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Lin Zhengyan, A self-normalized Chung type law of the iterated logarithm, *Теория вероятн. и ее примен.*, 1996, том 41, выпуск 4, 934–942

DOI: <https://doi.org/10.4213/tvp3285>

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Поступила в редакцию
25.VIII.1994

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A SELF-NORMALIZED CHUNG TYPE LAW OF THE ITERATED LOGARITHM¹⁾

В [3] установлен закон повторного логарифма, в котором использована неклассическая нормировка. В данной статье получен закон повторного логарифма в форме Чжуна, обобщающий результат [3].

Ключевые слова и фразы: закон повторного логарифма в форме Чжуна, область притяжения нормального закона.

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¹⁾Project supported by the National and Zhejiang Provincial Natural Science Foundations of China.

1. Introduction and Theorem. For a sequence of independent identically distributed (i.i.d.) random variables, the finite variance is a sufficient and necessary condition for obeying the law of the iterated logarithm (LIL). This is the well known Hartman and Wintner LIL and its converse by Strassen [7]. Griffin and Kuelbs [3] obtained the following self-normalized LIL for all distributions in the domain of attraction of the normal distribution.

Theorem A. Let $X, X_1, X_2, \dots$ be nondegenerate i.i.d. random variables with $EX = 0$ and

$$\lim_{x \rightarrow \infty} \frac{x^2 P\{|X| \geq x\}}{EX^2 I\{|X| \leq x\}} = 0. \quad (1.1)$$

Then

$$\limsup_{n \rightarrow \infty} \frac{|S_n|}{V_n(2 \log \log n)^{1/2}} = 1 \quad \text{a.s.}, \quad (1.2)$$

where $S_n = \sum_{i=1}^n X_i$, $V_n^2 = \sum_{i=1}^n X_i^2$.

Csörgő, Lin and Shao [2] generalized Theorem A to the case of studentized increments of partial sums.

Similarly, the finite variance also is a sufficient and necessary condition for obeying the Chung LIL (see [4], [1]). Let $X, X_1, X_2, \dots$ be nondegenerate i.i.d. random variables with $EX = 0$. Then

$$\liminf_{n \rightarrow \infty} \left(\frac{\log \log n}{n\sigma^2} \right)^{1/2} \max_{1 \leq i \leq n} |S_i| = \frac{\pi}{\sqrt{8}} \quad \text{a.s.} \quad (1.3)$$

if and only if $\sigma^2 := EX^2 < \infty$. A natural question is to find an analogue of Theorem A for the Chung LIL. We succeeded in doing this by assuming only condition (1.1).

For $x, t > 0$, let

$$K(x) = x^{-2} EX^2 I\{|X| \leq x\}, \quad d(t) = \inf \left\{ s \geq b+1 : K(s) \leq \frac{1}{t} \right\},$$

where $b = \inf\{x \geq 1 : K(x) > 0\}$. For $\varepsilon > 0$, let

$$d_n = d\left(\frac{\varepsilon n}{\log \log n}\right), \quad U_n^2 = \sum_{i=1}^n X_i^2 I\{|X_i| \leq d_n\}.$$

In the sequel, we always assume that ε is small enough.

Theorem. Let $X, X_1, X_2, \dots$ be nondegenerate i.i.d. random variables with $EX = 0$. Suppose that condition (1.1) is satisfied. Then

$$\liminf_{n \rightarrow \infty} \max_{1 \leq i \leq n} \frac{|S_i|}{U_n/(\log \log n)^{1/2}} = \frac{\pi}{\sqrt{8}} \quad \text{a.s.} \quad (1.4)$$

2. Proof. In order to prove the Theorem, we need some lemmas. Lemma 2.1 is a consequence of Lemma 3.4 in [3] (cf. [2]).

Lemma 2.1. Let $\xi, \xi_1, \xi_2, \dots$ be i.i.d. random variables. Then, for any $b > 0$, $\sigma^2 \geq E\xi^2 I\{|\xi| \leq b\}$ and $t > 0$

$$\begin{aligned} & P \left\{ \left| \sum_{i=1}^n \xi_i I\{|\xi_i| \leq b\} - E \sum_{i=1}^n \xi_i I\{|\xi_i| \leq b\} \right| \right. \\ & \quad \left. \geq \frac{1}{2} \left(1 + \exp \left(\frac{bt}{\sqrt{n\sigma^2}} \right) \right) t \sqrt{n\sigma^2} \right\} \leq 2e^{-t^2/2}. \end{aligned}$$

The following three lemmas are from Lemma 2.2, Lemma 2.4 and Lemma 2.5 of [6] respectively.

Lemma 2.2. Let $\{W(t), t \geq 0\}$ be a standard Wiener process. Put $M^+(t) = \sup_{0 \leq s \leq t} W(s)$, $M^-(t) = -\inf_{0 \leq s \leq t} W(s)$. Define

$$G(a, b) = P\{M^+(t) < a\sqrt{t}, M^-(t) < b\sqrt{t}\}, \quad a > 0, \quad b > 0.$$

Then for $0 < a + b \leq 2\pi$

$$\frac{a \wedge b}{2(a+b)} \exp\left(-\frac{\pi^2}{2(a+b)^2}\right) \leq G(a, b) \leq \frac{8(a \wedge b)}{a+b} \exp\left(-\frac{\pi^2}{2(a+b)^2}\right).$$

Lemma 2.3. Let $\{\xi_n, n \geq 1\}$ be a sequence of independent random variables with $E\xi_n = 0$, $E\xi_n^2 < \infty$, and $\sigma_n^2 = \sum_{i=1}^n E\xi_i^2$. Assume that for any $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} \frac{1}{\sigma_n^2} \sum_{j=1}^n E\xi_j^2 I\{|\xi_j| \geq \varepsilon \sigma_n\} = 0.$$

Then with $\zeta_n = \sum_{j=1}^n \xi_j$.

$$\lim_{n \rightarrow \infty} \sup_{|x| < y \leq A} \left| P\left\{ \max_{1 \leq i \leq n} \left| \frac{\zeta_i}{\sigma_n} + x \right| < y \right\} - P\left\{ \sup_{0 \leq t \leq 1} |W(t) + x| < y \right\} \right| = 0 \quad (2.1)$$

for any $A > 0$.

Lemma 2.4. Let $\{x_n\}$ be a sequence of positive numbers. Assume that the conditions of Lemma 2.3 and the following ones are satisfied:

$$x_n \rightarrow \infty, \quad \frac{\sigma_n}{x_n} \rightarrow \infty, \quad \max_{1 \leq j \leq n} \frac{E\xi_j^2}{x_n^2} \rightarrow 0$$

as $n \rightarrow \infty$ and for any $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} \max_{\substack{1 \leq i, i+j \leq n, \\ \sigma_j^2 - \sigma_i^2 \geq \varepsilon x_n^2}} \frac{\sum_{k=i+1}^{i+j} E\xi_k^2 I\{|\xi_k| \geq \varepsilon \sigma_n\}}{\sigma_j^2} = 0.$$

Then as $n \rightarrow \infty$

$$\ln P\left\{ \max_{1 \leq i \leq n} |\zeta_i| \leq x_n \right\} \sim -\frac{\pi^2 \sigma_n^2}{8x_n^2}.$$

Proof of the Theorem. We point out that there is a fact for $d(\cdot)$ as follows (cf. Lemma 3.1 of [5]): under condition (1.1), for any given $\theta > 0$, $p = 2/(1+\theta)$, and for $y > x$ and x large enough

$$1 \leq \frac{d(y)}{d(x)} \leq \left(\frac{y}{x} (1+\theta) \right)^{1/p}. \quad (2.2)$$

Noting $K(d(x)) = 1/x$ for $x > 2+b$, we have

$$K(d_n) = \frac{\log \log n}{\varepsilon n}, \quad (2.3)$$

$$EU_n^2 = n d_n^2 K(d_n) = d_n^2 \frac{\log \log n}{\varepsilon}. \quad (2.4)$$

First of all we observe U_n^2 . Put $\xi = X^2$, $b = d_n^2$, $\sigma^2 = d_n^4 K(d_n)$, $t = 2\sqrt{\log \log n}$ in Lemma 2.1. Then, by (2.3) we have

$$\frac{bt}{\sqrt{n\sigma^2}} = 2\sqrt{\varepsilon}, \quad t\sqrt{n\sigma^2} = 2\sqrt{\varepsilon} EU_n^2,$$

and hence Lemma 2.1 implies

$$\mathbf{P}\left\{U_n^2 > (1 + 4\sqrt{\varepsilon}) \mathbf{E}U_n^2\right\} \leq 2 \exp(-2 \log \log n) = 2(\log n)^{-2}.$$

Consequently, putting $N_k = [\theta^k]$ for some $\theta > 1$, we obtain

$$\limsup_{k \rightarrow \infty} \frac{U_{N_k}^2}{\mathbf{E}U_{N_k}^2} \leq 1 + 4\sqrt{\varepsilon} \quad \text{a.s.} \quad (2.5)$$

Moreover, (2.4) and (2.2) together imply

$$\begin{aligned} 1 &\leq \frac{\mathbf{E}U_{N_{k+1}}^2}{\mathbf{E}U_{N_k}^2} = \frac{d_{N_{k+1}}^2}{d_{N_k}^2} \frac{\log \log N_{k+1}}{\log \log N_k} \\ &\leq (1 + \theta)^{1/p} \left(\frac{N_{k+1} \log \log N_k}{N_k \log \log N_{k+1}} \right)^{2/p} \frac{\log \log N_{k+1}}{\log \log N_k} \leq 1 + \varepsilon \end{aligned} \quad (2.6)$$

for k large enough and θ near one enough.

At first, we prove

$$\liminf_{n \rightarrow \infty} \max_{1 \leq i \leq n} \frac{|S_i|}{U_n / (\log \log n)^{1/2}} \geq \frac{\pi}{\sqrt{8}} \quad \text{a.s.} \quad (2.7)$$

By (2.5) and (2.6),

$$\begin{aligned} \liminf_{n \rightarrow \infty} \max_{1 \leq i \leq n} \frac{|S_i|}{U_n / (\log \log n)^{1/2}} &\geq \liminf_{k \rightarrow \infty} \max_{1 \leq i \leq N_k} \frac{|S_i|}{U_{N_{k+1}} / (\log \log N_k)^{1/2}} \\ &\geq \liminf_{k \rightarrow \infty} \max_{1 \leq i \leq N_k} \frac{|S_i|}{(\mathbf{E}U_{N_k}^2 / \log \log N_k)^{1/2} (1 + 4\sqrt{6})^{1/2} (1 + \varepsilon)^{1/2}}. \end{aligned}$$

Thus it suffices for proving (2.7) to show

$$\liminf_{k \rightarrow \infty} \max_{1 \leq i \leq N_k} \frac{|S_i|}{(\mathbf{E}U_{N_k}^2 / \log \log N_k)^{1/2}} \geq \frac{\pi}{\sqrt{8}} \quad \text{a.s.} \quad (2.8)$$

Put

$$\beta_n = \left(\frac{\pi^2 \mathbf{E}U_n^2}{8 \log \log n} \right)^{1/2}, \quad n_j = j \left\lceil \frac{n}{\sigma^2 \log \log n} \right\rceil,$$

where $j = 1, \dots, M = [\varepsilon^2 \log \log n]$. Noting the independence we have

$$\begin{aligned} \mathbf{P}\left\{\max_{1 \leq i \leq n} |S_i| \leq (1 - \varepsilon) \beta_n\right\} &\leq \mathbf{P}\left\{\max_{1 \leq i \leq n_M} |S_i| \leq (1 - \varepsilon) \beta_n\right\} \\ &= \mathbf{E}I\left\{\max_{1 \leq i \leq n_{M-1}} |S_i| \leq (1 - \varepsilon) \beta_n\right\} I\left\{\max_{n_{M-1} < i \leq n_M} |S_i| \leq (1 - \varepsilon) \beta_n\right\} \\ &= \mathbf{E}\left(I\left\{\max_{1 \leq i \leq n_{M-1}} |S_i| \leq (1 - \varepsilon) \beta_n\right\} \right. \\ &\quad \times \mathbf{E}\left(I\left\{\max_{n_{M-1} < i \leq n_M} |S_i| \leq (1 - \varepsilon) \beta_n\right\} \middle| S_i, i = 1, \dots, n_{M-1}\right)\Bigg) \\ &= \int_{-(1-\varepsilon)\beta_n}^{(1-\varepsilon)\beta_n} \mathbf{P}\left\{\max_{n_{M-1} < i \leq n_M} |S_i - S_{n_{M-1}} + y| \leq (1 - \varepsilon) \beta_n\right\} \\ &\quad \times d\mathbf{P}\left\{\max_{1 \leq i \leq n_{M-1}} |S_i| \leq (1 - \varepsilon) \beta_n, S_{n_{M-1}} < y\right\}. \end{aligned} \quad (2.9)$$

Let $Y(n), Y_1(n), Y_2(n), \dots$ be i.i.d. random variables with the distribution function $G(\cdot)$, which is that of X conditioned on $|X| \leq d_n$. Then

$$\begin{aligned} \mathbf{E}Y^2(n) &= \frac{\mathbf{E}X^2 I\{|X| \leq d_n\}}{\mathbf{P}\{|X| \leq d_n\}} = \frac{d_n^2 K(d_n)}{\mathbf{P}\{|X| \leq d_n\}} \\ &= (1 + o(1)) \frac{d_n^2 \log \log n}{\varepsilon n} \quad \text{as } n \rightarrow \infty \end{aligned} \quad (2.10)$$

by (2.3). Moreover, since

$$\left(\mathbf{E}XI\{|X| \leq d_n\}\right)^2 = o\left(\mathbf{E}X^2 I\{|X| \leq d_n\}\right)$$

by the assumptions $\mathbf{E}X = 0$ and (1.1), we obtain

$$\left(\mathbf{E}Y(n)\right)^2 = o\left(\mathbf{E}Y^2(n)\right) \quad \text{as } n \rightarrow \infty. \quad (2.11)$$

Put

$$\begin{aligned} T_n(i) &= \sum_{j=1}^i \left(Y_j(n) - \mathbf{E}Y_j(n)\right), \\ R_n^2(i) &= \sum_{j=1}^i \left(Y_j(n) - \mathbf{E}Y_j(n)\right)^2. \end{aligned}$$

Then by (2.4), (2.10) and (2.11),

$$\mathbf{E}R_n^2(n) = (1 + o(1)) \mathbf{E}U_n^2 \quad \text{as } n \rightarrow \infty. \quad (2.12)$$

Put

$$\gamma_n = \left(\frac{\pi^2 \mathbf{E}R_n^2(n)}{8 \log \log n}\right)^{1/2}.$$

Condition (1.1) implies (cf. Lemma 6.2 in [3])

$$\mathbf{E}|X| I\{|X| \geq d_n\} = o(d_n K(d_n)) \quad \text{as } n \rightarrow \infty.$$

So we have

$$\sum_{j=n_{M-1}+1}^{n_M} |\mathbf{E}Y_j(n)| = o\left(\frac{nd_n K(d_n)}{\log \log n}\right) = o(d_n) = o(\gamma_n) \quad \text{as } n \rightarrow \infty. \quad (2.13)$$

Therefore for $|y| \leq (1 - \varepsilon) \beta_n$

$$\begin{aligned} &\mathbf{P}\left\{\max_{n_{M-1} < i \leq n_M} |S_i - S_{n_{M-1}} + y| \leq (1 - \varepsilon) \beta_n\right\} \\ &\leq \mathbf{P}\left\{\max_{n_{M-1} < i \leq n_M} |S_i - S_{n_{M-1}} + y| \leq (1 - \varepsilon) \beta_n, \max_{n_{M-1} < j \leq n_M} |X_j| \leq d_n\right\} \\ &\quad + \mathbf{P}\left\{\max_{n_{M-1} < j \leq n_M} |X_j| > d_n\right\} \\ &\leq \mathbf{P}\left\{\max_{n_{M-1} < i \leq n_M} |T_n(i) - T_n(n_{M-1}) + y| \leq (1 - 2\varepsilon) \gamma_n\right\} \\ &\quad + \mathbf{P}\left\{\max_{n_{M-1} < j \leq n_M} |X_j| > d_n\right\} \end{aligned} \quad (2.14)$$

for all large n . In term of (1.1) and (2.3), we obtain

$$nP\{|X| > d_n\} = o(\log \log n) \quad \text{as } n \rightarrow \infty.$$

Thus

$$\begin{aligned} P\left\{\max_{n_{M-1} < j \leq n_M} |X_j| \leq d_n\right\} &= \left(1 - P\{|X| > d_n\}\right)^{[n/(\varepsilon^2 \log \log n)]} \\ &\geq \exp\left\{-2 \frac{n}{\varepsilon^2 \log \log n} P\{|X| > d_n\}\right\} \geq \exp(-e^{-2/\varepsilon^2}) \end{aligned} \quad (2.15)$$

provided n is large enough, and hence

$$P\left\{\max_{n_{M-1} < j \leq n_M} |X_j| > d_n\right\} \leq e^{-2/\varepsilon^2}. \quad (2.16)$$

Moreover, it is easy to verify that the conditions in Lemmas 2.3 and 2.2 with $\xi_j = Y_j(n)$, $j = 1, \dots, n$, are satisfied. So using these lemmas we obtain

$$\begin{aligned} P\left\{\max_{n_{M-1} < i \leq n_M} |T_n(i) - T_n(n_{M-1}) + y| \leq (1 - 2\varepsilon) \gamma_n\right\} &\leq \exp\left(-\frac{2}{\varepsilon^2}\right) \\ &+ P\left\{\sup_{0 \leq t \leq 1} |W(t) + y(\mathbf{E}(T_n(n_M) - T_n(n_{M-1}))^2)^{-1/2}| \right. \\ &\quad \left. \leq (1 - 2\varepsilon) \gamma_n (\mathbf{E}(T_n(n_M) - T_n(n_{M-1}))^2)^{-1/2}\right\} \\ &\leq \exp\left(-\frac{2}{\varepsilon^2}\right) + 4 \exp\left(-\frac{1}{\varepsilon^2(1 - 2\varepsilon)^2}\right) \leq \exp\left(-\frac{1}{\varepsilon^2(1 - 2\varepsilon)}\right) \end{aligned} \quad (2.17)$$

provided ε is small enough. Inserting (2.16) and (2.17) into (2.14) yields

$$P\left\{\max_{n_{M-1} < i \leq n_M} |S_i - S_{n_{M-1}} + y| \leq (1 - \varepsilon) \beta_n\right\} \leq \exp\left(-\frac{1}{\varepsilon^2(1 - \varepsilon)}\right).$$

Therefore, (2.9) implies

$$\begin{aligned} P\left\{\max_{1 \leq i \leq n} |S_i| \leq (1 - \varepsilon) \beta_n\right\} &\leq \exp\left(-\frac{1}{\varepsilon^2(1 - \varepsilon)}\right) \\ &\times P\left\{\max_{1 \leq i \leq n_{M-1}} |S_i| \leq (1 - \varepsilon) \beta_n\right\}. \end{aligned}$$

Repeating the previous procedure we obtain

$$\begin{aligned} P\left\{\max_{1 \leq i \leq n} |S_i| \leq (1 - \varepsilon) \beta_n\right\} &\leq \exp\left(-\frac{M}{\varepsilon^2(1 - \varepsilon)}\right) \\ &\leq \exp\left(-\frac{1}{1 - \varepsilon/2} \log \log n\right) \leq (\log n)^{-(1+\varepsilon/2)}. \end{aligned} \quad (2.18)$$

Now (2.18) and the Borel-Cantelli lemma together imply (2.8), and hence (2.7) is proved.

Next we prove the inverse inequality. Let $N_k = [\theta k^{1+\varepsilon}]$ for some $\theta > 1$. Obviously it is enough to show

$$\liminf_{k \rightarrow \infty} \max_{1 \leq i \leq N_k} \frac{|S_i|}{U_{N_k}/(\log \log N_k)^{1/2}} \leq \frac{\pi}{\sqrt{8}} \quad \text{a.s.} \quad (2.19)$$

Write

$$\begin{aligned} & \liminf_{k \rightarrow \infty} \max_{1 \leq i \leq N_{k+1}} \frac{|S_i|}{U_{N_{k+1}}/(\log \log N_{k+1})^{1/2}} \\ & \leq \liminf_{k \rightarrow \infty} \max_{N_k < i \leq N_{k+1}} \frac{|S_i - S_{N_k}|}{U_{N_{k+1}}/(\log \log N_{k+1})^{1/2}} \\ & \quad + 2 \limsup_{k \rightarrow \infty} \max_{1 \leq i \leq N_k} \frac{|S_i|}{U_{N_{k+1}}/(\log \log N_{k+1})^{1/2}} =: m_1 + m_2. \end{aligned} \quad (2.20)$$

We consider m_2 first. From the proof of Theorem 1 in [2], we have

$$\limsup_{k \rightarrow \infty} \max_{1 \leq i \leq N_k} \frac{|S_i|}{U_{N_k}/(\log \log N_k)^{1/2}} \leq 1 \quad \text{a.s.} \quad (2.21)$$

Noting monotonicity of the function $d(\cdot)$, we have

$$\begin{aligned} & \frac{U_{N_{k+1}}^2 / \log \log N_{k+1}}{U_{N_k}^2 \log \log N_k} \geq \frac{\sum_{j=1}^{N_{k+1}} X_j^2 I\{|X_j| \leq d_{N_{k+1}}\}}{\sum_{j=1}^{N_k} X_j^2 I\{|X_j| \leq d_{N_k}\}} \frac{1}{(\log \log N_{k+1})^2} \\ & \geq \frac{\sum_{j=1}^{N_{k+1}} X_j^2 I\{|X_j| \leq d_{N_k}\}}{\sum_{j=1}^{N_k} X_j^2 I\{|X_j| \leq d_{N_k}\}} \frac{1}{(\log \log N_{k+1})^2}. \end{aligned} \quad (2.22)$$

Furthermore, checking the proof of Proposition 5.2 in [3] with $M = \exp(k^{\varepsilon/2})$ and

$$W_j = \sum_{i=(j-1)N_k+1}^{jN_k} X_i^2 I\{|X_i| \leq d_{N_k}\}$$

instead of $W_j = \sum_{i=(j-1)N_k+1}^{jN_k} X_i^2$, $j = 1, \dots, l_k := [N_{k+1}/N_k]$, we can find

$$\lim_{k \rightarrow \infty} \sum_{j=1}^{N_{k+1}} X_j^2 I\{|X_j| \leq d_{N_k}\} \left(\exp(k^{\varepsilon/2}) \sum_{j=1}^{N_k} X_j^2 I\{|X_j| \leq d_{N_k}\} \right)^{-1} = \infty \quad \text{a.s.} \quad (2.23)$$

Hence the left hand side of (2.22) tends to infinity with probability one. Therefore (2.21) implies

$$m_2 = 0 \quad \text{a.s.} \quad (2.24)$$

We turn to m_1 now. Put

$$T_k(i) = \sum_{j=N_k+1}^i \left(Y_j(N_{k+1}) - \mathbf{E}Y_j(N_{k+1}) \right), \quad i = N_k + 1, \dots, N_{k+1}.$$

It is easy to know (recalling (2.12) and (2.13)) that

$$(\gamma'_{N_{k+1}})^2 := \mathbf{E}T_k^2(N_{k+1}) = (1 + o(1)) N_{k+1} d_{N_{k+1}}^2 K(d_{N_{k+1}}) (1 + o(1)) \mathbf{E}U_{N_{k+1}}^2$$

and

$$\sum_{j=N_k+1}^{N_{k+1}} |\mathbf{E}Y_j(N_{k+1})| = o(\gamma'_{N_{k+1}}) \quad \text{as } k \rightarrow \infty.$$

Similarly to the proof of (2.5), it follows that

$$\liminf_{k \rightarrow \infty} \frac{U_{N_k}^2}{\mathbf{E}U_{N_k}^2} \geq 1 - 4\sqrt{\varepsilon} \quad \text{a.s.}$$

Hence

$$m_1 \leq \frac{1}{1-4\sqrt{\varepsilon}} \liminf_{k \rightarrow \infty} \max_{N_k < i \leq N_{k+1}} \frac{|S_i - S_{N_k}|}{\gamma'_{N_{k+1}} / (\log \log N_{k+1})^{1/2}} \quad \text{a.s.}$$

Thus, using Lemma 5.1 of [3], we have

$$\begin{aligned} & \mathbf{P} \left\{ \max_{N_k < i \leq N_{k+1}} |S_i - S_{N_k}| \leq (1+2\varepsilon) \gamma'_{N_{k+1}} \right\} \\ & \geq \mathbf{P} \left\{ \max_{N_k < i \leq N_{k+1}} |S_i - S_{N_k}| \leq (1+2\varepsilon) \gamma'_{N_{k+1}}, \max_{N_k < j \leq N_{k+1}} |X_j| \leq d_{N_{k+1}} \right\} \\ & \geq \mathbf{P} \left\{ \max_{N_k < i \leq N_{k+1}} |T_k(i)| \leq \left(1 + \frac{3}{2}\varepsilon\right) \gamma'_{N_{k+1}} \right\} \\ & \quad \times \mathbf{P} \left\{ \max_{N_k < j \leq N_{k+1}} |X_j| \leq d_{N_{k+1}} \right\}. \end{aligned} \quad (2.25)$$

Similarly to (2.15),

$$\begin{aligned} \mathbf{P} \left\{ \max_{N_k < j \leq N_{k+1}} |X_j| \leq d_{N_{k+1}} \right\} & \geq \exp \left\{ -2N_{k+1} \mathbf{P} \left\{ |X| > d_{N_{k+1}} \right\} \right\} \\ & \geq (\log N_{k+1})^{-\varepsilon/2} \end{aligned} \quad (2.26)$$

for large k . As for another probability in the right hand side of (2.25), we need Lemma 2.4. It is easy to see that the conditions of Lemma 2.4 with $\xi_j = X_j I\{|X_j| \leq d_{N_{k+1}}\}$, $j = N_k + 1, \dots, N_{k+1}$, are satisfied. Therefore

$$\begin{aligned} & \mathbf{P} \left\{ \max_{N_k < i \leq N_{k+1}} |T_k(i)| \leq \left(1 + \frac{3}{2}\varepsilon\right) \gamma'_{N_{k+1}} \right\} \\ & \geq \exp \left\{ -\frac{1}{1+3\varepsilon} \log \log N_{k+1} \right\} \geq (\log N_{k+1})^{-(1-2\varepsilon)}. \end{aligned} \quad (2.27)$$

Inserting (2.26) and (2.27) into (2.25) yields

$$\begin{aligned} & \mathbf{P} \left\{ \max_{N_k < i \leq N_{k+1}} |S_i - S_{N_k}| \leq (1+2\varepsilon) \gamma'_{N_{k+1}} \right\} \\ & \geq (\log N_{k+1})^{-(1-3\varepsilon/2)} = (k^{1+\varepsilon} \log \theta)^{-(1-3\varepsilon/2)}, \end{aligned}$$

which, in combination with the Borel–Cantelli lemma, implies

$$m_1 \leq \frac{1+2\varepsilon}{1-4\sqrt{\varepsilon}} \quad \text{a.s.} \quad (2.28)$$

Inserting (2.24) and (2.28) into (2.20) yields (2.19). The Theorem is proved.

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Поступила в редакцию
5.XII.1994

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SZYSZKOWSKI I.*

WEAK CONVERGENCE OF STOCHASTIC INTEGRALS¹⁾

Пусть $W_n = \{W_n(t): t \geq 0\}$, $n \geq 0$ есть последовательность согласованных càdlàg случайных процессов таких, что для каждого $n \geq 0$ процесс W_n является семимартингалом и имеет место слабая сходимость в пространстве Скорохода $D(\mathbf{R}_+; \mathbf{R}^2): (W_n, [W_n]) \rightarrow (W_0, V_0)$ при $n \rightarrow \infty$, где V_0 — некоторый почти наверное неубывающий процесс. Мы показываем, что $Z_n(\cdot) = \int_0^\cdot f(W_n(t-)) dW_n(t)$ слабо сходится в $D(\mathbf{R}_+, \mathbf{R})$ к процессу $\int_0^\cdot f(W_0(t-)) dW_0(t) + 2^{-1} \int_0^\cdot f'(W_0(t-)) \times d[W_0](t) - 2^{-1} \int_0^\cdot f'(W_0(t-)) dV_0(t)$ для любой аналитической функции f на $\mathbf{R}$. (Здесь $[W_n]$ обозначает квадратическую вариацию семимартингала W_n .)

Ключевые слова и фразы: слабая сходимость, стохастический интеграл, принцип инвариантности, семимартингал.

1. The main result. For each $n \geq 0$ let $W_n = \{(W_n(t), (\mathcal{F}_t^n)): t \geq 0\}$ be an $\mathbf{R}$ -valued càdlàg stochastic process adapted to a right-continuous complete filtration $(\mathcal{F}_t^n)_{t \geq 0}$ on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$. For simplicity of consideration we will always assume that $W_n(0) = 0$ a.s., $n \geq 0$.

The symbol $D^k[0, \infty)$ will denote the topological space of $\mathbf{R}^k$ -valued functions on $[0, \infty)$ that are right-continuous and have left-hand limits, endowed with J_1 -Skorokhod topology (cf. [1], [2]).

By $\mathcal{D}^n(\mathbf{R}^k)$ (abbreviated $\mathcal{D}^n$ or $\mathcal{D}$) we will denote the set of all stochastic processes with paths in the space $D^k[0, \infty)$, adapted to the filtration $(\mathcal{F}_t^n)_{t \geq 0}$. On $\mathcal{D}$ we put the topology of uniform convergence on compact sets in probability, abbreviated ucp. Note that this topology is metrizable, and the metric

$$d(X, Y) = \sum_{n=1}^{\infty} 2^{-n} \mathbf{E} \left(1 \wedge \sup_{0 \leq t \leq n} \|X(t) - Y(t)\| \right), \quad X, Y \in \mathcal{D},$$

($\|\cdot\|$ denotes the Euclidean norm in $\mathbf{R}$) makes $\mathcal{D}$ into a complete metric space compatible with ucp.

The following limit theorem will be proved in Section 2.

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¹⁾This research was supported in part by the Rhodes University Research Grant.