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ON THE RATES OF THE CHUNG-TYPE LAW OF LOGARITHM¹⁾

Пусть $\{X, X_n; n \geq 1\}$ — последовательность независимых одинаково распределенных случайных величин. Положим $S_n = X_1 + X_2 + \dots + X_n$, $M_n = \max_{k \leq n} |S_k|$, $n \geq 1$. Используя метод сильной аппроксимации, мы получаем, что если $\mathbf{E} X = 0$, $\mathbf{E} X^2 = \sigma^2 < \infty$ и $\mathbf{E} X^2 (\ln |X|)^{b+1} < \infty$ для некоторого фиксированного $b > -1$, то

$$\begin{aligned} \lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{n=1}^{\infty} \frac{(\ln n)^b}{n} \mathbf{P} \left(M_n \leq \varepsilon \sigma \sqrt{\frac{\pi^2 n}{8 \ln n}} \right) \\ = \frac{4}{\pi} \Gamma(b+1) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2b+3}}. \end{aligned}$$

Обсуждается также сходимость моментов, сходимость в L_2 и сходимость почти наверное.

Ключевые слова и фразы: закон логарифма типа Чжуна, сильная аппроксимация, сходимость моментов, сходимость в L_2 , сходимость почти наверное, независимые одинаково распределенные случайные величины.

1. Introduction and main results. Let $\{X, X_n; n \geq 1\}$ be a sequence of i.i.d. random variables with common distribution function F , mean 0, and positive finite variance σ^2 . Set $S_n = \sum_{k=1}^n X_k$, $M_n = \max_{k \leq n} |S_k|$, $n \geq 1$, and let $\ln n = \ln(n \vee e)$. Hsu and Robbins [11] and Erdős [7] established the well-known complete convergence

$$\sum_{n=1}^{\infty} \mathbf{P}(|S_n| \geq \varepsilon n) < \infty, \quad \varepsilon > 0, \quad (1.1)$$

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if and only if $\mathbf{E} X = 0$ and $\mathbf{E} X^2 < \infty$. Baum and Katz [1] generalized (1.1) and obtained that if $1 \leq p < 2$ and $r > p$, then

$$\sum_{n=1}^{\infty} n^{r-2} \mathbf{P}(|S_n| \geq \varepsilon n^{1/p}) < \infty, \quad \varepsilon > 0, \quad (1.2)$$

if and only if $\mathbf{E} X = 0$ and $\mathbf{E} |X|^{rp} < \infty$.

From then on, many authors considered various extensions of the results of (1.1) and (1.2). Some [9], [10], [15], [16], [19] studied the precise asymptotics. Among of them, Rozovsky [16] proved the following theorem.

Theorem A. Suppose that $\mathbf{E} X = 0$ and $\mathbf{E} X^2 = \sigma^2 < \infty$. Then, for $b \geq 0$,

$$\lim_{\varepsilon \searrow 0} \varepsilon^{2b+2} \sum_{n=1}^{\infty} \frac{(\ln n)^b}{n} \mathbf{P}(S_n \geq \varepsilon \sqrt{n \ln n}) = \frac{\mu^{(2b+2)}}{2(b+1)} \sigma^{2b+2}$$

if and only if $\mathbf{E} X^2 (\ln |X|)^{b-1} < \infty$, where $\mu^{(2b+2)}$ is the $(2b+2)$ th absolute moment of the standard normal distribution.

R e m a r k. In fact, Theorem A is a special case of Corollary 3 in [16], the reader is referred to [15], [16] for more general results.

Moreover, Zhang [20] considered the Chung-type law of logarithm and his result is stated as follows.

Theorem B. Let $r > 1$ and $a > -1$, and let $a_n(\varepsilon)$ be a function of ε such that

$$a_n(\varepsilon) \ln n \rightarrow \tau \text{ as } n \rightarrow \infty \text{ and } \varepsilon \nearrow \frac{1}{\sqrt{r-1}}. \quad (1.3)$$

Assume that

$$\begin{aligned} \mathbf{E} X &= 0, & \mathbf{E} X^2 &= \sigma^2 \quad (0 < \sigma < \infty), \\ \mathbf{E} |X|^{2+\alpha} &< \infty, & \text{for some } 0 < \alpha < 1. \end{aligned} \quad (1.4)$$

Then

$$\begin{aligned} \lim_{\varepsilon \nearrow 1/\sqrt{r-1}} [\varepsilon^{-2} - (r-1)]^{a+1} \sum_{n=1}^{\infty} n^{r-2} (\ln n)^a \mathbf{P}\left(M_n \leq \sigma(\varepsilon + a_n(\varepsilon)) \sqrt{\frac{\pi^2 n}{8 \ln n}}\right) \\ = \frac{4}{\pi} \exp\{2\tau(r-1)^{3/2}\} \Gamma(a+1). \end{aligned} \quad (1.5)$$

Conversely, if (1.5) holds for some $r > 1$ and $a > -1$, then $\mathbf{E} X = 0$ and $\mathbf{E} X^2 = \sigma^2$.

The main goal of this paper is to consider the case of $r = 1$ in Theorem B under the weaker moment condition. Moreover, the moment convergence, L_2 -convergence and a.s. convergence are also discussed. We state our results as follows.

Theorem 1.1. Suppose that $\mathbf{E} X = 0$, $\mathbf{E} X^2 = \sigma^2$ ($0 < \sigma < \infty$), and $\mathbf{E} X^2 (\ln |X|)^{b+1} < \infty$ for some fixed $b > -1$. Then we have

$$\begin{aligned} \lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{n=1}^{\infty} \frac{(\ln n)^b}{n} \mathbf{P}\left(M_n \leq \varepsilon \sigma \sqrt{\frac{\pi^2 n}{8 \ln n}}\right) \\ = \frac{4}{\pi} \Gamma(b+1) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2b+3}}. \end{aligned} \quad (1.6)$$

Conversely, if (1.6) holds for some $b > -1$ and $\sigma > 0$, then $\mathbf{E} X = 0$ and $\mathbf{E} X^2 = \sigma^2$.

Theorem 1.2. Suppose that $\mathbf{E} X = 0$, $\mathbf{E} X^2 = \sigma^2$ ($0 < \sigma < \infty$), and $\mathbf{E} X^2 (\ln |X|)^{b+1/2} < \infty$ for some fixed $b > -1/2$. Then we have

$$\begin{aligned} \lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{n=1}^{\infty} \frac{(\ln n)^b}{n^{3/2}} \mathbf{E} \left\{ \sigma \varepsilon \sqrt{\frac{\pi^2 n}{8 \ln n}} - M_n \right\}_+ \\ = \frac{\sigma}{\sqrt{2(b+1)}} \Gamma\left(b + \frac{1}{2}\right) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2b+2}}. \end{aligned} \quad (1.7)$$

Conversely, if (1.7) holds for some $b > -1/2$ and $\sigma > 0$, then $\mathbf{E}X = 0$ and $\mathbf{E}X^2 = \sigma^2$.

Theorem 1.3. Suppose that $\mathbf{E}X = 0$, $\mathbf{E}X^2 = \sigma^2$ ($0 < \sigma < \infty$), and $\mathbf{E}X^2(\ln|X|)^{b+1} < \infty$ for some fixed $-1 < b < -3/4$. Then we have

$$\begin{aligned} & \lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{n=1}^{\infty} \frac{(\ln n)^b}{n} I\left\{M_n \leq \varepsilon \sigma \sqrt{\frac{\pi^2 n}{8 \ln n}}\right\} \\ &= \frac{4}{\pi} \Gamma(b+1) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2b+3}} \text{ in } L_2 \text{ and a.s.} \end{aligned} \quad (1.8)$$

Conversely, if (1.8) holds in the sense of in probability for some $-1 < b < -3/4$ and $\sigma > 0$, then $\mathbf{E}X = 0$ and $\mathbf{E}X^2 = \sigma^2$.

The proofs of Theorems 1.1, 1.2, and 1.3 are given in Sections 2, 3, and 4, respectively. Throughout this paper, we let A, C etc. denote positive constants whose values can differ in different places. The writing $a_n \sim b_n$ means that $a_n/b_n \rightarrow 1$ as $n \rightarrow \infty$.

2. The proof of Theorem 1.1 . First, we verify (1.6) under the assumption that F is the normal distribution. Let $\{W(t); t \geq 0\}$ be a standard Wiener process and let N be a standard normal variable.

Proposition 2.1. For any fixed $b > -1$, we have

$$\begin{aligned} & \lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{n=1}^{\infty} \frac{(\ln n)^b}{n} \mathbf{P}\left(\sup_{0 \leq s \leq 1} |W(s)| \leq \varepsilon \sqrt{\frac{\pi^2}{8 \ln n}}\right) \\ &= \frac{4}{\pi} \Gamma(b+1) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2b+3}}. \end{aligned} \quad (2.1)$$

The following lemmas will be used in the proof.

Lemma 2.1. Let $\{W(t); t \geq 0\}$ be a standard Wiener process. Then for all $x > 0$,

$$\mathbf{P}\left(\sup_{0 \leq s \leq 1} |W(s)| \leq x\right) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \exp\left\{-\frac{\pi^2(2k+1)^2}{8x^2}\right\}. \quad (2.2)$$

In particular,

$$\begin{aligned} & \frac{2}{\pi} \exp\left\{-\frac{\pi^2}{8x^2}\right\} \leq \mathbf{P}\left(\sup_{0 \leq s \leq 1} |W(s)| \leq x\right) \leq \frac{4}{\pi} \exp\left\{-\frac{\pi^2}{8x^2}\right\}, \\ & \mathbf{P}\left(\sup_{0 \leq s \leq 1} |W(s)| \leq x\right) \sim \frac{4}{\pi} \exp\left\{-\frac{\pi^2}{8x^2}\right\}, \quad x \rightarrow 0. \end{aligned}$$

P r o o f. It is well known, see [5].

Now, we turn to prove Proposition 2.1. By (2.2) and noting that the limit in (2.1) does not depends on any finite terms of the infinite series, we have

$$\begin{aligned} & \lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{n=1}^{\infty} \frac{(\ln n)^b}{n} \mathbf{P}\left(\sup_{0 \leq s \leq 1} |W(s)| \leq \varepsilon \sqrt{\frac{\pi^2}{8 \ln n}}\right) \\ &= \lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{n=1}^{\infty} \frac{(\ln n)^b}{n} \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \exp\left\{-\frac{(2k+1)^2 \ln n}{\varepsilon^2}\right\} \\ &= \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \int_e^{\infty} \frac{(\ln x)^b}{x} \exp\left\{-\frac{(2k+1)^2 \ln x}{\varepsilon^2}\right\} dx \\ &= \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \int_{(2k+1)^2/\varepsilon^2}^{\infty} \left[\frac{s\varepsilon^2}{(2k+1)^2}\right]^b \exp\{-s\} \frac{\varepsilon^2}{(2k+1)^2} ds \end{aligned}$$

$$\begin{aligned} & \text{(by letting } \frac{(2k+1)^2 \ln x}{\varepsilon^2} = s) \\ &= \frac{4}{\pi} \Gamma(b+1) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2b+3}}. \end{aligned}$$

In the sequel, without loss of generality, we assume $\sigma = 1$. For each n and $1 \leq j \leq n$, we let

$$\begin{aligned} X'_{nj} &= X_j I\left\{|X_j| \leq \sqrt{\frac{n}{\ln n}}\right\}, \quad X_{nj}^{(1)} = X'_{nj} - \mathbf{E} X'_{nj}, \\ S_{nj}^{(1)} &= \sum_{i=1}^j X_{ni}^{(1)}, \quad M_n^{(1)} = \max_{1 \leq k \leq n} |S_{nk}^{(1)}|, \quad B_n = \sum_{j=1}^n \mathbf{D} X_{nj}^{(1)}. \end{aligned}$$

Then we have the following result.

Proposition 2.2. *Suppose $\mathbf{E} X = 0$, $\mathbf{E} X^2 = 1$, and $\mathbf{E} X^2 (\ln |X|)^{b+1} < \infty$ for some fixed $b > -1$. Then for any $0 < \alpha < 1$ and $x > 0$, there exists a sequence of positive numbers $\{q_n\}$ such that*

$$\begin{aligned} & \mathbf{P}\left(\sup_{0 \leq s \leq 1} |W(s)| \leq x - \varepsilon^{1-\alpha} \frac{3}{\sqrt{\ln n}}\right) - q_n \leq \mathbf{P}(M_n \leq x\sqrt{B_n}) \\ & \leq \mathbf{P}\left(\sup_{0 \leq s \leq 1} |W(s)| \leq x + \varepsilon^{1-\alpha} \frac{3}{\sqrt{\ln n}}\right) + q_n \end{aligned} \quad (2.3)$$

holds for large n , where $q_n \geq 0$ satisfies

$$\lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{n=1}^{\infty} \frac{(\ln n)^b}{n} q_n = 0. \quad (2.4)$$

The following lemmas will be used in the proof.

Lemma 2.2. *Let $\{W(t); t \geq 0\}$ be a standard Wiener process. For any $\varepsilon > 0$ there exists a constant $C = C(\varepsilon) > 0$ such that*

$$\mathbf{P}\left(\sup_{0 \leq s \leq 1-h} \sup_{0 < t \leq h} |W(s+t) - W(s)| \geq \nu\sqrt{h}\right) \leq \frac{C}{h} e^{-\nu^2/(2+\varepsilon)}$$

for every positive ν and $h < 1$.

P r o o f. It is Lemma 1.1.1 of [6].

Lemma 2.3. *Let $\{\xi_n; n \geq 1\}$ be a sequence of independent random variables with mean zero and finite variance. Then we can redefine $\{\xi_n; n \geq 1\}$ on a richer probability space together with a sequence of independent normal random variables $\{\eta_n; n \geq 1\}$ with $\mathbf{E} \eta_n = 0$ and $\mathbf{E} \eta_n^2 = \mathbf{E} \xi_n^2$ such that for all $q > 2$ and $y > 0$,*

$$\mathbf{P}\left(\max_{k \leq n} \left|\sum_{i=1}^k \xi_i - \sum_{i=1}^k \eta_i\right| \geq y\right) \leq (Aq)^q y^{-q} \sum_{i=1}^n \mathbf{E} |\xi_i|^q,$$

where A is a universal constant.

P r o o f. See [17], [18].

Now, we turn to the proof of Proposition 2.2. By Lemma 2.3, there exist a sequence of independent normal variables $\{\eta_i; i \geq 1\}$ with $\mathbf{E} \eta_i = 0$ and $\mathbf{E} \eta_i^2 = \mathbf{D} X_{ni}^{(1)}$ in the same and richer probability space with $\{X_i; i \geq 1\}$, then for any $0 \leq t \leq 1$, we define

$$W_n(tB_n) = \sum_{i=1}^{[nt]} \eta_i + \eta_{[nt]+1}(nt - [nt]).$$

Let

$$\begin{aligned} q_n &= \mathbf{P}\left(\left|\frac{M_n}{\sqrt{B_n}} - \frac{\sup_{0 \leq s \leq 1} W_n(sB_n)}{\sqrt{B_n}}\right| \geq \varepsilon^{1-\alpha} \frac{3}{\sqrt{\ln n}}\right), \\ p_n &= \mathbf{P}\left(\left|\frac{M_n^{(1)}}{\sqrt{B_n}} - \frac{\sup_{0 \leq s \leq 1} W_n(sB_n)}{\sqrt{B_n}}\right| \geq \varepsilon^{1-\alpha} \frac{1}{\sqrt{\ln n}}\right). \end{aligned}$$

Obviously, q_n satisfies (2.3) since $\{W_n(tB_n)/\sqrt{B_n}; t \geq 0\} \stackrel{\mathcal{D}}{=} \{W(t); t \geq 0\}$ for each n and

$$q_n \leq \mathbf{P}\left(\Delta_n \geq \varepsilon^{1-\alpha} \sqrt{\frac{n}{\ln n}}\right) + p_n \quad (2.5)$$

for large n , where we define $\Delta_n = \max_{k \leq n} |S_{nk}^{(1)} - S_k|$. By noting that $B_n/n \rightarrow 1$ as $n \rightarrow \infty$ and Lemma 2.3, for any fixed $q > 2$,

$$\begin{aligned} p_{n1} &:= \mathbf{P}\left(\max_{k \leq n} \left|S_{nk}^{(1)} - W_n\left(\frac{k}{n}B_n\right)\right| \geq \frac{\varepsilon^{1-\alpha}}{2} \sqrt{\frac{B_n}{\ln n}}\right) \\ &= \mathbf{P}\left(\max_{k \leq n} \left|\sum_{i=1}^k X_{ni}^{(1)} - \sum_{i=1}^k \eta_i\right| \geq \frac{\varepsilon^{1-\alpha}}{2} \sqrt{\frac{B_n}{\ln n}}\right) \\ &\leq C\varepsilon^{-q(1-\alpha)} (\ln n)^{q/2} n^{-q/2} n \mathbf{E}|X|^q I\left\{|X| \leq \sqrt{\frac{n}{\ln n}}\right\}. \end{aligned}$$

So we have

$$\begin{aligned} &\lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{n=1}^{\infty} \frac{(\ln n)^b}{n} p_{n1} \\ &\leq C \lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)-q(1-\alpha)} \sum_{n=1}^{\infty} \frac{(\ln n)^{b+q/2}}{n^{q/2}} \mathbf{E}|X|^q I\left\{|X| \leq \sqrt{\frac{n}{\ln n}}\right\} \\ &= C \lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)-q(1-\alpha)} \sum_{n=1}^{\infty} \frac{(\ln n)^{b+q/2}}{n^{q/2}} \\ &\quad \times \sum_{j=1}^n \mathbf{E}|X|^q I\left\{\sqrt{\frac{j-1}{\ln(j-1)}} < |X| \leq \sqrt{\frac{j}{\ln j}}\right\} \\ &\leq C \lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)-q(1-\alpha)} \sum_{j=1}^{\infty} \mathbf{E}|X|^q I\left\{\sqrt{\frac{j-1}{\ln(j-1)}} < |X| \leq \sqrt{\frac{j}{\ln j}}\right\} \\ &\quad \times j^{1-q/2} (\ln j)^{b+q/2} \\ &\leq C \lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)-q(1-\alpha)} \mathbf{E} X^2 (\ln |X|)^{b+1} = 0. \end{aligned} \quad (2.6)$$

On the other hand, by Lemma 2.2,

$$\begin{aligned} p_{n2} &:= \mathbf{P}\left(\sup_{0 \leq s \leq 1} \left|W_n(sB_n) - W_n\left(\frac{[ns]}{n}B_n\right)\right| \geq \frac{\varepsilon^{1-\alpha}}{2} \sqrt{\frac{B_n}{\ln n}}\right) \\ &= \mathbf{P}\left(\sup_{0 \leq s \leq 1} \left|W_n(s) - W_n\left(\frac{[ns]}{n}\right)\right| \geq \frac{\varepsilon^{1-\alpha}}{2} \sqrt{\frac{n}{\ln n}} \sqrt{\frac{1}{n}}\right) \\ &\leq \mathbf{P}\left(\sup_{0 \leq s' \leq 1-1/n} \sup_{0 < t \leq 1/n} |W_n(s'+t) - W_n(s')| \geq \frac{\varepsilon^{1-\alpha}}{2} \sqrt{\frac{n}{\ln n}} \sqrt{\frac{1}{n}}\right) \\ &\leq Cn \exp\left\{-\frac{n\varepsilon^{2-2\alpha}}{12 \ln n}\right\}. \end{aligned}$$

So we have

$$\begin{aligned} \lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{n=1}^{\infty} \frac{(\ln n)^b}{n} p_{n2} &\leq C \lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{n=1}^{\infty} (\ln n)^b \exp \left\{ -\frac{n\varepsilon^{2-2\alpha}}{12 \ln n} \right\} \\ &\leq C \lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)-2+2\alpha} = 0. \end{aligned} \quad (2.7)$$

Using (2.6) and (2.7) we conclude

$$\lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{n=1}^{\infty} \frac{(\ln n)^b}{n} p_n \leq \lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{n=1}^{\infty} \frac{(\ln n)^b}{n} (p_{n1} + p_{n2}) = 0. \quad (2.8)$$

Moreover, let

$$\alpha_n := n \mathbf{E} |X| I\{|X| > \sqrt{n/\ln n}\} \quad \text{and} \quad \Xi := \{n: \alpha_n \leq \frac{1}{2} \varepsilon^{1-\alpha} \sqrt{n/\ln n}\}.$$

Then for $n \in \Xi$, we have

$$\max_{k \leq n} \left| \sum_{i=1}^k \mathbf{E} X'_{ni} \right| \leq \alpha_n \leq \frac{\varepsilon^{1-\alpha}}{2} \sqrt{\frac{n}{\ln n}}. \quad (2.9)$$

Hence, for $n \in \Xi$,

$$\begin{aligned} \left\{ \Delta_n \geq \varepsilon^{1-\alpha} \sqrt{\frac{n}{\ln n}} \right\} &\subset \left\{ \max_{k \leq n} \left| \sum_{i=1}^k (X'_{ni} - X_i) \right| + \max_{k \leq n} \left| \sum_{i=1}^k \mathbf{E} X'_{ni} \right| \right. \\ &\quad \left. \geq \varepsilon^{1-\alpha} \sqrt{\frac{n}{\ln n}} \right\} \\ &\subset \left\{ \max_{k \leq n} \left| \sum_{i=1}^k (X'_{ni} - X_i) \right| + \alpha_n \geq \varepsilon^{1-\alpha} \sqrt{\frac{n}{\ln n}} \right\} \\ &\subset \left\{ \max_{k \leq n} \left| \sum_{i=1}^k (X'_{ni} - X_i) \right| \geq \frac{\varepsilon^{1-\alpha}}{2} \sqrt{\frac{n}{\ln n}} > 0 \right\} \\ &\subset \bigcup_{j=1}^n \{X_j \neq X'_{nj}\}. \end{aligned} \quad (2.10)$$

So for any $\varepsilon > 0$, we have

$$\begin{aligned} \lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{n \in \Xi} \frac{(\ln n)^b}{n} \mathbf{P} \left(\Delta_n \geq \varepsilon^{1-\alpha} \sqrt{\frac{n}{\ln n}} \right) \\ \leq \lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{n=1}^{\infty} (\ln n)^b \mathbf{P} \left(|X| > \sqrt{\frac{n}{\ln n}} \right) \\ \leq C \lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{j=1}^{\infty} \mathbf{P} \left(\sqrt{\frac{j}{\ln j}} < |X| \leq \sqrt{\frac{j+1}{\ln(j+1)}} \right) j (\ln j)^b \\ \leq C \lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \mathbf{E} X^2 (\ln |X|)^{b+1} = 0. \end{aligned} \quad (2.11)$$

If $n \notin \Xi$, we have

$$\begin{aligned} \lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{n \notin \Xi} \frac{(\ln n)^b}{n} \mathbf{P} \left(\Delta_n \geq \varepsilon^{1-\alpha} \sqrt{\frac{n}{\ln n}} \right) \\ \leq \lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{n \notin \Xi} \frac{(\ln n)^b}{n} \end{aligned}$$

$$\begin{aligned}
&\leq \lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{n \notin \Xi} \frac{(\ln n)^b}{n} \frac{2\alpha_n}{\varepsilon^{1-\alpha}} \sqrt{\frac{\ln n}{n}} \\
&\leq 2 \lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)+\alpha-1} \sum_{n=1}^{\infty} \frac{(\ln n)^{b+1/2}}{\sqrt{n}} \\
&\quad \times \sum_{j=n}^{\infty} \mathbf{E} |X| I \left\{ \sqrt{\frac{j}{\ln j}} < |X| \leq \sqrt{\frac{j+1}{\ln(j+1)}} \right\} \\
&\leq C \lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)+\alpha-1} \sum_{j=1}^{\infty} \mathbf{E} |X| I \left\{ \sqrt{\frac{j}{\ln j}} < |X| \leq \sqrt{\frac{j+1}{\ln(j+1)}} \right\} \sqrt{j} (\ln j)^{b+1/2} \\
&\leq C \lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)+\alpha-1} \mathbf{E} X^2 (\ln |X|)^{b+1} = 0.
\end{aligned} \tag{2.12}$$

We obtain (2.4) from (2.5), (2.8), (2.11), and (2.12). The proposition is proved.

Now we turn to prove Theorem 1.1. For any $0 < a, \alpha < 1$, define

$$\begin{cases} \varepsilon_1 := \varepsilon - 3\varepsilon^{1-\alpha} \sqrt{\frac{8}{\pi^2}} \sim \varepsilon \\ \varepsilon_2 := (1+a)\varepsilon + 3\varepsilon^{1-\alpha} \sqrt{\frac{8}{\pi^2}} \sim \varepsilon \end{cases} \quad \text{as } \varepsilon \nearrow \infty \text{ and } a \searrow 0. \tag{2.13}$$

Noting that $n > B_n \sim n$ and using Proposition 2.2, for any $0 < a < 1$ and n large enough, we have

$$\begin{aligned}
&\mathbf{P} \left(\sup_{0 \leq s \leq 1} |W(s)| \leq \varepsilon_1 \sqrt{\frac{\pi^2}{8 \ln n}} \right) - q_n \\
&= \mathbf{P} \left(\sup_{0 \leq s \leq 1} |W(s)| \leq \varepsilon \sqrt{\frac{\pi^2}{8 \ln n}} - 3\varepsilon^{1-\alpha} \frac{1}{\sqrt{\ln n}} \right) - q_n \\
&\leq \mathbf{P} \left(M_n \leq \varepsilon \sqrt{\frac{\pi^2 B_n}{8 \ln n}} \right) \leq \mathbf{P} \left(M_n \leq \varepsilon \sqrt{\frac{\pi^2 n}{8 \ln n}} \right) \\
&\leq \mathbf{P} \left(M_n \leq (1+a)\varepsilon \sqrt{\frac{\pi^2 B_n}{8 \ln n}} \right) \\
&\leq \mathbf{P} \left(\sup_{0 \leq s \leq 1} |W(s)| \leq (1+a)\varepsilon \sqrt{\frac{\pi^2}{8 \ln n}} + 3\varepsilon^{1-\alpha} \frac{1}{\sqrt{\ln n}} \right) + q_n \\
&= \mathbf{P} \left(\sup_{0 \leq s \leq 1} |W(s)| \leq \varepsilon_2 \sqrt{\frac{\pi^2}{8 \ln n}} \right) + q_n.
\end{aligned}$$

Then (1.6) is proved by the definitions of ε_1 and ε_2 and Proposition 2.1.

Let us prove the converse part. From [8] (see also [14, p. 74, (2.70)]) it is easy to see that for all $m \geq 1$,

$$\mathbf{P}(|S_m| \leq 2\sqrt{m}) \leq K \left(\int_{-2\sqrt{m}}^{2\sqrt{m}} x^2 d\tilde{F}(x) \right)^{-1/2},$$

where $\tilde{F}(x)$ is the distribution function of the symmetrized X , and K is an absolute constant. So if $\mathbf{E} X^2 = \infty$, then for any $M > 2$ we can choose m_0 large enough such that

$$\mathbf{P}(|S_m| \leq 2\sqrt{m}) \leq e^{-2M}, \quad m \geq m_0. \tag{2.14}$$

For $\varepsilon > 0$, let $m = [\varepsilon^2 n / \ln n]$ and $N = [n/m]$ (here $[a]$ means the integer part of a), then for all $n \geq m_0^2$ and $\varepsilon \geq 1$,

$$\mathbf{P} \left(M_n \leq \varepsilon \sqrt{\frac{n}{\ln n}} \right) \leq \mathbf{P}(|S_{km} - S_{(k-1)m}| \leq 2\sqrt{m}, k = 1, \dots, N)$$

$$\leq e^{-2MN} \leq \exp \left\{ -M \frac{\ln n}{\varepsilon^2} \right\}. \quad (2.15)$$

So for any $b > -1$, there exists a constant $C = C(b, \varepsilon)$ such that

$$\begin{aligned} & \limsup_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{n=1}^{\infty} \frac{(\ln n)^b}{n} \mathbf{P} \left(M_n \leq \varepsilon \sqrt{\frac{n}{\ln n}} \right) \\ & \leq \lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{n=1}^{\infty} \frac{(\ln n)^b}{n} \exp \left\{ -M \frac{\ln n}{\varepsilon^2} \right\} \\ & \leq C \lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \int_e^{\infty} \frac{(\ln x)^b}{x} \exp \left\{ -M \frac{\ln x}{\varepsilon^2} \right\} dx \\ & = C \lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \int_{M/\varepsilon^2}^{\infty} \left(\frac{\varepsilon^2 s}{M} \right)^b e^{-s} \frac{\varepsilon^2}{M} ds \\ & = CM^{-(b+1)} \Gamma(b+1) \rightarrow 0 \end{aligned} \quad (2.16)$$

as $M \rightarrow \infty$, which implies that (1.6) cannot hold, i.e., we have $\mathbf{E} X^2 < \infty$. If $\mathbf{E} X = \mu \neq 0$, then (2.14), and further (2.16) also hold since $|S_m/\sqrt{m}| \rightarrow \infty$ a.s. as $m \rightarrow \infty$. So we conclude $\mathbf{E} X = 0$.

Suppose that $\mathbf{E} X = 0$, $\mathbf{E} X^2 < \infty$, and (1.6) holds for some positive constant σ . The direct part of Theorem 1.1 guarantees that (1.6) holds with $\mathbf{E} X^2$ instead of σ^2 . This is obviously a contradiction if $\mathbf{E} X^2 \neq \sigma^2$.

3. The proof of Theorem 1.2. The proof is similar to that of Theorem 1.1, so we only give the key steps. First, we verify (1.7) under the assumption that F is the normal distribution.

Proposition 3.1. Suppose $\mathbf{E} X = 0$, $\mathbf{E} X^2 = 1$, and $\mathbf{E} X^2 (\ln |X|)^{b+1} < \infty$ for some fixed $b > -1/2$. Then we have

$$\begin{aligned} & \lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{n=1}^{\infty} \frac{(\ln n)^b}{n} \mathbf{E} \left\{ \varepsilon \sqrt{\frac{\pi^2}{8 \ln n}} - \sup_{0 \leq s \leq 1} |W(s)| \right\}_+ \\ & = \frac{\Gamma(b+1/2)}{\sqrt{2}(b+1)} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2b+2}}. \end{aligned} \quad (3.1)$$

P r o o f is omitted, since it is similar to that in Theorem 1.1.

Next, using the same truncation method as in Section 2, we have the following result.

Proposition 3.2. Suppose $\mathbf{E} X = 0$, $\mathbf{E} X^2 = 1$, and $\mathbf{E} X^2 (\ln |X|)^{b+1/2} < \infty$ for some fixed $b > -1/2$. Then for any $0 < \alpha < 1$, there exist a sequence of positive numbers $\{p_n\}$ such that

$$\begin{aligned} & \sqrt{\frac{B_n}{n}} \mathbf{E} \left\{ x - \varepsilon^{1-\alpha} \frac{1}{\sqrt{\ln n}} - \sup_{0 \leq s \leq 1} |W(s)| \right\}_+ - p_n \\ & \leq \frac{1}{\sqrt{n}} \mathbf{E} \left\{ x \sqrt{B_n} - M_n^{(1)} \right\}_+ \\ & \leq \sqrt{\frac{B_n}{n}} \mathbf{E} \left\{ x + \varepsilon^{1-\alpha} \frac{1}{\sqrt{\ln n}} - \sup_{0 \leq s \leq 1} |W(s)| \right\}_+ + p_n \end{aligned} \quad (3.2)$$

for large n , where $p_n \geq 0$ satisfies

$$\lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{n=1}^{\infty} \frac{(\ln n)^b}{n} p_n = 0. \quad (3.3)$$

P r o o f. By Lemma 2.3 for any $q > 2$,

$$\begin{aligned} p_{n1} &:= \frac{1}{\sqrt{n}} \mathbf{E} \left\{ \max_{k \leq n} \left| \sum_{j=1}^k X_{nj}^{(1)} - W_n \left(\frac{k}{n} B_n \right) \right| - \frac{1}{2} \varepsilon^{1-\alpha} \sqrt{\frac{B_n}{\ln n}} \right\}_+ \\ &= \frac{1}{\sqrt{n}} \int_{\varepsilon^{1-\alpha} \sqrt{B_n / \ln n} / 2}^{\infty} \mathbf{P} \left(\max_{k \leq n} \left| \sum_{j=1}^k X_{nj}^{(1)} - W_n \left(\frac{k}{n} B_n \right) \right| \geq t \right) dt \\ &\leq C \sqrt{n} \mathbf{E} |X|^q I \left\{ |X| \leq \sqrt{\frac{n}{\ln n}} \right\} \int_{\varepsilon^{1-\alpha} \sqrt{B_n / \ln n} / 2}^{\infty} t^{-q} dt \\ &\leq C \varepsilon^{(1-\alpha)(1-q)} n^{1-q/2} (\ln n)^{(q-1)/2} \mathbf{E} |X|^q I \left\{ |X| \leq \sqrt{\frac{n}{\ln n}} \right\}, \end{aligned}$$

then we have

$$\begin{aligned} \lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{n=1}^{\infty} \frac{(\ln n)^b}{n} p_{n1} &\leq C \lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)+(1-\alpha)(1-q)} n^{-q/2} (\ln n)^{b+(q-1)/2} \mathbf{E} |X|^q I \left\{ |X| \leq \sqrt{\frac{n}{\ln n}} \right\} \\ &\leq C \lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)+(1-\alpha)(1-q)} \sum_{j=1}^{\infty} \mathbf{E} |X|^q I \left\{ \sqrt{\frac{j-1}{\ln(j-1)}} < |X| \leq \sqrt{\frac{j}{\ln j}} \right\} \\ &\quad \times j^{1-q/2} (\ln j)^{b+(q-1)/2} \\ &\leq C \lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)+(1-\alpha)(1-q)} \mathbf{E} X^2 (\ln |X|)^{b+(q-1)/2+(2-q)/2} = 0. \end{aligned} \quad (3.4)$$

On the other hand,

$$\begin{aligned} p_{n2} &:= \frac{1}{\sqrt{n}} \mathbf{E} \left\{ \max_{0 \leq s \leq 1} \left| W_n(s B_n) - W_n \left(\frac{[ns] B_n}{n} \right) \right| - \frac{1}{2} \varepsilon^{1-\alpha} \sqrt{\frac{B_n}{\ln n}} \right\}_+ \\ &\leq C \mathbf{E} \left\{ \max_{0 \leq s \leq 1} \left| W_n(s) - W_n \left(\frac{[ns]}{n} \right) \right| - \frac{1}{2} \varepsilon^{1-\alpha} \frac{1}{\sqrt{\ln n}} \right\}_+ \\ &= C \int_0^{\infty} \mathbf{P} \left(\max_{0 \leq s \leq 1} \left| W_n(s) - W_n \left(\frac{[ns]}{n} \right) \right| \geq t + \frac{1}{2} \varepsilon^{1-\alpha} \frac{1}{\sqrt{\ln n}} \right) dt \\ &\leq C \mathbf{P} \left(\max_{0 \leq s \leq 1} \left| W_n(s) - W_n \left(\frac{[ns]}{n} \right) \right| \geq \frac{1}{2} \varepsilon^{1-\alpha} \frac{1}{\sqrt{\ln n}} \right) \\ &\quad + C \int_1^{\infty} \mathbf{P} \left(\max_{0 \leq s \leq 1} \left| W_n(s) - W_n \left(\frac{[ns]}{n} \right) \right| \geq t \right) dt \\ &=: p_{n21} + p_{n22}. \end{aligned} \quad (3.5)$$

Then by Lemma 2.2,

$$\lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{n=1}^{\infty} \frac{(\ln n)^b}{n} p_{n21} \leq \lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{n=1}^{\infty} (\ln n)^b \exp \left\{ -\frac{n \varepsilon^{2-2\alpha}}{12 \ln n} \right\} = 0, \quad (3.6)$$

and

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(\ln n)^b}{n} p_{n22} &\leq C \sum_{n=1}^{\infty} (\ln n)^b \int_1^{\infty} \exp \left\{ -\frac{nt^2}{3} \right\} dt \\ &\leq C \int_e^{\infty} (\ln x)^b \int_1^{\infty} \exp \left\{ -\frac{xt^2}{3} \right\} dt dx \leq C \int_e^{\infty} \frac{(\ln x)^b}{\sqrt{x}} \int_{\sqrt{x/3}}^{\infty} e^{-y^2} dy dx \\ &\leq C \int_{\sqrt{e/3}}^{\infty} \int_e^{3y^2} \frac{(\ln x)^b}{\sqrt{x}} e^{-y^2} dx dy < \infty. \end{aligned} \quad (3.7)$$

Notice that

$$\begin{aligned} & \mathbf{E} \left\{ x\sqrt{B_n} - \max_{k \leq n} \left| \sum_{j=1}^k X_{nj}^{(1)} \right| \right\}_+ \\ &= \mathbf{E} \left\{ x\sqrt{B_n} - \max_{0 \leq s \leq 1} \left| \sum_{j=1}^{[ns]} X_{nj}^{(1)} - W_n(sB_n) + W_n(sB_n) \right| \right\}_+, \\ & \mathbf{E} \left\{ x\sqrt{B_n} - \max_{0 \leq s \leq 1} |W_n(sB_n)| \right\}_+ = \sqrt{B_n} \mathbf{E} \left\{ x - \sup_{0 \leq s \leq 1} |W_n(s)| \right\}_+. \end{aligned}$$

Let

$$p_n = \mathbf{E} \left\{ \sup_{0 \leq s \leq 1} \left| \sum_{k=1}^{[ns]} X_{nk}^{(1)} - W_n(sB_n) \right| - \varepsilon^{1-\alpha} \sqrt{\frac{B_n}{\ln n}} \right\}_+,$$

then obviously (3.2) holds and $p_n = p_{n1} + p_{n2}$ satisfies (3.3) by (3.4)–(3.7). The proof is completed.

Now we turn to the proof of the direct part of Theorem 1.2. By $\mathbf{E} X = 0$, we have

$$\begin{aligned} & \frac{1}{\sqrt{n}} \mathbf{E} \left\{ \varepsilon \sqrt{\frac{\pi^2 B_n}{8 \ln n}} - M_n \right\}_+ \\ &= \frac{1}{\sqrt{n}} \mathbf{E} \left\{ \varepsilon \sqrt{\frac{\pi^2 B_n}{8 \ln n}} - \max_{k \leq n} \left| \sum_{i=1}^k (X_{ni}^{(1)} + X_i - X_{ni}^{(1)}) \right| \right\}_+ \\ &= \frac{1}{\sqrt{n}} \mathbf{E} \left\{ \varepsilon \sqrt{\frac{\pi^2 B_n}{8 \ln n}} - \max_{k \leq n} \left| \sum_{i=1}^k \left(X_{ni}^{(1)} + X_i I \left\{ |X_i| > \sqrt{\frac{n}{\ln n}} \right\} \right. \right. \right. \\ & \quad \left. \left. \left. - \mathbf{E} X_i I \left\{ |X_i| > \sqrt{\frac{n}{\ln n}} \right\} \right) \right| \right\}_+ \end{aligned} \quad (3.8)$$

and

$$\begin{aligned} q_n &:= \frac{1}{\sqrt{n}} \mathbf{E} \left\{ \max_{k \leq n} \sum_{i=1}^k \mathbf{E} |X_i| I \left\{ |X_i| > \sqrt{\frac{n}{\ln n}} \right\} \right\} \\ &\leq \sqrt{n} \mathbf{E} |X| I \left\{ |X| > \sqrt{\frac{n}{\ln n}} \right\}. \end{aligned} \quad (3.9)$$

Clearly,

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{(\ln n)^b}{\sqrt{n}} \mathbf{E} |X| I \left\{ |X| > \sqrt{\frac{n}{\ln n}} \right\} \\ &= \sum_{n=1}^{\infty} \frac{(\ln n)^b}{\sqrt{n}} \sum_{j=n}^{\infty} \mathbf{E} |X| I \left\{ \sqrt{\frac{j}{\ln j}} < |X| \leq \sqrt{\frac{j+1}{\ln(j+1)}} \right\} \\ &\leq C \sum_{j=1}^{\infty} \mathbf{E} |X| I \left\{ \sqrt{\frac{j}{\ln j}} < |X| \leq \sqrt{\frac{j+1}{\ln(j+1)}} \right\} \sqrt{j} (\ln j)^b \\ &\leq C \mathbf{E} X^2 (\ln |X|)^{b+1/2} < \infty. \end{aligned} \quad (3.10)$$

At the same time, we have

$$\begin{aligned} & \frac{1}{\sqrt{n}} \mathbf{E} \left\{ \varepsilon \sqrt{\frac{\pi^2 B_n}{8 \ln n}} - M_n^{(1)} \right\}_+ - 2q_n \leq \frac{1}{\sqrt{n}} \mathbf{E} \left\{ \varepsilon \sqrt{\frac{\pi^2 B_n}{8 \ln n}} - M_n \right\}_+ \\ &\leq \frac{1}{\sqrt{n}} \mathbf{E} \left\{ \varepsilon \sqrt{\frac{\pi^2 B_n}{8 \ln n}} - M_n^{(1)} \right\}_+ + 2q_n. \end{aligned} \quad (3.11)$$

For any $0 < \alpha, a < 1$, denote

$$\begin{cases} \varepsilon_1 := \varepsilon - \varepsilon^{1-\alpha} \sqrt{\frac{8}{\pi^2}} \sim \varepsilon \\ \varepsilon_2 := \varepsilon(1+a) - \varepsilon^{1-\alpha} \sqrt{\frac{8}{\pi^2}} \sim \varepsilon \end{cases} \quad \text{as } \varepsilon \nearrow \infty \text{ and } a \searrow 0. \quad (3.12)$$

By Proposition 3.2 and (3.11), for n large enough we have

$$\begin{aligned} & \sqrt{\frac{B_n}{n}} \mathbf{E} \left\{ \varepsilon_1 \sqrt{\frac{\pi^2}{8 \ln n}} - \sup_{0 \leq s \leq 1} |W(s)| \right\}_+ - p_n - 2q_n \\ & \leq \sqrt{\frac{B_n}{n}} \mathbf{E} \left\{ \varepsilon \sqrt{\frac{\pi^2}{8 \ln n}} - \varepsilon^{1-\alpha} \frac{1}{\sqrt{\ln n}} - \sup_{0 \leq s \leq 1} |W(s)| \right\}_+ - p_n - 2q_n \\ & \leq \frac{1}{\sqrt{n}} \mathbf{E} \left\{ \varepsilon \sqrt{\frac{\pi^2 B_n}{8 \ln n}} - M_n \right\}_+ \leq \frac{1}{\sqrt{n}} \mathbf{E} \left\{ \varepsilon \sqrt{\frac{\pi^2 n}{8 \ln n}} - M_n \right\}_+ \\ & \leq \frac{1}{\sqrt{n}} \mathbf{E} \left\{ \varepsilon(1+a) \sqrt{\frac{\pi^2 B_n}{8 \ln n}} - M_n \right\}_+ \\ & \leq \sqrt{\frac{B_n}{n}} \mathbf{E} \left\{ \varepsilon(1+a) \sqrt{\frac{\pi^2}{8 \ln n}} + \varepsilon^{1-\alpha} \frac{1}{\sqrt{\ln n}} - \sup_{0 \leq s \leq 1} |W(s)| \right\}_+ + p_n + 2q_n \\ & = \sqrt{\frac{B_n}{n}} \mathbf{E} \left\{ \varepsilon_2 \sqrt{\frac{\pi^2}{8 \ln n}} - \sup_{0 \leq s \leq 1} |W(s)| \right\}_+ + p_n + 2q_n. \end{aligned} \quad (3.13)$$

Then the proof of the direct part is completed by Propositions 3.1, 3.2, inequality (3.10), the definitions of ε_1 , and ε_2 and relation $B_n \sim n$.

Consider the converse part. If $\mathbf{E} X^2 = \infty$, then by (2.15) we have

$$\begin{aligned} & \limsup_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{n=1}^{\infty} \frac{(\ln n)^b}{n^{3/2}} \mathbf{E} \left\{ \varepsilon \sqrt{\frac{n}{\ln n}} - M_n \right\}_+ \\ & = \limsup_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{n=1}^{\infty} \frac{(\ln n)^b}{n^{3/2}} \int_0^{\varepsilon \sqrt{n/\ln n}} \mathbf{P} \left(\varepsilon \sqrt{\frac{n}{\ln n}} - M_n \geq x \right) dx \\ & \leq \limsup_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{n=1}^{\infty} \frac{(\ln n)^b}{n^{3/2}} \int_0^{\varepsilon \sqrt{n/\ln n}} \mathbf{P} \left(M_n \leq \varepsilon \sqrt{\frac{n}{\ln n}} \right) dx \\ & \leq \limsup_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)+1} \sum_{n=1}^{\infty} \frac{(\ln n)^b}{n^{3/2}} \sqrt{\frac{n}{\ln n}} \exp \left\{ -M \frac{\ln n}{\varepsilon^2} \right\} \\ & \leq C \limsup_{\varepsilon \nearrow \infty} \varepsilon^{-2b-1} \int_e^{\infty} \frac{(\ln x)^{b-1/2}}{x} \exp \left\{ -M \frac{\ln x}{\varepsilon^2} \right\} dx \\ & = C \limsup_{\varepsilon \nearrow \infty} \varepsilon^{-2b-1} \int_{M/\varepsilon^2}^{\infty} \left(\frac{\varepsilon^2 s}{M} \right)^{b-1/2} e^{-s} \frac{\varepsilon^2}{M} ds \\ & = C M^{-(b+1/2)} \Gamma \left(b + \frac{1}{2} \right) \rightarrow 0 \end{aligned}$$

as $M \rightarrow \infty$, which is a contradiction to (1.7). So we conclude $\mathbf{E} X^2 < \infty$. The proofs of equalities $\mathbf{E} X = 0$ and $\mathbf{E} X^2 = \sigma^2$ are similar to those in Theorem 1.1, so we also omit them.

4. The proof of Theorem 1.3. We need the following proposition in order to prove Theorem 1.3.

Proposition 4.1. Suppose f is a real function with $|f(x)| \leq A$ and $|f'(x)| \leq A$. Then for any $b < -3/4$ we have

$$\mathbf{D}\left\{\sum_{n=m}^l \frac{(\ln n)^b}{n} f\left(\frac{M_n}{\varepsilon} \sqrt{\frac{\ln n}{n}}\right)\right\} \leq C \frac{(\ln m)^{2b}}{m} + \frac{C}{\varepsilon} (\ln m)^{2b+3/2}. \quad (4.1)$$

P r o o f. Denote $M_{ij}^* = \max_{i \leq k < j} |S_k - S_i|$, then M_{ij}^* and M_i are independent and $|M_j - M_{ij}^*| \leq M_i$. So for any $j > i$,

$$\begin{aligned} & \text{cov} \left(f\left(\frac{M_i}{\varepsilon} \sqrt{\frac{\ln i}{i}}\right), f\left(\frac{M_j}{\varepsilon} \sqrt{\frac{\ln j}{j}}\right) \right) \\ &= \text{cov} \left(f\left(\frac{M_i}{\varepsilon} \sqrt{\frac{\ln i}{i}}\right), f\left(\frac{M_j}{\varepsilon} \sqrt{\frac{\ln j}{j}}\right) - f\left(\frac{M_{ij}^*}{\varepsilon} \sqrt{\frac{\ln j}{j}}\right) \right) \\ &\leq 2A \mathbf{E} \left| f\left(\frac{M_j}{\varepsilon} \sqrt{\frac{\ln j}{j}}\right) - f\left(\frac{M_{ij}^*}{\varepsilon} \sqrt{\frac{\ln j}{j}}\right) \right| \\ &\leq 2A^2 \frac{1}{\varepsilon} \sqrt{\frac{\ln j}{j}} \mathbf{E} |M_j - M_{ij}^*| \leq 2A^2 \frac{1}{\varepsilon} \sqrt{\frac{\ln j}{j}} \mathbf{E} M_i \leq C \frac{\sqrt{i}}{\varepsilon} \sqrt{\frac{\ln j}{j}}. \end{aligned}$$

Further, for $b < -3/4$,

$$\begin{aligned} & \mathbf{D}\left\{\sum_{n=m}^l \frac{(\ln n)^b}{n} f\left(\frac{M_n}{\varepsilon} \sqrt{\frac{\ln n}{n}}\right)\right\} = \sum_{n=m}^l \frac{(\ln n)^{2b}}{n^2} \mathbf{D}\left\{f\left(\frac{M_n}{\varepsilon} \sqrt{\frac{\ln n}{n}}\right)\right\} \\ &+ 2 \sum_{i=m}^l \sum_{j=i+1}^l \frac{(\ln i)^b (\ln j)^b}{ij} \text{cov} \left(f\left(\frac{M_i}{\varepsilon} \sqrt{\frac{\ln i}{i}}\right), f\left(\frac{M_j}{\varepsilon} \sqrt{\frac{\ln j}{j}}\right) \right) \\ &\leq C \sum_{n=m}^l \frac{(\ln n)^{2b}}{n^2} + C \sum_{i=m}^l \sum_{j=i+1}^l \frac{(\ln i)^b (\ln j)^b}{ij} \frac{\sqrt{i}}{\varepsilon} \sqrt{\frac{\ln j}{j}} \\ &\leq C \frac{(\ln m)^{2b}}{m} + \frac{C}{\varepsilon} (\ln m)^{2b+3/2}. \end{aligned} \quad (4.2)$$

Now, we turn to prove Theorem 1.3. Fix $M > 4$ and $0 < \delta < 1/2$. Let $B(\varepsilon) = \exp\{\varepsilon^2/M\}$ and let f be a real function which satisfies

$$I\{|x| < 1 - \delta\} \leq f(x) \leq I\{|x| < 1\} \quad \text{and} \quad \sup_x |f'(x)| < \infty.$$

Define $\varepsilon_k = [k \ln k]$, $k > M > 4$. Then by Proposition 4.1,

$$\begin{aligned} & \mathbf{D}\left\{\sum_{n>B(\varepsilon_k)} \frac{(\ln n)^b}{n} f\left(\frac{M_n}{\varepsilon_{k-1}} \sqrt{\frac{\ln n}{n}}\right)\right\} \\ &\leq C \frac{[\ln B(\varepsilon_k)]^{2b}}{B(\varepsilon_k)} + \frac{C}{\varepsilon_{k-1}} [\ln B(\varepsilon_k)]^{2b+3/2} \\ &= C \left(\frac{[k \ln k]^2}{M}\right)^{2b} \exp\left\{-\frac{[k \ln k]^2}{M}\right\} + \frac{C}{[(k-1) \ln(k-1)]} \left(\frac{[k \ln k]^2}{M}\right)^{2b+3/2} \\ &\leq C \frac{[k \ln k]^{4b}}{M^{2b}} + \frac{C}{M^{2b+3/2}} [k \ln k]^{4b+2}. \end{aligned} \quad (4.3)$$

By $-1 < b < -3/4$ and the Borel–Cantelli lemma we can conclude that

$$\sum_{n>B(\varepsilon_k)} \frac{(\ln n)^b}{n} \left\{ f\left(\frac{M_n}{\varepsilon_{k-1}} \sqrt{\frac{\ln n}{n}}\right) - \mathbf{E} f\left(\frac{M_n}{\varepsilon_{k-1}} \sqrt{\frac{\ln n}{n}}\right) \right\} \rightarrow 0 \text{ in } L_2 \text{ and a.s.}$$

as $k \rightarrow \infty$. Then using Theorem 1.1 and the inequality

$$\lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{n \leq B(\varepsilon)} \frac{(\ln n)^b}{n} \leq \frac{1}{b+1} \lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} [\ln B(\varepsilon)]^{b+1} = \frac{1}{b+1} M^{-(b+1)},$$

we obtain

$$\begin{aligned} & \liminf_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{n > B(\varepsilon)} \frac{(\ln n)^b}{n} I \left\{ M_n \leq \varepsilon \sqrt{\frac{\pi^2 n}{8 \ln n}} \right\} \\ & \geq \liminf_{k \rightarrow \infty} \varepsilon_k^{-2(b+1)} \sum_{n > B(\varepsilon_k)} \frac{(\ln n)^b}{n} I \left\{ M_n \leq \varepsilon_{k-1} \sqrt{\frac{\pi^2 n}{8 \ln n}} \right\} \\ & \geq \liminf_{k \rightarrow \infty} \varepsilon_k^{-2(b+1)} \sum_{n > B(\varepsilon_k)} \frac{(\ln n)^b}{n} f \left(\frac{M_n}{\varepsilon_{k-1} \sqrt{\pi^2 n / (8 \ln n)}} \right) \\ & \geq \liminf_{k \rightarrow \infty} \varepsilon_k^{-2(b+1)} \sum_{n > B(\varepsilon_k)} \frac{(\ln n)^b}{n} \mathbf{E} f \left(\frac{M_n}{\varepsilon_{k-1} \sqrt{\pi^2 n / (8 \ln n)}} \right) \\ & \geq \liminf_{k \rightarrow \infty} \varepsilon_k^{-2(b+1)} \sum_{n > B(\varepsilon_k)} \frac{(\ln n)^b}{n} \mathbf{P} \left(M_n \leq \varepsilon_{k-1} (1 - \delta) \sqrt{\frac{\pi^2 n}{8 \ln n}} \right) \\ & \geq (1 - \delta)^{-2(b+1)} \frac{4}{\pi} \Gamma(b+1) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2b+3}} \\ & \quad - \frac{1}{b+1} M^{-(b+1)} \text{ in } L_2 \text{ and a.s.} \end{aligned} \quad (4.4)$$

By the arbitrariness of δ and M , we have

$$\begin{aligned} & \liminf_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{n > B(\varepsilon)} \frac{(\ln n)^b}{n} I \left\{ M_n \leq \varepsilon \sqrt{\frac{\pi^2 n}{8 \ln n}} \right\} \\ & \geq \frac{4}{\pi} \Gamma(b+1) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2b+3}} \text{ in } L_2 \text{ and a.s.} \end{aligned} \quad (4.5)$$

Similarly, we have

$$\begin{aligned} & \limsup_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{n > B(\varepsilon)} \frac{(\ln n)^b}{n} I \left\{ M_n \leq \varepsilon \sqrt{\frac{\pi^2 n}{8 \ln n}} \right\} \\ & \leq \frac{4}{\pi} \Gamma(b+1) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2b+3}} \text{ in } L_2 \text{ and a.s.} \end{aligned} \quad (4.6)$$

Inequalities (4.5) and (4.6) complete the proof of the direct part.

Next we will prove the converse part. We show $\mathbf{E} X^2 < \infty$ first. If $\mathbf{E} X^2 = \infty$, then by the proof of Theorem 1.1 we know that

$$\lim_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{n=1}^{\infty} \frac{(\ln n)^b}{n} \mathbf{P} \left(M_n \leq \varepsilon \sqrt{\frac{\pi^2 n}{8 \ln n}} \right) = 0. \quad (4.7)$$

On the other hand, if (1.8) holds in the sense of in probability, then by the Fatou lemma,

$$\begin{aligned} & \limsup_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{n=1}^{\infty} \frac{(\ln n)^b}{n} \mathbf{P} \left(M_n \leq \varepsilon \sqrt{\frac{\pi^2 n}{8 \ln n}} \right) \\ & = \sigma^{-2(b+1)} \limsup_{\varepsilon \nearrow \infty} \varepsilon^{-2(b+1)} \sum_{n=1}^{\infty} \frac{(\ln n)^b}{n} \mathbf{P} \left(M_n \leq \varepsilon \sigma \sqrt{\frac{\pi^2 n}{8 \ln n}} \right) \end{aligned}$$

$$\begin{aligned}
&\geq \sigma^{-2(b+1)} \liminf_{k \rightarrow \infty} \mathbf{E} \left[\varepsilon_k^{-2(b+1)} \sum_{n=1}^{\infty} \frac{(\ln n)^b}{n} I \left\{ M_n \leq \varepsilon_k \sigma \sqrt{\frac{\pi^2 n}{8 \ln n}} \right\} \right] \\
&\geq \sigma^{-2(b+1)} \mathbf{E} \left[\liminf_{k \rightarrow \infty} \varepsilon_k^{-2(b+1)} \sum_{n=1}^{\infty} \frac{(\ln n)^b}{n} I \left\{ M_n \leq \varepsilon_k \sigma \sqrt{\frac{\pi^2 n}{8 \ln n}} \right\} \right] \\
&= \sigma^{-2(b+1)} \frac{4}{\pi} \Gamma(b+1) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2b+3}} > 0,
\end{aligned} \tag{4.8}$$

which is a contradiction to (4.7). The proof of $\mathbf{E} X^2 = \sigma^2$ and $\mathbf{E} X = 0$ are similar to these in Theorem 1.1, and the details are omitted here. The proof is completed.

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