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SZYSZKOWSKI I.*

WEAK CONVERGENCE OF STOCHASTIC INTEGRALS¹⁾

Пусть $W_n = \{W_n(t): t \geq 0\}$, $n \geq 0$ есть последовательность согласованных càdlàg случайных процессов таких, что для каждого $n \geq 0$ процесс W_n является семимартингалом и имеет место слабая сходимость в пространстве Скорохода $D(\mathbf{R}_+; \mathbf{R}^2): (W_n, [W_n]) \rightarrow (W_0, V_0)$ при $n \rightarrow \infty$, где V_0 — некоторый почти наверное неубывающий процесс. Мы показываем, что $Z_n(\cdot) = \int_0^\cdot f(W_n(t-)) dW_n(t)$ слабо сходится в $D(\mathbf{R}_+, \mathbf{R})$ к процессу $\int_0^\cdot f(W_0(t-)) dW_0(t) + 2^{-1} \int_0^\cdot f'(W_0(t-)) \times d[W_0](t) - 2^{-1} \int_0^\cdot f'(W_0(t-)) dV_0(t)$ для любой аналитической функции f на $\mathbf{R}$. (Здесь $[W_n]$ обозначает квадратическую вариацию семимартингала W_n .)

Ключевые слова и фразы: слабая сходимость, стохастический интеграл, принцип инвариантности, семимартингал.

1. The main result. For each $n \geq 0$ let $W_n = \{(W_n(t), (\mathcal{F}_t^n)): t \geq 0\}$ be an $\mathbf{R}$ -valued càdlàg stochastic process adapted to a right-continuous complete filtration $(\mathcal{F}_t^n)_{t \geq 0}$ on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$. For simplicity of consideration we will always assume that $W_n(0) = 0$ a.s., $n \geq 0$.

The symbol $D^k[0, \infty)$ will denote the topological space of $\mathbf{R}^k$ -valued functions on $[0, \infty)$ that are right-continuous and have left-hand limits, endowed with J_1 -Skorokhod topology (cf. [1], [2]).

By $\mathcal{D}^n(\mathbf{R}^k)$ (abbreviated $\mathcal{D}^n$ or $\mathcal{D}$) we will denote the set of all stochastic processes with paths in the space $D^k[0, \infty)$, adapted to the filtration $(\mathcal{F}_t^n)_{t \geq 0}$. On $\mathcal{D}$ we put the topology of uniform convergence on compact sets in probability, abbreviated ucp. Note that this topology is metrizable, and the metric

$$d(X, Y) = \sum_{n=1}^{\infty} 2^{-n} \mathbf{E} \left(1 \wedge \sup_{0 \leq t \leq n} \|X(t) - Y(t)\| \right), \quad X, Y \in \mathcal{D},$$

($\|\cdot\|$ denotes the Euclidean norm in $\mathbf{R}$) makes $\mathcal{D}$ into a complete metric space compatible with ucp.

The following limit theorem will be proved in Section 2.

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Theorem. Let $\{W_n, n \geq 0\}$ be a sequence of semimartingales such that for each $n \geq 0$, $W_n \in \mathcal{D}^n(\mathbf{R})$. If

$$(W_n, [W_n]) \longrightarrow (W_0, V_0) \text{ weakly, in } D^2[0, \infty) \text{ as } n \rightarrow \infty, \quad (1)$$

for some a.s. non-decreasing process $V_0 \in \mathcal{D}^0(\mathbf{R})$, $V_0 = 0$ a.s., then

$$\begin{aligned} \int_0^\cdot f(W_n(t-)) dW_n(t) &\longrightarrow \int_0^\cdot f(W_0(t-)) dW_0(t) \\ &+ 2^{-1} \int_0^\cdot f'(W_0(t-)) d[W_0](t) - 2^{-1} \int_0^\cdot f'(W_0(t-)) dV_0(t) \end{aligned} \quad (2)$$

weakly, in $D^1[0, \infty)$ as $n \rightarrow \infty$, for any analytic function f ($[X]$ stands for the quadratic variation process of X).

Theorem given above generalizes or extends the latest results presented by Strasser [3], Janson and Wichura [4], Yoshihara [5], Jakubowski, Memin and Pages [6], Kurtz and Protter [7], Szyszkowski [8]–[10] (where the case $V_0 = [W_0]$ was discussed), and Takahata [11].

Note that, in general, V_0 can be different from $[W_0]$.

However for some specific sequences $\{W_n, n \geq 0\}$ of semimartingales condition (1) implies $V_0 = [W_0]$ (see e.g. [12]).

2. The proof (sketch). Note that by utilising [13, Theorems 7.12 and 8.2] and [14, Theorem 4.2], we can reduce our main Theorem to the following “discrete version”.

Proposition. Let $\{X_{nk}, k \geq 1, n \geq 1\}$ be a double array of random variables defined on a common complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and let for each $n \geq 1$, $(k_n(t))_{t \geq 0}$ be a right-continuous, non-decreasing deterministic integer-valued process taking all the values between zero and $k_n(\infty)$. Define

$$W_n(t) = S_{n, k_n(t)}, \quad t \geq 0,$$

where $S_{n, k} = \sum_{i=1}^k X_{ni}$, $k \geq 1$, $n \geq 1$, $\mathcal{F}_t^n = \sigma(X_{ni}: i = 1, 2, \dots, k_n(t))$. Assume (1) holds for some semimartingales W_0 and V_0 . Then (2) holds for any analytic function f on $\mathbf{R}$.

Remark 1. By (1) it follows that, for each $T \geq 0$,

$$\lim_{C \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{P} \left\{ \sup_{0 \leq t \leq T} |W_n(t)| > C \right\} = 0.$$

Hence to prove our Proposition it is enough to show that (2) holds when $\sup_{0 \leq t \leq T} |W_n(t)|$ is a.s. bounded, for each $n \geq 1$ and $T > 0$.

Proof of the Proposition. Observe that it is enough to prove our Proposition for the case $T = 1$ only (cf. [1]). Thus in the sequel we restrict our considerations to the interval $[0, 1]$, using notation and some basic facts contained in the well-known Billingsley's book [14, Ch. 2 and 3].

Let Γ_T denote the set of all finite non-random partitions $\gamma: 0 = t_0 < t_1 < \dots < t_k = T$ of the interval $[0, T]$ and

$$|\gamma| = \max_{1 \leq i \leq k} (t_i - t_{i-1}).$$

For $\gamma = (t_i) \in \Gamma_1$ and $X \in \mathcal{D}$, we adopt the following notation:

$$\begin{aligned} \int_0^t g^\gamma(X) dX &= \sum_{t_i \in \gamma} g(X(t_i)) (X(t_{i+1} \wedge t) - X(t_i \wedge t)), \\ [X]^\gamma(t) &= [X]_t^\gamma = \sum_{t_i \in \gamma} (X(t_i \wedge t) - X(t_{i-1} \wedge t))^2. \end{aligned}$$

For each $\gamma \in \Gamma_1$ and $t \in [0, 1]$, the stochastic integral

$$Z_n(t) = \int_0^t f(W_n) dW_n = \sum_{j=0}^{k_n(t)-1} f(S_{nj}) X_{nj+1}$$

can be expressed as follows:

$$Z_n(t) = \int_0^t f^\gamma(W_n) dW_n + \sum_{t_i \in \gamma} \sum_{j \in B(t_i, n, t)} [f(S_{nj}) - f(W_n(t_i))] X_{nj+1},$$

where $B(t_i, n, t) = \{j: [k_n(t_i \wedge t)] \leq j < [k_n(t_{i+1} \wedge t)]\}$.

Moreover, we have

$$\begin{aligned} & \sum_{t_i \in \gamma} \sum_{j \in B(t_i, n, t)} [f(S_{nj}) - f(W_n(t_i))] X_{nj+1} \\ &= \sum_{t_i \in \gamma} \sum_{j \in B(t_i, n, t)} \sum_{k=1}^{\infty} \left[\frac{df^k}{dx^k}(W_n(t_i)) \frac{(S_{nj} - W_n(t_i))^k}{k!} \right] X_{nj+1} \\ &= \sum_{t_i \in \gamma} \sum_{k=1}^{\infty} \sum_{j \in B(t_i, n, t)} \left[\frac{df^k}{dx^k}(W_n(t_i)) \frac{(S_{nj} - W_n(t_i))^k}{k!} \right] X_{nj+1} \\ &= \sum_{k=1}^{\infty} \sum_{t_i \in \gamma} \sum_{j \in B(t_i, n, t)} \left[\frac{df^k}{dx^k}(W_n(t_i)) \frac{(S_{nj} - W_n(t_i))^k}{k!} \right] X_{nj+1}, \end{aligned}$$

where the interchanges of the summation signs are justified because, on each event of the form $\{\sup_{0 \leq t \leq 1} |W_n(t)| \leq C\}$, $C > 0$, the series under consideration is absolutely convergent (see also Remark 1).

Next, we have

$$\begin{aligned} & \sum_{t_i \in \gamma} \sum_{j \in B(t_i, n, t)} \frac{df}{dx}(W_n(t_i))(S_{nj} - W_n(t_i)) X_{nj+1} \\ &= \sum_{t_i \in \gamma} \frac{df}{dx}(W_n(t_i)) \sum_{j \in B(t_i, n, t)} (S_{nj} - W_n(t_i)) X_{nj+1} \\ &= 2^{-1} \left(\int_0^t \frac{df^\gamma}{dx}(W_n) d[W_n]^\gamma - \int_0^t \frac{df^\gamma}{dx}(W_n) d[W_n] \right) \\ &= \int_0^t \frac{df^\gamma}{dx}(W_n) d\{2^{-1}([W_n]^\gamma - [W_n])\} \\ &= \int_0^t \frac{df^\gamma}{dx}(W_n) dD_{n\gamma} \quad (\text{say}), \quad n \geq 1. \end{aligned}$$

Note that

$$\begin{aligned} & \sum_{t_i \in \gamma} \sum_{j \in B(t_i, n, t)} \frac{df^k}{dx^k}(W_n(t_i))(S_{nj} - W_n(t_i))^{k-1} X_{nj+1} \\ &= \int_0^t \Delta_\gamma^k(W_n) dD_{n\gamma}, \quad t \in [0, 1], \quad n \geq 1, \end{aligned}$$

where

$$\Delta_\gamma^k(W_n)(s) = \frac{df^k}{dx^k}(W_n(t_i)) \frac{(W_n(s) - W_n(t_i))^{k-1}}{k!},$$

for $t_i \leq s < t_{i+1}$, $k \geq 1$.

Now, combining the above results, we obtain for each $\gamma \in \Gamma_1$:

$$Z_n(\cdot) = \int_0^\cdot f^\gamma(W_n) dW_n + \sum_{k=1}^{\infty} \int_0^\cdot \Delta_\gamma^k(W_n) dD_{n\gamma} = I_{n\gamma}(\cdot) + I_{n\gamma}(\cdot) \quad (\text{say}),$$

where

$$D_{n\gamma}(s) = 2^{-1} \left([W_n]^\gamma(s) - [W_n](s) \right).$$

Without loss of generality we may and do assume that all the partitions γ under consideration consist of continuity points of the limiting process (W_0, V_0) .

It is easy to show that, for each $\gamma \in \Gamma_1$, the following weak limit theorem holds in the Skorokhod space $D^2[0, 1]$:

$$\lim_{n \rightarrow \infty} (I_{n\gamma}(\cdot), J_{n\gamma}(\cdot)) = \left(\int_0^\cdot f^\gamma(W_0(t)) dW_0(t), \sum_{k=1}^{\infty} \int_0^\cdot \Delta_\gamma^k(W_0) dD_{0\gamma} \right).$$

Hence, for each $\gamma \in \Gamma_1$,

$$\lim_{n \rightarrow \infty} Z_n(\cdot) = \int_0^\cdot f^\gamma(W_0(t)) dW_0(t) + \sum_{k=1}^{\infty} \int_0^\cdot \Delta_\gamma^k(W_0) dD_{0\gamma},$$

weakly in $D[0, 1]$. But

$$\lim_{|\gamma| \rightarrow 0} \int_0^\cdot f^\gamma(W_0(t)) dW_0(t) = \int_0^\cdot f(W_0(t-)) dW_0(t),$$

uniformly on $[0, 1]$, in probability.

Moreover, the following limit theorems hold with probability one in the space $D[0, 1]$:

$$\begin{aligned} \lim_{|\gamma| \rightarrow 0} D_{0\gamma}(\cdot) &= D_0(\cdot) = 2^{-1} \{ [W_0](\cdot) - V_0(\cdot) \}, \\ \lim_{|\gamma| \rightarrow 0} \int_0^\cdot \Delta_\gamma^1(W_0) dD_{0\gamma} &= 2^{-1} \left\{ \int_0^\cdot f'(W_0(t-)) d[W_0](t) - \int_0^\cdot f'(W_0(t-)) dV_0(t) \right\} \end{aligned}$$

and, for each $k \geq 2$,

$$\lim_{|\gamma| \rightarrow 0} \int_0^\cdot \Delta_\gamma^k(W_0) dD_{0\gamma} \equiv 0.$$

Finally

$$\begin{aligned} \lim_{n \rightarrow \infty} Z_n(\cdot) &= \lim_{|\gamma| \rightarrow 0} \int_0^\cdot f^\gamma(W_0(t)) dW_0(t) + \lim_{|\gamma| \rightarrow 0} \sum_{k=1}^{\infty} \int_0^\cdot \Delta_\gamma^k(W_0) dD_{0\gamma} \\ &= \int_0^\cdot f(W_0(t-)) dW_0(t) \\ &\quad + 2^{-1} \left\{ \int_0^\cdot f'(W_0(t-)) d[W_0](t) - \int_0^\cdot f'(W_0(t-)) dV_0(t) \right\} \end{aligned}$$

weakly in $D[0, 1]$, what completes the proof of our Proposition.

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