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© 2002 г. *BARNDORFF-NIELSEN O.E.*, *PÉREZ-ABREU V.***EXTENSIONS OF TYPE G AND MARGINAL INFINITE
DIVISIBILITY¹⁾

Случайный элемент из евклидова пространства мы называем маргинально безгранично делимым относительно некоторого класса линейных отображений этого пространства, если распределение образа этого случайного элемента при любом отображении из указанного класса является безгранично делимым. Частные случаи данного понятия применяются для обобщения на многомерный случай понятия вероятностного закона типа G . Центральная роль принадлежит неотрицательным матрицам.

Ключевые слова и фразы: обратное распределение Уишарта, матричный обратно-гауссовский закон, многомерный гауссовский\обратно-гауссовский закон, многомерные устойчивые законы, случайные положительно определенные матрицы, саморазложимость.

1. Introduction

Marcus [28] introduced the concept of type G random variables and processes and this has been further studied by Rosinski [30] (see also [27] and references therein).

Recall that a random variable x is said to be of type G if in law x is of the form $z\sqrt{s}$, where z and $s > 0$ are independent random variables with s being infinitely divisible and z having the standard normal distribution. We refer to $z\sqrt{s}$ as a G -representation of x .

There is no single extension of type G to higher dimensions. In this paper we present two extensions that we called $\text{mult}G$ and $\text{ext}G$. The first extension is to random vectors x of the form $zS^{1/2}$, where z is an m -dimensional standard normal vector independent of the infinitely divisible nonnegative

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random $m \times m$ matrix S . The second, more general, class consists of those random vectors whose one-dimensional projections are of type G .

In independent, concurrent, work [25], [26] (for some related earlier work, see [30]) Maejima and Rosinsky have proposed and studied an extension of type G that, as it has turned out, is a special case of what in this paper is called $\text{mult}G$. Their results are largely different from those presented here.

Part of the interest in type G stems from its relation to the notion of subordination of Brownian motion. From this point of view a natural generalization is to multivariate subordination as studied in [5]. This latter approach is different from those mentioned above.

Section 2 introduces general concepts of marginal infinite divisibility and marginal selfdecomposability and considers these more closely in the special case of nonnegative random matrices, showing in particular that several kinds of random matrices, commonly used in statistics, though not infinitely divisible do have the property of marginal infinite divisibility. In that section we also review some useful general facts about (fully) infinitely divisible nonnegative random matrices and present some examples, including a matrix version of the Inverse Gaussian law. Section 3 introduces the $\text{mult}G$ distributions and derive some properties of these, and Section 4 provides various examples, including a new representation of the characteristic function of the symmetric multivariate stable law as well a multivariate extension of the Normal Inverse Gaussian law. Then, in Section 5, the concept of $\text{ext}G$ laws, which includes that of type $\text{mult}G$ as a special case, is defined and some elementary properties of $\text{ext}G$ are established. In Section 6 we return to the concept of marginal infinite divisibility of positive definite random matrices and discuss its relation to $\text{ext}G$. Section 7 extends the discussion to stochastic processes of type $\text{ext}G$ and includes a variety of examples of this type. The concluding Section 8 indicates some open problems.

In the sequel we shall use the following notation for characteristic functions and cumulant transforms of a random variate x

$$\chi\{\zeta \nmid x\} = \mathbf{E}\{e^{i\zeta x}\}, \quad C\{\zeta \nmid x\} = \ln \mathbf{E}\{e^{i\zeta x}\}, \quad \bar{K}\{\theta \nmid x\} = \ln \mathbf{E}\{e^{-\theta x}\}$$

with straightforward extensions of the notation to more general random variates. Vectors will be taken as row vectors.

2. Random matrices and infinite divisibility

In Subsection 2.2 below we summarize, for later use, some known facts about infinitely divisible random matrices and we present some examples of such matrices. First, however, we shall propose concepts of marginal infinite divisibility and of marginal selfdecomposability of probability laws and apply them to the special case of random matrices.

The results discussed here will be drawn upon in the subsequent sections of the paper, that deal with the extended concepts of type G .

2.1. Marginal infinite divisibility and marginal selfdecomposability. We first consider marginal infinite divisibility.

Definition 2.1. Let E be a Euclidean space and let Ψ be a collection of linear maps, each element ψ of Ψ mapping E into some Euclidean space. A random variate x in E is said to be marginal infinitely divisible relative to Ψ if the law of $\psi(x)$ is infinitely divisible for all $\psi \in \Psi$.

Trivially, infinite divisibility implies marginal infinite divisibility, whatever the class Ψ .

We recall in this connection that infinite divisibility of all linear combinations $c \cdot x = c_1 x_1 + \cdots + c_m x_m$ of an m -dimensional random vector x is not sufficient to imply infinite divisibility of x itself, see [17] and [24, p. 224].

In the following we apply the above concept of marginal infinite divisibility to random vectors in $\mathbf{R}^m$ with Ψ the class of linear maps $c \cdot x$ and to the space of random $m \times m$ matrices M with Ψ being the class of mappings of the form

$$\psi: M \rightarrow cMc^T$$

for $c \in \mathbf{R}^m$. In the rest of the paper only these two types of marginal infinite divisibility are considered, so when we speak of marginal infinite divisibility below the relevant class Ψ is understood. A similar remark applies to the concept of marginal selfdecomposability introduced at the end of the present subsection.

Many useful nonnegative definite random matrices are not infinitely divisible, but have the weaker property just introduced.

Example 2.1. Wishart distribution. It was shown by Lévy [22] that the Wishart distribution is not infinitely divisible (see also [14] and [37]). However, the distribution is marginal infinitely divisible. To show this it is enough to consider the standard Wishart distribution. Thus let $z_1, \dots, z_n$ be independent and m -dimensional random vectors having the standard normal distribution and let $W_n = z_1^T z_1 + \cdots + z_n^T z_n$. To see that W_n is marginal infinitely divisible it is enough to prove that $z_1^T z_1$ is marginal infinitely divisible, which follows since for any $c \in \mathbf{R}^m$ we have $cz_1^T z_1 c^T = (cz_1^T)^2$, which obeys a Gamma law.

Example 2.2. Inverse Wishart distribution. The Inverse Wishart law is marginal infinitely divisible. In fact, if S is a nondegenerate nonnegative definite random $m \times m$ matrix such that S^{-1} follows a Wishart distribution, i.e., S itself has an inverse Wishart law, then for any $r \times m$ matrix C of rank r ($r \leq m$) the law of CSC^T is inverse Wishart, cf. for instance, [13, p. 330]. In particular then, if $C = c$, a vector of dimension m , we have that cSc^T has a reciprocal Gamma distribution and this is infinitely divisible (cf. [3], see also [16]).

R e m a r k 2.1. It was pointed out to us by M. Eaton, that a nonnegative definite random matrix S which is orthogonally invariant ($OSO^T \stackrel{\text{law}}{=} S$ for every orthonormal matrix O) is marginal infinitely divisible if and only if the first diagonal element s_{11} of S is infinitely divisible. Then, since the Wishart and the Inverse Wishart distributions are orthogonally invariant, this provides another proof of the marginal infinite divisibility of these laws.

Recall that a random element x is selfdecomposable if for every $\rho \in (0, 1)$ it is possible to represent x in law as

$$x \stackrel{\text{law}}{=} \rho x + x_\rho$$

for some random element x_ρ independent of x .

We define marginal selfdecomposability in complete analogy to marginal infinite divisibility.

D e f i n i t i o n 2.2. Let E be a Euclidean space and let Ψ be a collection of linear maps, each element ψ of Ψ mapping E into some Euclidean space. A random variate x in E is said to be marginally selfdecomposable relative to Ψ if the law of $\psi(x)$ is selfdecomposable for all $\psi \in \Psi$.

Clearly, selfdecomposability implies marginal selfdecomposability, whatever the class Ψ .

The Wishart and Inverse Wishart distributions are not only marginal infinitely divisible, as shown above, but in fact marginal selfdecomposable, as is easily verified.

There exist multivariate infinitely divisible random vectors x such that every linear transformation of x to a lower dimension is selfdecomposable without the distributions being themselves selfdecomposable, see [34]. A rather similar remark applies to stable laws, cf. [15].

2.2. Infinitely divisible random matrices. The study of infinitely divisible nonnegative random matrices can be carried out in the context of infinitely divisible random elements taking values in cones. A nondegenerate cone $\mathcal{K}$ of a finite dimensional Euclidean space $\mathcal{X}$ is a convex closed set containing the points λx , $\lambda \geq 0$, along with each point $x \in \mathcal{K}$.

A probability measure Q on a cone $\mathcal{K}$ is infinitely divisible if for any $n \geq 1$, there exists a probability measure Q_n on $\mathcal{K}$ such that $Q = Q_n^{*n}$, i.e., Q is the n -fold convolution of Q_n with itself.

In the sequel $\mathcal{X} = \mathbf{M}_m$, the Euclidean space of $m \times m$ symmetric matrices with inner product $\langle A, B \rangle = \text{tr}(AB)$, and $\mathcal{K} = \mathbf{M}_m^+$, the cone of symmetric positive definite matrices in $\mathbf{M}_m$. We write $\mathbf{S}$ for the unit sphere of $\mathbf{M}_m$ and $\mathbf{SM}_m = \mathbf{S} \cap \mathbf{M}_m$ and $\mathbf{SM}_m^+ = \mathbf{S} \cap \mathbf{M}_m^+$.

The following result can be obtained from [19] (cf. also [38, p. 156–157] for a closely related result).

Proposition 2.1. *The distribution of a random element S taking values in the cone $\mathbf{M}_m^+$ is infinitely divisible if and only if its cumulant function*

allows the representation

$$\overline{K}\{\Theta \nmid S\} = -\langle A, \Theta \rangle - \int_{\mathbf{M}_m^+} (1 - e^{-\langle \Theta, \Sigma \rangle}) V(d\Sigma), \quad (1)$$

where A (the drift) is in $\mathbf{M}_m^+$ and V is a measure on $\mathbf{M}_m^+$ such that

$$\int_{\mathbf{M}_m^+} \min(1, |\Sigma|) V(d\Sigma) < \infty$$

and $V(\mathbf{M}_m \setminus \mathbf{M}_m^+) = 0$. (Here $|\Sigma|$ denotes the trace norm of Σ , i.e., $|\Sigma| = (\text{tr } \Sigma^2)^{1/2}$.)

Below we present some important examples of infinitely divisible non-negative definite random matrices.

Example 2.3. Matrix α -stable laws. We now consider results on stable random matrices. An $\mathbf{M}_m$ -valued random element S is said to have an α -stable distribution (see [32, Remark 3, p. 66]) if and only if $0 < \alpha \leq 2$ and for every $\Sigma \in \mathbf{M}_m$

$$C\{\Sigma \nmid S\} = \int_{\mathbf{SM}_m} \int_0^\infty \left[e^{ir \text{tr}(\Sigma \Theta)} - 1 - ir \frac{\text{tr}(\Sigma \Theta)}{1 + r^2} \right] \frac{dr}{r^{1+\alpha}} \Gamma(d\Theta) + i \text{tr}(\Sigma \Theta_0),$$

where $\Theta_0 \in \mathbf{M}_m$ and Γ is a finite measure on $\mathbf{SM}_m$.

For $0 < \alpha < 1$, one has $\int_0^\infty (1 + r^2)^{-1} r^{-\alpha} dr < \infty$, and the centering term can be absorbed in the constant term. Supposing further that Γ is concentrated on the unit sphere $\mathbf{SM}_m^+$ of the closed cone $\mathbf{M}_m^+$, as a special case of the above formula for $C\{\Sigma \nmid S\}$ we therefore have for every $\Sigma \in \mathbf{M}_m^+$

$$\begin{aligned} C\{\Sigma \nmid S\} &= \int_{\mathbf{SM}_m^+} \int_0^\infty (e^{ir \text{tr}(\Sigma \Theta)} - 1) \frac{dr}{r^{1+\alpha}} \Gamma(d\Theta) \\ &= \int_{\mathbf{M}_m^+} (e^{i \text{tr}(\Sigma \Lambda)} - 1) V(d\Lambda), \end{aligned} \quad (2)$$

where $V(d\Lambda) = r^{-(1+\alpha)} dr \Gamma(d\Theta)$, $\Lambda = r\Theta$, $r = \|\Lambda\| = \{\text{tr}(\Lambda\Lambda)\}^{1/2}$, $\Theta = \Lambda/r$ and V is such that $\int_{\|\Lambda\| \leq 1} \|\Lambda\| V(d\Lambda) < \infty$. Then, from Proposition 2.1, (2) is the cumulant function of an infinitely divisible symmetric nonnegative definite random matrix S such that for $\Sigma \in \mathbf{M}_m^+$

$$\overline{K}\{\Sigma \nmid S\} = - \int_{\mathbf{SM}_m^+} \int_0^\infty (1 - e^{-r \text{tr}(\Sigma \Theta)}) \frac{dr}{r^{1+\alpha}} \Gamma(d\Theta).$$

Finally, by usual integration techniques we have

$$\int_0^\infty \left[e^{-r \text{tr}(\Sigma \Theta)} - 1 \right] \frac{dr}{r^{1+\alpha}} = -[\text{tr}(\Sigma \Theta)]^\alpha.$$

Thus we have shown that for any $0 < \alpha < 2$ there exists an $\alpha/2$ -stable — and hence infinitely divisible — nonnegative definite symmetric random matrix S such that for $\Sigma \in \mathbf{M}_m^+$

$$\overline{K}\{\Sigma \nmid S\} = - \int_{\mathbf{SM}_m^+} [\text{tr}(\Sigma \Theta)]^{\alpha/2} \Gamma(d\Theta). \quad (3)$$

We observe that when Γ is concentrated on $\{m^{-1/2}\mathbf{I}\}$, $\text{tr}(S)$ follows a one-dimensional nonnegative $\alpha/2$ -stable distribution.

Example 2.4. *Multivariate extension of the tempered stable laws.* The class of tempered ($\alpha/2$ -) stable laws are obtained from the positive $\alpha/2$ -stable laws by exponential tilting. This class has recently been discussed in [4], [11], see also the contribution of Rosinski to the discussion of [9] and [12].

A matrix extension of the tempered stable distributions may now be obtained as follows. Let S be an $\alpha/2$ -stable nonnegative definite random matrix with cumulant function (3), let $\Sigma \in \mathbf{M}_m^+$ and let R be a random matrix with density relative to the law of S given by

$$p(r; \Sigma) = \exp \left\{ -\text{tr}(r\Sigma) - \overline{K}\{\Sigma \sharp S\} \right\} \tag{4}$$

and cumulant transform

$$\overline{K}\{\Phi \sharp R\} = \overline{K}\{\Phi + \Sigma \sharp S\} - \overline{K}\{\Sigma \sharp S\}. \tag{5}$$

In other words, the law of R is determined by exponential tilting from the law of S .

For any natural exponential family of probability laws $\mathscr{P}$, an element of $\mathscr{P}$ is infinitely divisible if and only if all elements of $\mathscr{P}$ are infinitely divisible. Hence R is an infinitely divisible nonnegative definite symmetric random matrix. Furthermore,

$$\begin{aligned} \overline{K}\{\Phi \sharp R\} &= \int_{\mathbf{SM}_m^+} [\text{tr}(\Sigma\Theta)]^{\alpha/2} \Gamma(d\Theta) - \int_{\mathbf{SM}_m^+} \left[\text{tr}((\Sigma + \Phi)\Theta) \right]^{\alpha/2} \Gamma(d\Theta) \\ &= \int_{\mathbf{SM}_m^+} \int_0^\infty \left(1 - e^{-r\text{tr}(\Sigma\Theta)} \right) \frac{dr}{r^{1+\alpha/2}} \Gamma(d\Theta) \\ &\quad - \int_{\mathbf{SM}_m^+} \int_0^\infty \left(1 - e^{-r\text{tr}((\Sigma+\Phi)\Theta)} \right) \frac{dr}{r^{1+\alpha/2}} \Gamma(d\Theta) \\ &= - \int_{\mathbf{SM}_m^+} \int_0^\infty \left(1 - e^{-r\text{tr}(\Phi\Theta)} \right) e^{-r\text{tr}(\Sigma\Theta)} \frac{dr}{r^{1+\alpha/2}} \Gamma(d\Theta) \\ &= - \int_{\mathbf{M}_m^+} \left(1 - e^{-\text{tr}(\Phi\Omega)} \right) V(d\Omega), \end{aligned} \tag{6}$$

where

$$V(d\Omega) = \frac{e^{-r\text{tr}(\Sigma\Theta)}}{r^{1+\alpha/2}} dr \Gamma(d\Theta)$$

is a Lévy measure, with $r^2 = \text{tr}(\Omega^T\Omega)$ and $\Theta = \Omega/r$.

In the case $\alpha = 1$, R follows a matrix extension of the Inverse Gaussian distribution, where $\Sigma \in \mathbf{M}_m^+$ and Γ , a finite measure on $\mathbf{SM}_m^+$, are parameters. We note that if C is a nonsingular $m \times m$ deterministic matrix then the random matrix CRC^T follows again this matrix extension of the Inverse Gaussian distribution but now with parameters $(C^{-1})^T\Sigma C^{-1}$ and $\Gamma(C^{-1}d \cdot (C^{-1})^T)$. This generalizes an elementary property of the one-dimensional Inverse Gaussian distribution.

2.3. Selfdecomposable random matrices. We recall that a nonnegative random matrix S is selfdecomposable if for any $0 < \rho < 1$, there exists a random matrix S_ρ independent of S such that $S \stackrel{\text{law}}{=} \rho S + S_\rho$.

A characterization of selfdecomposable random matrices was given by Sato [33] (for extensions to Banach space valued elements, see [20] and [21]).

Proposition 2.2. *A nonnegative random matrix S is selfdecomposable if and only if its Lévy measure V is given by*

$$V(A) = \int_{\mathbf{SM}_m^+} \Gamma(d\omega) \int_0^\infty 1_A(t\omega) h(t, \omega) t^{-1} dt, \quad (7)$$

for A in the family of Borel sets $\mathbf{B}(\mathbf{M}_m^+ \setminus \{0\})$, where Γ is a finite measure on $\mathbf{SM}_m^+$ and the function $h(t, \omega)$ is nonnegative, nonincreasing in t and Borel measurable. Moreover, Γ and h satisfy

$$\int_{\mathbf{SM}_m^+} \Gamma(d\omega) \int_0^\infty \min(t, 1) h(t, \omega) t^{-1} dt < \infty.$$

P r o o f. This result follows from Theorem 6.1 in [20] (taking $U_t x = tx$) and Proposition 2.1 above.

Proposition 2.3. *Let S be a selfdecomposable random matrix. Then, for any $0 < \rho < 1$, the random matrix S_ρ in the decomposition $S \stackrel{\text{law}}{=} \rho S + S_\rho$ is a nonnegative and infinitely divisible random matrix.*

P r o o f. From (1) and (7) (without loss of generality we take $A = 0$)

$$\begin{aligned} \overline{K}\{\Theta \nmid S_\rho\} &= \overline{K}\{\Theta \nmid S\} - \overline{K}\{\Theta \nmid \rho S\} \\ &= - \int_{\mathbf{M}_m^+} \left(1 - e^{-\text{tr}(\Theta\Sigma)}\right) V(d\Sigma) + \int_{\mathbf{M}_m^+} \left(1 - e^{-\rho\text{tr}(\Theta\Sigma)}\right) V(d\Sigma) \\ &= - \int_{\mathbf{SM}_m^+} \Gamma(d\omega) \int_0^\infty \left(1 - e^{-t\text{tr}(\omega\Sigma)}\right) \{h(t, \omega) - h(\rho^{-1}t, \omega)\} t^{-1} dt. \end{aligned}$$

From Proposition 2.2 we have that h is nonincreasing in t which implies that $h_\rho(t, \omega) = h(t, \omega) - h(\rho^{-1}t, \omega)$ is nonnegative. Defining

$$V_\rho(A) = \int_{\mathbf{SM}_m^+} \Gamma(d\omega) \int_0^\infty 1_A(t\omega) h_\rho(t, \omega) t^{-1} dt \quad (8)$$

we have that $V_\rho(A) \geq 0$ and

$$\int_{\mathbf{M}_m^+} \min(1, |\Sigma|) V_\rho(d\Sigma) = \int_{\mathbf{SM}_m^+} \Gamma(d\omega) \int_0^\infty \min(t, 1) h_\rho(t, \omega) t^{-1} dt < \infty.$$

Therefore, $\overline{K}\{\Theta \nmid S_\rho\} = - \int_{\mathbf{M}_m^+} (1 - e^{-\text{tr}(\Theta\Sigma)}) V_\rho(d\Sigma)$. The result now follows from Proposition 2.1.

Example 2.5. Matrix tempered stable laws. The matrix tempered stable laws, introduced above, are selfdecomposable. This follows from Example 2.4 and Proposition 2.2, with $h(t, \omega) = t^{-\alpha} e^{-t\text{tr}(\Sigma\omega)}$ which is positive and decreasing in $t > 0$. (Since stability implies selfdecomposability, the matrix α -stable laws provide further examples of selfdecomposable random matrices.)

3. multG laws

Let S be a nonnegative definite random $m \times m$ matrix with law Q and let $S^{1/2}$ be its symmetric square root. Then, for z a standard normal vector in $\mathbf{R}^m$ independent of S , a random vector x such that $x \stackrel{\text{law}}{=} zS^{1/2}$ will have distribution function

$$P(x) = \int_{\mathbf{M}_m^+} \Phi_m(x; \Sigma) Q(d\Sigma), \quad (9)$$

where $\Phi_m(x; \Sigma)$ denotes the distribution function of the m -dimensional normal distribution with mean 0 and covariance matrix Σ . Furthermore,

$$C\{\zeta \nmid x\} = \overline{K} \left\{ \frac{1}{2} \zeta^T \zeta \nmid S \right\}. \quad (10)$$

Definition 3.1. An m -dimensional random vector x is said to be of type multG if x is representable in law as $zS^{1/2}$ with S a nonnegative definite and infinitely divisible random matrix and z a standard normal vector in $\mathbf{R}^m$ independent of S .

In this case, which constitutes an immediate generalization of type G, we say that x has G -representation $zS^{1/2}$. Note that if x has G -representation $zS^{1/2}$ with $S = sI$, for s a nonnegative infinite divisible random variable, then the law of $c \cdot x$ depends on c through $|c|^2$ only.

Proposition 3.1. Let x be an m -dimensional multG random vector with G -representation $zS^{1/2}$, and suppose that the infinitely divisible nonnegative definite random matrix S with law Q has Lévy measure V , concentrated on the space on positive definite matrices. Then x is infinitely divisible with probability density p of the form

$$p(x) = \int_{\mathbf{M}_m^+} \varphi_m(x; \Sigma) Q(d\Sigma)$$

and Lévy density u given by

$$u(x) = \int_{\mathbf{M}_m^+} \varphi_m(x; \Sigma) V(d\Sigma), \quad (11)$$

where $\varphi_m(x; \Sigma)$ denotes the density function of the m -dimensional normal distribution with mean 0 and variance matrix Σ .

P r o o f. Since S is infinitely divisible there exists an $\mathbf{M}_m^+$ -valued Lévy process $S(t)$ such that $S(1) \stackrel{\text{law}}{=} S$. For each $t > 0$, by formula (10)

$$tC\{\zeta \nmid x\} = t\overline{K} \left\{ \frac{1}{2} \zeta^T \zeta \nmid S \right\} = \overline{K} \left\{ \frac{1}{2} \zeta^T \zeta \nmid S(t) \right\} = C\{\zeta \nmid x(t)\},$$

where $x(t) \stackrel{\text{law}}{=} zS(t)^{1/2}$, for z a standard normal vector independent of $S(t)$. This verifies the infinite divisibility of x .

In proving (11), without loss of generality we may assume that x has no Gaussian component, i.e., the characteristic triplet of x is of the

form $(0, 0, U)$. Now, on the one hand, since x is a symmetric infinitely divisible random vector we have

$$C\{c \dagger x\} = \int_{\mathbf{R}^m} (\cos(c \cdot x) - 1) U(dx). \quad (12)$$

On the other hand, by Proposition 2.1,

$$\overline{K}\left\{\frac{1}{2} c^T c \dagger S\right\} = - \int_{\mathbf{M}_m^+} \left(1 - e^{-\text{tr}(c^T c \Sigma)/2}\right) V(d\Sigma).$$

Recalling further that for a given $\Sigma \in \mathbf{M}_m^+$

$$e^{-c \Sigma c^T / 2} = \int_{\mathbf{R}^m} \cos(c \cdot x) \varphi_m(x; \Sigma) dx \quad (13)$$

which implies $1 - e^{-c \Sigma c^T / 2} = \int_{\mathbf{R}^m} (1 - \cos(c \cdot x)) \varphi_m(x; \Sigma) dx$ we find

$$\overline{K}\left\{\frac{1}{2} c^T c \dagger S\right\} = - \int_{\mathbf{M}_m^+} \int_{\mathbf{R}^m} (1 - \cos(c \cdot x)) \varphi_m(x; \Sigma) dx V(d\Sigma) \quad (14)$$

$$= \int_{\mathbf{R}^m} (\cos(c \cdot x) - 1) \int_{\mathbf{M}_m^+} \varphi_m(x; \Sigma) V(d\Sigma) dx. \quad (15)$$

The result then follows from (10), (12), (14) and the uniqueness of the Lévy measure.

R e m a r k 3.1. a) In general, when the support of V includes non-negative definite matrices, x is still infinitely divisible with distribution function (9) and Lévy measure U given by

$$U(dx) = \int_{\mathbf{M}_m^+} \Phi_m(dx; \Sigma) V(d\Sigma). \quad (16)$$

b) Let x be an m -dimensional random vector having a G -representation $zS^{1/2}$. Then for any $n \times m$ matrix C the random vector $zS^{1/2}C^T$ has the G -representation $z_n S_{n,C}^{1/2}$, where $S_{n,C} = CSC^T$ is an infinitely divisible $n \times n$ nonnegative definite symmetric random matrix.

c) Using radial components in $\mathbf{M}_m$, we have that (16) can be written in the following useful representation

$$U(dx) = \int_{\mathbf{SM}_m^+} \int_0^\infty \Phi_m(dx; t\Theta) dt \Gamma(d\Theta), \quad (17)$$

where Γ is a finite measure on $\mathbf{SM}_m^+$.

d) Another useful representation for the Lévy density u , when V is concentrated on positive definite matrices, is the following. Let $\mathbf{DM}_m^+$ be the set of elements of $\mathbf{M}_m^+$ whose determinant is one. Then

$$u(x) = \int_{\mathbf{DM}_m^+} \int_0^\infty \varphi_m(x; t\Theta) dt \Delta(d\Theta), \quad (18)$$

where Δ is a finite measure on $\mathbf{DM}_m^+$.

e) A random vector $x = (x_1, \dots, x_m)$ having a G -representation $zS^{1/2}$ has independent components if and only if the measure Γ in (17) is concentrated on $\{D_1, \dots, D_m\}$, where $D_i = \text{diag}(0, \dots, 1, \dots, 0)$ is the diagonal matrix with zero diagonal elements apart for the i -th element which is equal to 1. This fact follows using (17) and exercise E12.10 in [35].

Finally, we give necessary and sufficient conditions for selfdecomposability of a mult G random vector, extending the one-dimensional result by Rosinski [30, Corollary 3].

Corollary 3.1. *Let x be a mult G random vector. Then x is selfdecomposable if and only if the function*

$$k_\omega(r) = r^m \int_{\mathbf{M}_m^+} \varphi_m(r\omega; \Sigma) V(d\Sigma) \quad (19)$$

is decreasing in $r > 0$, for each ω in the unit sphere $\mathbf{S}^m$ in $\mathbf{R}^m$.

P r o o f. From (11) and using radial components in $\mathbf{R}^m$ we have

$$U(A) = \int_{\mathbf{S}^m} \lambda(d\omega) \int_0^\infty r^{m-1} 1_A(r\omega) \int_{\mathbf{M}_m^+} \varphi_m(r\omega; \Sigma) V(d\Sigma) dr,$$

where $\lambda(d\omega)$ is a finite measure on $\mathbf{S}^m$ the unit sphere in $\mathbf{R}^m$. From Theorem 15.10 in [35], it follows that x has a selfdecomposable distribution if and only if the function $k_\omega(r)$ is decreasing in $r > 0$.

Furthermore, as in the one-dimensional case (see [36]), we now prove that selfdecomposability of S implies selfdecomposability of x . Moreover, for each $0 < \rho < 1$, x_ρ in the representation $x = \rho x + x_\rho$ is mult G .

Proposition 3.2. *Let x be a mult G random vector with G -representation $zS^{1/2}$. If S is selfdecomposable then x is selfdecomposable. Moreover, for each $0 < \rho < 1$, x_ρ in the representation $x = \rho x + x_\rho$ is mult G .*

P r o o f. Let $0 < \rho < 1$. From Proposition 2.3, $S \stackrel{\text{law}}{=} \rho S + S_\rho$, where S_ρ is a nonnegative (infinitely divisible) random matrix independent of S . Then, from (10)

$$\begin{aligned} C\{\zeta \dagger x\} &= \overline{K} \left\{ \frac{1}{2} \zeta^T \zeta \dagger S \right\} = \overline{K} \left\{ \frac{1}{2} \zeta^T \zeta \dagger \rho S \right\} + \overline{K} \left\{ \frac{1}{2} \zeta^T \zeta \dagger S_\rho \right\} \\ &= C\{\zeta \dagger \rho x\} + C\{\zeta \dagger x_\rho\}, \end{aligned} \quad (20)$$

where $x_\rho = z_\rho S_\rho^{1/2}$ and z_ρ is a standard normal vector independent of (z, S, S_ρ) . The result follows.

4. Examples of mult G laws

In this section we present some important examples of mult G random vectors.

Example 4.1. *Multivariate symmetric α -stable distributions.* The multivariate symmetric α -stable laws have a G -representation, as we now show.

Let S be an $\alpha/2$ -stable nonnegative definite symmetric random matrix with cumulant function (3) and take $x = zS^{1/2}$ for z a standard normal vector in $\mathbf{R}^m$ independent of S . Then, by Proposition 3.1, x is a mult G random vector and, by (10) and (3), we find that x is a multivariate symmetric α -stable random vector such that for every $\zeta \in \mathbf{R}^m$

$$C\{\zeta \dagger x\} = \overline{K}\left\{\frac{1}{2}\zeta^T\zeta \dagger S\right\} = -\frac{1}{2^{\alpha/2}} \int_{\mathbf{SM}_m^+} [\zeta\Theta\zeta^T]^{\alpha/2} \Gamma(d\Theta). \quad (21)$$

In particular, when S is of the form sI (i.e., Γ is concentrated on $\{m^{-1/2}I\}$), where s an $\alpha/2$ -stable positive variable, we obtain the multivariate sub-Gaussian law as presented in [32, Section 2.5].

We observe that (21) gives a new parametrization (see also [23, p. 124–125]) of the multivariate symmetric α -stable distributions in terms of the measure Γ on $\mathbf{SM}_m^+$, as an alternative to the usual representation

$$C\{\zeta \dagger x\} = -\frac{1}{2^{\alpha/2}} \int_{\mathbf{S}^m} |\zeta \cdot \theta|^\alpha \sigma(d\theta),$$

where σ is a symmetric measure on the unit sphere $\mathbf{S}^m$ of $\mathbf{R}^m$ (see [32, Theorem 2.4.3]).

Trivially, multivariate stable laws are selfdecomposable.

Example 4.2. *Multivariate extension of the symmetric Normal Inverse Gaussian law*²⁾. A matrix extension of the Inverse Gaussian distribution was considered in Section 2. Using Proposition 3.1 we have that if z is a standard normal vector in $\mathbf{R}^m$ independent of R , then $x = zR^{1/2}$ has a multivariate type mult G distribution with cumulant function

$$C\{\zeta \dagger x\} = \int_{\mathbf{SM}_m^+} [\text{tr}(\Sigma\Theta)]^{\alpha/2} \Gamma(d\Theta) - \int_{\mathbf{SM}_m^+} \left[\text{tr} \left(\left(\Sigma + \frac{1}{2} \zeta^T \zeta \right) \Theta \right) \right]^{\alpha/2} \Gamma(d\Theta) \quad (22)$$

and Lévy density

$$u(x; \Sigma, \Gamma) = \int_{\mathbf{SM}_m^+} \int_0^\infty \varphi_m(x; r\Theta) \frac{e^{-r\text{tr}(\Sigma\Theta)}}{r^{1+\alpha/2}} dr \Gamma(d\Theta). \quad (23)$$

The resulting law constitutes a generalization of the symmetric Normal Inverse Gaussian distribution to dimensions greater than one.

We observe that for a real τ

$$\begin{aligned} C\{\tau \dagger c \cdot x\} &= \int_{\mathbf{SM}_m^+} [\text{tr}(\Sigma\Theta)]^{\alpha/2} \Gamma(d\Theta) \\ &\quad - \int_{\mathbf{SM}_m^+} \left[\text{tr} \left(\left(\Sigma + \frac{1}{2} \tau^2 c^T c \right) \Theta \right) \right]^{\alpha/2} \Gamma(d\Theta). \end{aligned} \quad (24)$$

²⁾ For information on the one-dimensional normal inverse Gaussian distributions see for instance, [8], [9] and references therein.

In particular, taking Γ concentrated on the matrix $m^{-1/2}\mathbf{I}$ (i.e., $\Gamma(\{m^{-1/2}\mathbf{I}\}) = \delta m^{-1/4}$), $\Sigma = \gamma^2\mathbf{I}$ and $\alpha = 1$ we obtain

$$C\{\tau \dagger c \cdot x\} = \delta\gamma \left[1 - \left\{1 + \frac{1}{2}\gamma^{-2}cc^T\tau^2\right\}^{1/2}\right], \tag{25}$$

which shows that in this case the one-dimensional marginal distributions are Normal Inverse Gaussian (in other words, $x \stackrel{\text{law}}{=} zs^{1/2}$, where s has a one-dimensional Inverse Gaussian distribution independent of the standard normal variate z).

Finally, from Example 2.5 and Proposition 3.2 we have that the multi-variate Normal Inverse Gaussian law is selfdecomposable.

5. extG laws

We proceed to consider a weakened version of multG.

Definition 5.1. We shall say that a random vector $x = (x_1, \dots, x_m)$ and its distribution are of type extG if for any deterministic vector $c = (c_1, \dots, c_m)$ the law of $c \cdot x$ is of type G .

Every type multG variate x is necessarily of type extG. In fact, with S as the infinitely divisible nonnegative definite random matrix entering the G -representation of x we have that conditionally on S the law of $c \cdot x$ is $N(0, cSc^T)$ and writing $s_c = cSc^T$ and noting that the infinite divisibility of S implies infinite divisibility of s_c we obtain that $c \cdot x$ is extG with a G -representation $z\sqrt{s_c}$.

Let $\text{ext}G_m$ denote the family of m -dimensional extG laws and let $\text{ext}G_\infty = \bigcup_1^\infty \text{ext}G_m$.

The following proposition summarizes elementary properties of type extG laws.

Proposition 5.1. a) *Linear transformations of extG distributions are again type extG.*

b) *Let $x \in \text{ext}G_m$ and $y \in \text{ext}G_n$ and suppose that x and y are independent. Then $(x, y) \in \text{ext}G_{m+n}$.*

c) *Sums of independent extG variates are type extG.*

d) *The class $\text{ext}G_\infty$ is closed under convergence in law.*

P r o o f. The proofs of a) and b) are trivial.

c) It suffices to consider the sum of just two variates, x and y , of class $\text{ext}G_m$. We must show that $c \cdot (x + y) = c \cdot x + c \cdot y$ is of class $\text{ext}G_1$ and one sees that it is enough to prove that if x and y are one-dimensional with independent G -representations $z_1\sqrt{s_1}$ and $z_2\sqrt{s_2}$, respectively, then $x + y$ is type extG. But

$$x + y \stackrel{\text{law}}{=} z_1\sqrt{s_1} + z_2\sqrt{s_2} \stackrel{\text{law}}{=} z(s_1 + s_2)^{1/2}$$

with z a normal standard random variable independent of (s_1, s_2) .

d) Consider a sequence of vectors x_n of class $\text{ext}G_m$, converging in law to a random vector x , and for any $c \in \mathbf{R}^m$ let $z\sqrt{s_n(\zeta)}$ be a G -representation of $e(\zeta) \cdot x$, where $e(\zeta)$ denotes the unit vector in the direction of ζ . (We let $e(\zeta) = 0$ if $\zeta = 0$.) Then, by assumption, $\chi\{\zeta \dagger x_n\} \rightarrow \chi\{\zeta \dagger x\}$ for all $\zeta \in \mathbf{R}^m$ and on the other hand we have

$$C\{\zeta \dagger x_n\} = \overline{K} \left\{ \frac{1}{2} |\zeta|^2 \dagger s_n(\zeta) \right\}.$$

It follows that, given ζ , $\overline{K}\{\theta \dagger s_n(\zeta)\}$ converges for all $\theta \geq 0$ to some function $\kappa(\theta) = \kappa(\theta; \zeta)$, where $\kappa(0) = C\{0 \dagger x\} = 0$. Hence κ is necessarily the cumulant function of a nonnegative random variable $s(\zeta)$, i.e., $\kappa(\theta) = \overline{K}\{\theta \dagger s(\zeta)\}$, which implies

$$C\{\zeta \dagger x\} = \overline{K} \left\{ \frac{1}{2} |\zeta|^2 \dagger s(\zeta) \right\},$$

i.e., x is of type $\text{ext}G$.

Note that if a multivariate distribution D is of type $\text{ext}G$ and if for all $c \in \mathbf{R}^d$ the Lévy measure of s_c in the G -representation $z\sqrt{s_c}$ of $c \cdot x$ is known, then insight into the nature of D can be obtained by simulating the one-dimensional projected laws of D using the techniques discussed in the surveys given by Asmussen [1, Section VIII.2] and Rosinski [31].

Examples of $\text{ext}G$ laws are provided in the next section.

It is an open problem whether the $\text{ext}G$ property of a random vector x implies that x is infinitely divisible. (We recall in this connection the result due to Ibragimov [17] mentioned in Section 2.)

6. Marginal infinite divisibility and $\text{ext}G$

We now discuss the relevance of the concept of marginal infinite divisibility of positive definite infinitely divisible matrices to that of type $\text{ext}G$.

Proposition 6.1. *A multivariate random variable x is of type $\text{ext}G$ if it has the representation $x \stackrel{\text{law}}{=} zS^{1/2}$, where S is a marginal infinitely divisible random matrix with values in $\mathbf{M}_m^+$ and z is a standard normal vector in $\mathbf{R}^m$ independent of S .*

P r o o f. Since cSc^T is an infinitely divisible positive random variable, via conditioning on S we find

$$\begin{aligned} \chi\{\zeta \dagger x \cdot c\} &= \chi\{\zeta \dagger zS^{1/2}c^T\} = \mathbf{E} \exp \left\{ -\frac{1}{2} \zeta^2 cSc^T \right\} \\ &= \exp \left\{ \overline{K} \left\{ \frac{1}{2} \zeta^2 \dagger cSc^T \right\} \right\} \end{aligned}$$

which shows that $c \cdot x \stackrel{\text{law}}{=} w(cSc^T)^{1/2}$, for w a standard normal variable independent of S , and hence x is of type $\text{ext}G$.

Remark 6.1. The same line of argument shows that if S is marginally selfdecomposable then so is x .

Example 6.1. Wishart case. Let $S = W_n$ with W_n a Wishart matrix, as in Example 2.1. Then, by Proposition 6.1, $x = zW_n^{1/2}$ is of type $\text{ext}G$.

The cumulant function of W_n is

$$\overline{K}\{\Theta \nmid W_n\} = -\frac{n}{2} \ln \det(I_m + 2\Theta)$$

and hence for the cumulant function of x we find

$$C\{\zeta \nmid x\} = \overline{K}\left\{\frac{1}{2} \zeta^T \zeta \nmid W_n\right\} = -\frac{n}{2} \ln \det(I_m + \zeta^T \zeta) = -\frac{n}{2} \ln(1 + |\zeta|^2). \quad (26)$$

For $n = 2$ this is the cumulant function of the bivariate Laplace distribution.

Since (26) depends on ζ through $|\zeta|^2$ only and since the one-dimensional Laplace distribution is of type G , with a G -representation $z\sqrt{s}$, where s follows the generalized inverse Gaussian law $\text{GIG}(1, 0, 1)$ (cf. [2]), we find that x possesses a G -representation $zS^{1/2}$ with $S = sI_m$.

Example 6.2. Inverse Wishart case (multivariate t distribution). It follows from Example 2.2 that if W_n is Wishart then $x = zW_n^{-1/2}$ is of type $\text{ext}G$. The law of x is, in fact, that of the multivariate t distribution (as defined, for instance, in [18, p. 144]). Note that, by the remarks at the end of the previous example, x has a G -representation of the form $zS^{1/2}$ with $S = sI$.

Example 6.3. Let (w_1, w_2) be a two-dimensional Gaussian random vector with zero mean and covariance matrix Σ . It has been shown in [39] that the bivariate Gamma distribution of (w_1^2, w_2^2) is infinitely divisible. Let s be a positive $\frac{1}{2}$ -stable random variable and let $y = s^{1/2}(w_1^2, w_2^2)$. Let $y_1, \dots, y_n$ be n independent copies of y . Then $S = y_1^T y_1 + \dots + y_n^T y_n$ is a marginal infinitely divisible nonnegative definite random matrix. Indeed, it is enough to show that $cy^T yc^T = (c \cdot y)^2$ is infinitely divisible. Observing that $(c \cdot y)^2 = s(c_1 w_1^2 + c_2 w_2^2)^2 = sq^2$, where q is positive and infinitely divisible it follows, by a result of Barndorff-Nielsen, Pedersen and Sato [5], that the product sq^2 is infinitely divisible.

In its general form, the probability density function of the bivariate Gamma law is

$$\frac{(\psi_1 \psi_2 - 1)^\lambda}{2^{\lambda+1} \Gamma(\lambda)} I_{\lambda-1}(2\sqrt{x_1 x_2}) (x_1 x_2)^{-(\lambda-1)/2} e^{-\psi_1 x_1 - \psi_2 x_2},$$

where $I_{\lambda-1}$ is a Bessel function. The exponential form of this implies that if x is defined as $zS^{1/2}$ with S as above then the characteristic function of $c \cdot x$ does not depend on c solely through $|c|^2$ (in contrast to what was the case in Examples 6.1 and 6.2).

7. $\text{ext}G$ stochastic processes

An n -dimensional stochastic process $x(t)$ will be said to be of type $\text{ext}G$ if all finite dimensional laws of $x(t)$ are type $\text{ext}G$, in other words if for any $0 < t_1 < \dots < t_r$ we have that $(x(t_1), \dots, x(t_r)) \in \text{ext}G_r$.

7.1. Relations to Lévy processes. Multivariate subordinated processes are studied systematically in [5]. Let $\{T(t)\}_{t \geq 0}$ be a d -dimensional subordinator, i.e., T is a Lévy process with values in $\mathbf{R}_{0+}^d$ (where $\mathbf{R}_{0+} = [0, \infty)$). Furthermore, let $\{X_1(t)\}_{t \geq 0}, \dots, \{X_d(t)\}_{t \geq 0}$ be independent Lévy processes, of dimension $n_1, \dots, n_d$ respectively, and consider the d -dimensional process $\{X(t)\}_{t \geq 0}$ given by $X(t) = (X_1(t), \dots, X_d(t))$. Then

$$Y(t) = (X_1(T_1(t)), \dots, X_d(T_d(t)))$$

constitutes a Lévy process of dimension $n_1 + \dots + n_d$.

If $X_i(t)$ is n_i -dimensional Brownian motion, for $i = 1, \dots, d$, then the multivariate subordinated process $Y(t)$ is of type $\text{ext}G$. In fact, for each t , $Y(t)$ has a G -representation, as is easily checked, and hence $Y(t)$ is $\text{mult}G$.

As mentioned above, it is an open problem whether the type $\text{ext}G$ property of a random vector x implies that x is infinitely divisible. Suppose, however, that x is infinitely divisible, as well as $\text{ext}G$. Then there exists a Lévy process $Y(t)$ such that $Y(1)$ and x are identical in law, and the process $Y(t)$ is of type $\text{ext}G$. To show this, let $z\sqrt{s_c}$ be the G -representation of $c \cdot x$ and let $T_c(t)$ be the Lévy process generated by s_c , i.e., the Lévy process such that $T(1)$ has the same law as s_c . Consider first the one-dimensional case. For any $t \in \mathbf{R}_+$ we have, taking the process T to be independent of z ,

$$\begin{aligned} C\{\zeta \sharp Y(t)\} &= tC\{\zeta \sharp Y(1)\} = tC\{\zeta \sharp x\} = t\bar{K} \left\{ \frac{1}{2} \sharp s_{\zeta^2} \right\} \\ &= \bar{K} \left\{ \frac{1}{2} \sharp T_{\zeta^2}(t) \right\} = C\left\{ \zeta \sharp z \sqrt{T_{\zeta^2}(t)} \right\} \end{aligned}$$

showing that $Y(t)$ has G -representation $z\sqrt{T_{\zeta^2}(t)}$. The argument is extended in a straightforward manner to verify that $(Y(t_1), \dots, Y(t_r))$ is $\text{ext}G_r$ for any $t_1 < \dots < t_r$.

Other relations to Lévy processes are considered in [26].

7.2. Stationary processes of type $\text{ext}G$. In this section we consider a class of strictly stationary processes constructed in [6], showing that they are of type $\text{ext}G$.

Let F be an arbitrary one-dimensional distribution function and suppose (v, w) is a bivariate stochastic process of the form $y(F(\cdot))$, where $y(t) = (y_1(t), y_2(t))$ is a bivariate Lévy process such that $y(1)$ has a G -representation $(z_1, z_2)\sqrt{s}$, where s is one-dimensional. Then a process $x = \{x(t)\}_{t \in \mathbf{R}}$

is well defined by

$$x(t) = \int_{-\infty}^{\infty} \cos(\lambda t) \, v(d\lambda) + \int_{-\infty}^{\infty} \sin(\lambda t) \, w(d\lambda), \tag{27}$$

where the integrals are interpreted in the sense of integration with respect to independently scattered random measures. It was proved in [6] that the process x is strictly stationary and infinitely divisible and that for $t_1 < \cdots < t_m$

$$C\{(c_1, \dots, c_m) \ddagger (x(t_1), \dots, x(t_m))\} = \int_{-\infty}^{\infty} \overline{K} \left\{ \frac{1}{2} \psi(c, \lambda) \ddagger s \right\} F(d\lambda), \tag{28}$$

where $c = (c_1, \dots, c_m)$, $\psi(c, \lambda) = c\Omega_\lambda(t_1, \dots, t_m)c^T$ and

$$\Omega_\lambda(t_1, \dots, t_m) = \{\cos(\lambda(t_j - t_k)); \, i, j = 1, \dots, m\}$$

is a symmetric nonnegative definite matrix.

Moreover, for any one-dimensional distribution D of type G there exists a strictly stationary process x of the above kind having D as the one-dimensional marginal law.

We now show that the finite dimensional distributions of any such process are of type $\text{ext}G$. Let $x = (x(t_1), \dots, x(t_m))$, then by (28)

$$C\{\zeta \ddagger c \cdot x\} = \int_{-\infty}^{\infty} \overline{K} \left\{ \frac{1}{2} \zeta^2 \ddagger \psi(c, \lambda)s \right\} F(d\lambda) \tag{29}$$

and the result follows by Proposition 5.1.

Since $c \cdot x$ in (29) is type G it has a G -representation $z\sqrt{s_c}$. An expression for the Lévy measure V_c of s_c in terms of the Lévy measure V of s is obtained by noting that, writing θ for $\frac{1}{2}\zeta^2$,

$$\overline{K} \{ \theta \ddagger s \} = - \int_0^\infty (1 - e^{-\theta \xi}) \, V(d\xi)$$

which together with (29) yields

$$\begin{aligned} \overline{K} \{ \theta \ddagger s_c \} &= \int_{-\infty}^{\infty} \overline{K} \{ \psi(c, \lambda) \theta \ddagger s \} F(d\lambda) \\ &= - \int_{-\infty}^{\infty} \int_0^\infty (1 - e^{-\theta \psi(c, \lambda) \xi}) \, V(d\xi) F(d\lambda) \\ &= - \int_0^\infty (1 - e^{-\theta \xi}) \int_{-\infty}^{\infty} V([\psi(c, \lambda)]^{-1} d\xi) F(d\lambda) \end{aligned}$$

from which it follows that

$$V_c(d\xi) = \int_{-\infty}^{\infty} V([\psi(c, \lambda)]^{-1} d\xi) F(d\lambda). \tag{30}$$

By the usual Lamperti transformation, given a strictly stationary process x , one can construct an H -selfsimilar process $x^*(t) = t^H x(\ln t)$. Then if x is given by (27), the selfsimilar process x^* is itself of type $\text{ext}G$. For a suitable choice of the distribution function F , it was shown in [6] that the corresponding selfsimilar process x^* has second order stationary increments.

7.3. A general class and its relation to type G processes. We now present a class of examples that includes the type G processes studied by Rosinski [30] (see also references therein), showing that they are as well type ext G processes in our sense.

Let ν be a σ -finite measure on the space of nonnegative definite matrices $\mathbf{M}_m^+$ and let s be an infinitely divisible positive random variable with cumulant function $\bar{K}\{\cdot \dagger s\}$ and Lévy measure V . Suppose that there exists a random vector x in $\mathbf{R}^m$ whose cumulant transform is given by

$$C\{\zeta \dagger x\} = \int_{\mathbf{M}_m^+} \bar{K} \left\{ \frac{1}{2} \zeta \Omega \zeta^T \dagger s \right\} \nu(d\Omega). \quad (31)$$

Note that the expression (21) for the cumulant function of the multivariate symmetric α -stable law is a special case of (31).

We may rewrite (31) as

$$C\{\zeta \dagger x\} = \int_{\mathbf{M}_m^+} \bar{K} \left\{ \frac{1}{2} \zeta^T \zeta \dagger s \Omega \right\} \nu(d\Omega)$$

and, by the infinite divisibility of s , there must exist an infinitely divisible nonnegative definite random matrix S such that

$$\bar{K} \left\{ \frac{1}{2} \zeta^T \zeta \dagger S \right\} = \int_{\mathbf{M}_m^+} \bar{K} \left\{ \frac{1}{2} \zeta^T \zeta \dagger s \Omega \right\} \nu(d\Omega).$$

It follows that x has a G -representation $zS^{1/2}$ and is therefore of type ext G (cf. Proposition 2.1).

Furthermore, the Lévy density of x is given by

$$u(x) = \int_{\mathbf{M}_m^+} \int_0^\infty \varphi_m(x; \xi \Omega) V(d\xi) \nu(d\Omega).$$

In the case where ν is concentrated on matrices of the form $y^T y$, ν may be thought of as a measure on $\mathbf{R}^m$ and we then have

$$C\{\zeta \dagger x\} = \int_{\mathbf{R}^m} \bar{K} \left\{ \frac{1}{2} (\zeta y^T)^2 \dagger s \right\} \nu(dy), \quad (32)$$

which is the definition of a type G multivariate distribution as given by Rosinski [30]. The corresponding Lévy density of x is

$$u(x) = \int_{\mathbf{R}^m} \int_0^\infty \prod_{j=1}^m \varphi(x_j; \xi y_j^2) V(d\xi) \nu(dy).$$

8. Concluding remarks

It would be of some interest to extend part of the above discussion to concepts that may be denoted type G^+ , mult G^+ and ext G^+ , where, in particular, saying that a random variable x is of type G^+ means that x has structure

$$x \stackrel{\text{law}}{=} \beta s + z\sqrt{s}$$

with β an arbitrary constant, in other words the law of x is a normal variance-mean mixture. A recent interesting result on such random variables, due to Sato [36], states that if s is selfdecomposable then so is x . The study of these extended concepts would relate to stochastic processes of the form

$$Y(t) = \mu t + \beta S(t) + b(t)[S(t)]^{1/2},$$

where μt is a drift term, $b(t)$ is m -dimensional Brownian motion and $S(t)$ is a process of positive definite random $m \times m$ matrices. Processes of this kind occur, in particular, in financial econometrics, as multivariate stochastic volatility models; see [9], [10]. In these papers the authors considered in particular the case in which the process $S(t)$ has the factor structure

$$S(t) = \text{diag} [\tau_1(t), \dots, \tau_m(t)] + \phi^T \phi \tau_0(t),$$

where $\tau_0, \tau_1, \dots, \tau_m$ are positive independent processes of Ornstein-Uhlenbeck type and ϕ is a parameter vector.

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