cap S(n, \infty) = \emptyset$. Lawler [13] discusses loop-free points; the time n is *loop-free* for S if $S(-\infty, n] \cap S(n, \infty) = \emptyset$. These are obviously similar to cutpoints, and the following argument is based on the method used to establish the existence of loop-free points in Sections 3.2 and 7.7 of [13].

We will show that

$$\mathbf{P}\{S(-\infty, n)U \cap S(n, \infty) = \emptyset \text{ for infinitely many } n \geq 1\} = 1.$$

First we show that the event

$$D_n = \{S(-\infty, n)U \cap S(n, \infty) = \emptyset\}$$

has positive probability. This probability is independent of n and so we take $n = 0$.

Let

$$\mathcal{N} = \sum_{i,j=0}^{\infty} 1_{\{S_j \in S_{-i}U\}}$$

count the number of pairs (i, j) such that $i, j \geq 0$ and there is a possible transition from S_{-i} to S_j . Clearly $\mathcal{N} \geq 1$ (consider $(i, j) = (1, 0)$). The condition on p in (14) entails

$$\mathbf{E}\mathcal{N} = \sum_{i,j=0}^{\infty} \mathbf{P}\{S_{-i}^{-1}S_j \in U\} = \sum_{i,j=0}^{\infty} \sum_{u \in U} p_{i+j}(u) \leq C \sum_{i,j=0}^{\infty} (i+j+1)^{-5/2} < \infty$$

for some finite C . Hence $\mathcal{N}$ is finite a.s.

We define a pair (i_0, j_0) of non-negative integers to be **-last* if S_{j_0} is in $S_{-i_0}U$ and S_j is not in $S_{-i}U$ for $i \geq i_0 + 1, j \geq j_0$. There is at least one such pair: choose i_0 maximal so that there is a j with $S_j \in S_{-i_0}U$ and then choose j_0 maximal so that $S_{j_0} \in S_{-i_0}U$. Let $\mathcal{N}_*$ count the number of **-last* pairs. We have

$$1 \leq \mathbf{E}\mathcal{N}_* = \sum_{i_0, j_0=0}^{\infty} \mathbf{P}\{S_{-i_0}^{-1}S_{j_0} \in U \text{ and } S_{-i}^{-1}S_j \notin U \text{ for } i \geq i_0 + 1, j \geq j_0\}.$$

We condition on the value of $S_{-i_0}^{-1}S_{j_0}$ and use stationarity to obtain

$$\begin{aligned} \mathbf{E}\mathcal{N}_* &= \sum_{i_0, j_0=0}^{\infty} \sum_{u \in U} p_{i_0+j_0}(u) \mathbf{P}\{S_{-i}^{-1}uS_j \notin U \text{ for } i \geq 1, j \geq 0\} \\ &\leq \mathbf{E}\mathcal{N} \max_{u \in U} \mathbf{P}\{S_{-i}^{-1}uS_j \notin U \text{ for } i \geq 1, j \geq 0\}. \end{aligned}$$

At this point we modify the argument in Section 7.7 of [13] to deal with the maximum over U . Let Y have distribution p and be independent

of X_n , $-\infty < n < \infty$. Note that $\{S_{-i}^{-1}S_j: i \geq 1, j \geq 1\}$ has the same distribution as $\{S_{-i}^{-1}YS_{j-1}, i \geq 1, j \geq 1\}$. Replacing $j-1$ by j gives

$$P(D_0) = P\{S_{-i}^{-1}YS_j \notin U \text{ for } i \geq 1, j \geq 0\}.$$

By conditioning on the value of Y we obtain

$$P(D_0) \geq p_{\min} \max_{u \in U} P\{S_{-i}^{-1}uS_j \notin U \text{ for } i \geq 1, j \geq 0\}.$$

Hence

$$P(D_0) \geq p_{\min} \frac{EN_*}{EN} \geq \frac{p_{\min}}{EN} > 0.$$

We can now apply the ergodic theorem because $(X_n)_{n=-\infty}^{\infty}$ is an ergodic, stationary sequence. With probability 1

$$\frac{1}{N} \sum_{n=1}^N 1_{D_n} \rightarrow P(D_0).$$

Thus the set $\{n \geq 1 \mid n \text{ is a cut-epoch for } S\}$ almost surely has positive density which is at least $p_{\min}/EN$.

3. Exchangeable events

We return to the setting where $S_0, S_1, \dots$ is a transient Markov chain on the state space X . Recall that

$$V_n(x) = \sum_{i=0}^n 1_{S_i}(x)$$

and

$$V(x) = \lim_{n \rightarrow \infty} V_n(x)$$

for every $x \in X$.

Lemma 3.1. For $n \geq 1$ consider the event $C_n = \{S_n \text{ is a cutpoint}\}$. For $n \geq 1$ this event is in the σ -field $\sigma(V, S_0)$, and for trajectories in C_n the values of V_{n-1} and S_n are determined by the values of V and S_0 .

Proof. Let $(s_0, s_1, \dots)$ be a trajectory with only permitted transitions and let $v = 1_{s_0} + 1_{s_1} + \dots$ be the value of V for this trajectory. If $(s_0, s_1, \dots)$ lies in C_n , then v may be decomposed as

$$v = v_1 + 1_y + v_2,$$

where $v_1 = 1_{s_0} + \dots + 1_{s_{n-1}}$, in the middle term $y = s_n$, and $v_2 = 1_{s_{n+1}} + 1_{s_{n+2}} + \dots$. This decomposition has the following properties:

1. $v_1(y) = v_2(y) = 0$, $v_1(s_0) > 0$;
2. $\sum_{x \in X} v_1(x) = n$;
3. if $v_1(z_1) > 0$ and $v_2(z_2) > 0$, then $p(z_1, z_2) = 0$.

To establish the first property, we note if $s_i = s_n$ for some $i < n$ then $p(s_i, s_{n+1}) = p(s_n, s_{n+1}) > 0$, and we reason similarly if $v_2(s_n) > 0$. Properties 2 and 3 are obvious.

Conversely, suppose a decomposition $v = v_1 + 1_y + v_2$ with properties 1–3 is given. We show $v_1 = 1_{s_0} + \dots + 1_{s_{n-1}}$, $y = s_n$ and $v_2 = 1_{s_{n+1}} + 1_{s_{n+2}} + \dots$, and then Property 3 implies $(s_0, s_1, \dots)$ lies in $\mathcal{C}_n$. Now $y = s_k$ for some k , which is unique by Property 1. Choose $i_1, \dots, i_{n-1}$ such that

$$v_1 = 1_{s_0} + 1_{s_{i_1}} + \dots + 1_{s_{i_{n-1}}}.$$

The only arrangement consistent with Property 3 is that $\{i_1, \dots, i_{n-1}\} = \{1, \dots, n-1\}$ and $k = n$.

Thus $\mathcal{C}_n$ may be partitioned into subevents $\mathcal{C}_n \cap \{V_{n-1} = v_1, S_n = y\}$ which are in the σ -field $\sigma(V, S_0)$, and the lemma is proved.

PROOF OF PROPOSITION 1.1. First suppose that the distribution of S_0 is concentrated at one state, i.e., $\mathbf{P}\{S_0 = x_0\} = 1$ for some $x_0 \in X$. Then we can describe the conditional distribution of the process given the occupation field V . By hypothesis there is an infinite sequence of cut-epochs $n_1 < n_2 < \dots$; by Lemma 3.1, at these cut-epochs the values of $S_{n_1}, S_{n_2}, \dots$ are determined. The sequence of states $(S_{n_k}, \dots, S_{n_{k+1}-1})$ visited between successive cut-epochs is constrained to start at S_{n_k} and have occupation numbers $V_{n_{k+1}-1} - V_{n_k-1}$. For different k these sequences are mutually independent, and hence the tail σ -field of the process $\{S_n, V_{n-1}\}_{n \geq 0}$, given V , is trivial by the Kolmogorov 0–1 law. It then follows (see Theorem 1.2 in [10]) that the tail σ -field of the extended chain, which is identical to the exchangeable σ -field of the original chain, is generated by the occupation field V .

Now suppose S_0 is random. Consider two trajectories, $(x_0, x_1, \dots)$ and $(y_0, y_1, \dots)$, with the same occupation numbers. Then $y_0 = x_m$ for some m . If $n > m$ is a cut-epoch for $(x_0, x_1, \dots)$, then n is also a cut-epoch for $(y_0, y_1, \dots)$, we have $x_n = y_n$, and the initial segments $(x_0, \dots, x_{n-1})$ and $(y_0, \dots, y_{n-1})$ have the same occupation numbers. Indeed let

$$i = \max \left\{ k \mid y_k \in \{x_0, \dots, x_{n-1}\} \right\} \quad \text{and} \\ j = \min \left\{ k \mid y_k \in \{x_{n+1}, x_{n+2}, \dots\} \right\}.$$

Then $y_j \neq x_m = y_0$ so $j > 0$. If $y_{j-1} \in \{x_0, \dots, x_{n-1}\}$ then x_n is not a cutpoint. Thus $y_{j-1} = x_n$, and similarly $y_{i+1} = x_n$. Hence $y_n = x_n$ and $(y_0, \dots, y_{n-1})$ is a rearrangement of $(x_0, \dots, x_{n-1})$. This shows that for the conditioned extended chain there is a sequence of times $n_1 < n_2 < \dots$ such that any two trajectories will meet at all but finitely many of these times. Using the Markov property of the conditioned chain, this allows us to couple trajectories of the conditioned chain started at any two initial states. Thus the tail of the conditioned extended chain is trivial, and the exchangeable σ -field is generated by V .

R e m a r k. We can also give a more elementary, but longer, proof of the previous theorem that avoids considering the extended chain conditioned on V . Instead, one can verify that for every $k \geq 1$, the conditional expectation $E(f | \mathcal{E})$ of any function f that depends on $S_0, \dots, S_k$, given the exchangeable σ -field $\mathcal{E}$, is $\sigma(V)$ -measurable.

4. Concluding remarks

1. For random walk with mean 0 and bounded jumps in $\mathbf{Z}^d$, the argument in Section 2 actually implies that the number of approximate spheres that are visited only once by $\{S_1, \dots, S_n\}$, and never revisited by the RW, grows logarithmically in n . Following our work, G. Lawler in a recent preprint («Cut Times for Simple Random Walk» (1995)) has refined the methods in his book, and showed that for simple random walk in $\mathbf{Z}^3$, the total number of cutpoints among $\{S_1, \dots, S_n\}$ grows like a power of n . Thus the cutpoints described in our proof of Proposition 2.1 are non-typical.

2. Consider a transient Markov chain on a tree (i.e., a countable acyclic graph), where all permitted transitions are to nearest neighbors. James [8] shows that for such chains, the exchangeable σ -field is always generated by the occupation numbers, although for some of these chains the number of cutpoints is finite almost surely. Thus the existence of infinitely many cutpoints is sufficient, but not necessary, for the description of the exchangeable σ -field in Proposition 1.1.

3. The following question, due to Kaimanovich and Vershik (see [11], [10]), is still open.

Question. Let $\{S_n\}$ be a transient Markov chain on the state space X . Let $V_n(x)$ and $V(x)$ denote the visit counts and occupation numbers defined in the introduction. When is the tail of the chain $\{(S_n, V_{n-1} \bmod 2)\}$ generated by the random variables $\{V(x) \bmod 2: x \in X\}$?

Kaimanovich [10] gives a positive answer to this question when $\{S_n\}$ is a random walk on $\mathbf{Z}$ with non-zero drift. However, even the case where $\{S_n\}$ is simple random walk on $\mathbf{Z}^3$ is unresolved.

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