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SHARP MEAN-VARIANCE INEQUALITIES FOR QUANTILES OF DISTRIBUTIONS DETERMINED BY CONVEX AND STAR ORDERS¹⁾

Устанавливаются точные верхние границы отклонения квантилей от среднего, выраженного в единицах стандартного отклонения, для всех распределений, связанных с данным выпуклым и звездным отношениями порядка, и описываются распределения, на которых эти границы достигаются. Полученные результаты конкретизируются для семейств распределений: с монотонной плотностью, с монотонной интенсивностью отказа, с монотонной плотностью в среднем, с монотонной интенсивностью отказа в среднем.

Ключевые слова и фразы: квантиль, выпуклый порядок, звездный порядок, монотонная плотность (в среднем), монотонная интенсивность отказа (в среднем), проекция, выпуклый конус.

1. Introduction. Let F denote a distribution function of a real random variable X with finite mean μ_F and variance σ_F^2 . Define the respective right continuous quantile function by

$$F^{-1}(x) = \sup\{y: F(y) \leq x\}, \quad 0 < x < 1.$$

The aim of this paper is to determine sharp upper bounds on standardized upper quantiles $[F^{-1}(p) - \mu_F]/\sigma_F$ for various orders $0 < p < 1$ and all F belonging to restricted families of distributions determined by two partial stochastic orders: the convex and star ones. Note that the standardization makes the problem nontrivial, because simple location and/or scale transformations can make quantiles arbitrarily large. The mean and standard deviation are the most natural location and scale parameters. We first recall the optimal bound due to Hoeffding [4] for the family of arbitrary distributions

$$\frac{F^{-1}(p) - \mu_F}{\sigma_F} \leq \sqrt{\frac{p}{1-p}}, \quad 0 < p < 1, \quad (1.1)$$

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which is attained by the two-point distribution

$$\mathbf{P}\left\{X = \mu_F - \sigma_F \sqrt{\frac{1-p}{p}}\right\} = p = 1 - \mathbf{P}\left\{X = \mu_F + \sigma_F \sqrt{\frac{p}{1-p}}\right\}. \quad (1.2)$$

Using the specific form of the Chebyshev inequality

$$\mathbf{P}\{|X - \mu_F| \geq F^{-1}(p) - \mu_F\} \leq \frac{\sigma_F^2}{[F^{-1}(p) - \mu_F]^2} \quad (1.3)$$

for symmetrically distributed X , we obtain

$$\frac{F^{-1}(p) - \mu_F}{\sigma_F} \leq \frac{1}{\sqrt{2(1-p)}}, \quad \frac{1}{2} < p < 1, \quad (1.4)$$

which becomes equality for

$$\mathbf{P}\left\{X = \mu_F \pm \sigma_F \sqrt{\frac{2}{1-p}}\right\} = 1 - p, \quad \mathbf{P}\{X = \mu_F\} = 2p - 1.$$

For $p < \frac{1}{2}$ we have $F^{-1}(p) \leq [F^{-1}(\frac{1}{2}-) + F^{-1}(\frac{1}{2})]/2 = \mu_F$. Since $F^{-1}(p)$ is the upper quantile, the bound jumps at $\frac{1}{2}$ from 0 to 1. Bounds (1.1) and (1.3) coincide for the upper median. By the Gauss inequality

$$\mathbf{P}\{|X - \mu_F| \geq a\} \leq \frac{4\sigma_F^2}{9a^2} \quad (1.5)$$

that refines the Chebyshev one for the unimodal distributions, we immediately get

$$\frac{F^{-1}(p) - \mu_F}{\sigma_F} \leq \frac{1}{3} \sqrt{\frac{2}{1-p}}, \quad \frac{1}{2} < p < 1, \quad (1.6)$$

for symmetric unimodal distributions. There are some improvements and generalizations of (1.5) due to Vysochanskii and Petunin [9], and Dharmadhikari and Joag-dev [1] (see also [2, Section 1.5]), but they do not provide natural evaluations of standardized quantiles.

Here we consider four types of restricted families of distributions which either precede or succeed a fixed distribution function W with a density w in either of convex and star orders. We also assume that all distributions under study are absolutely continuous which makes the orders reflexive, transitive and antisymmetric (cf. [2, Theorem 9.1, p. 217]). We say that F succeeds W in the convex order ($F \succeq_c W$) if the composition $F^{-1}W$ is convex on the support of W which is an interval $[a_W, d_W)$, say. We shall write the end-points simply a and d , when no confusion can arise. The star relation $F \succeq_* W$ is satisfied if $F^{-1}W$ is starshaped, i.e., $F^{-1}W(x)/(x - a_W)$ is nondecreasing on the support of W . Clearly $F \succeq_c W$ implies $F \succeq_* W$. The convex order is location-scale invariant. Usually the star order is defined for life distributions with $a_F = a_W = 0$ for which this is merely scale invariant. Our

definition makes the order invariant under arbitrary affine transformations of the argument.

Bounds on quantiles of distributions determined by the convex and star orders are presented in Sections 2 and 3, respectively. In both the cases, the results are specified for the classes of distributions with extreme element W being uniform and exponential, written as $U(x) = x$, $0 \leq x \leq 1$, and $V(x) = 1 - \exp(-x)$, $x \geq 0$, respectively. They have natural interpretations and numerous applications in statistical inference and reliability. Condition $F \succeq_c U$ implies that F has a decreasing density (DD), say f , on its support, and $F \succeq_* V$ holds if F is a decreasing failure rate (DFR) distribution function. The reversed relations characterize distributions with increasing density and failure rate (ID and IFR), respectively. Relations $F \succeq_* V$ and $F \preceq_* V$ define distributions with decreasing and increasing failure rates in average (DFRA and IFRA), respectively. Less popular notions are the decrease and increase of density in average (DDA and IDA, respectively). This means that the function $x \mapsto (x - a_F)^{-1} \int_{a_F}^x f(y) dy$, $x > a_F$, is either decreasing or increasing which is alternatively defined by relations $F \succeq_* U$ and $F \preceq_* U$, respectively. The relations describe a tendency of increase of probability with decrease and increase of arguments for possibly multimodal distributions. Here we adhere to a common convention of calling a function decreasing (increasing) if it is merely nonincreasing (non-decreasing, respectively). We also point out that some bounds are attained by distributions with atoms. However, it is possible to construct sequences of absolutely continuous distributions that attain the bounds in limit. The sequences have the quantile functions which approach the quantile functions of optimal discontinuous distributions in properly defined Hilbert spaces. This follows from the method of proving the inequalities which is based on projecting linear functionals onto convex cones in Hilbert spaces. For details we refer the reader to Section 4. Numerical comparisons of the best inequalities on quantiles for various classes of distributions and some concluding remarks will be presented in Section 5.

2. Distributions determined by convex order. Take a fixed distribution function W with a density w , finite mean μ_W and variance σ_W^2 . We assume that the support of W is an interval with a finite left end-point $a = a_W$ and possibly infinite right one $d = d_W$. For a W -distributed random variable X and arbitrary $\beta \in (a, d)$, define the expectation and variance of X under condition of exceeding level β by

$$\mu_W(\beta) = \mathbf{E}_W(X \mid X > \beta) = \frac{\int_{\beta}^d xw(x) dx}{1 - W(\beta)}, \quad (2.1)$$

$$\sigma_W^2(\beta) = \mathbf{D}_W(X \mid X > \beta) = \frac{\int_{\beta}^d x^2 w(x) dx - \mu_W^2(\beta)}{1 - W(\beta)}. \quad (2.2)$$

We also define a distance between p -th quantile and the conditional mean in respective standard deviation units

$$\delta_W(\beta) = \frac{W^{-1}(p) - \mu_W(\beta)}{\sigma_W(\beta)}. \quad (2.3)$$

The notation is used below for describing mean-variance bounds on quantiles of distributions succeeding W in the convex order.

Theorem 1. Assume that $F \succeq_c W$.

If $W^{-1}(p) \leq \mu_W$, then $F^{-1}(p) \leq \mu_F$.

If $\mu_W < W^{-1}(p) \leq \mu_W + \sigma_W^2/(\mu_W - a_W)$, then

$$\frac{F^{-1}(p) - \mu_F}{\sigma_F} \leq \frac{W^{-1}(p) - \mu_W}{\sigma_W}, \quad (2.4)$$

and the equality holds for the location-scale transformations of W , i.e., for

$$F_1(x) = W\left(\mu_W + \sigma_W \frac{x - \mu_F}{\sigma_F}\right). \quad (2.5)$$

Finally, if $W^{-1}(p) > \mu_W + \sigma_W^2/(\mu_W - a_W)$, then there exists a unique solution $\beta = \beta(p) \in (a_W, W^{-1}(p))$ to equation

$$[W^{-1}(p) - \mu_W(\beta)] [\mu_W(\beta) - \beta] = \sigma_W^2(\beta), \quad (2.6)$$

and

$$\frac{F^{-1}(p) - \mu_F}{\sigma_F} \leq A = A_1(p) = \left[\frac{\delta_W^2(\beta) + W(\beta)}{1 - W(\beta)} \right]^{1/2}. \quad (2.7)$$

The bound in (2.7) is attained by

$$F_1(x) = W\left(\beta + [\mu_W(\beta) - \beta] [1 - W(\beta)] \times \left(1 + A \frac{x - \mu_F}{\sigma_F}\right)\right) \mathbf{1}_{[\mu_F - \sigma_F/A, \infty)}(x). \quad (2.8)$$

Distribution function (2.8) has a jump of height $W(\beta) < p$ at the left support end-point, and shares the shape of W on the right. This is a life distribution for $\mu_F = \sigma_F/A$. Mean-variance bounds for $F \succeq_c U, V$ are presented in Propositions 1, 2, respectively.

Proposition 1. Let F have a decreasing density function.

If $0 < p \leq \frac{1}{2}$, then $F^{-1}(p) \leq \mu_F$.

If $\frac{1}{2} < p \leq \frac{2}{3}$, then $[F^{-1}(p) - \mu_F]/\sigma_F \leq 2\sqrt{3}(p - \frac{1}{2})$, which becomes equality for the uniform distribution on $[\mu_F - \sqrt{3}\sigma_F, \mu_F + \sqrt{3}\sigma_F]$.

If $\frac{2}{3} < p < 1$, then

$$\frac{F^{-1}(p) - \mu_F}{\sigma_F} \leq \frac{1}{3} \left(\frac{9p - 5}{1 - p} \right)^{1/2}.$$

This is an equality for the mixture of the Dirac distribution concentrated at $\mu_F - 3\sigma_F[(1 - p)/(9p - 5)]^{1/2}$ and the uniform one on

$$\left[\mu_F - 3\sigma_F \left(\frac{1 - p}{9p - 5} \right)^{1/2}, \mu_F + 3\sigma_F \frac{3p - 1}{[(1 - p)(9p - 5)]^{1/2}} \right]$$

with weights $3p - 2$ and $3(1 - p)$, respectively.

Proposition 2. *Let F have a decreasing failure rate.*

If $0 < p \leq 1 - e^{-1} \approx 0.63212$, then $F^{-1}(p) \leq \mu_F$.

If $1 - e^{-1} < p \leq 1 - e^{-2} \approx 0.86466$, then bound $[F^{-1}(p) - \mu_F]/\sigma_F \leq -\ln(1 - p) - 1$, is attained by the exponential distribution function with location μ_F and scale σ_F .

If $1 - e^{-2} < p < 1$, then for $q = (1 - p)e^2 < 1$, we get $[F^{-1}(p) - \mu_F]/\sigma_F \leq (2/q - 1)^{1/2}$. This is an equality if F is the combination of an atom at $\mu_F - \sigma_F[q/(2 - q)]^{1/2}$, and the exponential distribution with location $\mu_F[q(2 - q)]^{1/2} - \sigma_F q$ and scale σ_F , with respective probabilities $1 - q$ and q .

We easily check that bounds presented in Propositions 1 and 2 are continuous in p , and tend to infinity if $p \nearrow 1$. The same conclusions hold for the general case. Note that the three bounds of Theorem 1 can be replaced by the single one

$$\frac{F^{-1}(p) - \mu_F}{\sigma_F} \leq \left[\frac{(\max\{\delta_W(\beta), 0\})^2 + W(\beta)}{1 - W(\beta)} \right]^{1/2}$$

with $\beta = -\infty$ and a_W for $b = W^{-1}(p) = \mu_W$ and $\mu_W + \sigma_W^2/(\mu_W - a_W)$, respectively. If $b = W^{-1}(p) > \mu_W$, then $\beta(W^{-1}(p))$ is strictly increasing. Indeed, using (2.6), we obtain

$$b(\beta) = \frac{\mathbf{E}_W(X(X - \beta) \mid X > \beta)}{\mathbf{E}_W(X - \beta \mid X > \beta)}, \tag{2.9}$$

$$b'(\beta) = \frac{\mathbf{D}_W(X \mid X > \beta)}{\mathbf{E}_W^2(X - \beta \mid X > \beta)} > 0. \tag{2.10}$$

Therefore $b(\beta)$ is strictly increasing, and so is its inverse. Moreover, $b(\beta) \nearrow d_W$ as $\beta \nearrow d_W$, and so does the inverse which implies that (2.7) becomes arbitrarily large.

In description of bounds for distributions determined by the reversed convex order relation, we use moments of right truncated W -distributed variables (cf. (2.1)–(2.3)):

$$\bar{\mu}_W(\gamma) = \mathbf{E}_W \min\{X, \gamma\}, \tag{2.11}$$

$$\bar{\sigma}_W^2(\gamma) = \mathbf{D}_W \min\{X, \gamma\}, \tag{2.12}$$

$$\bar{\delta}_W(\gamma) = \frac{W^{-1}(p) - \bar{\mu}_W(\gamma)}{\bar{\sigma}_W(\gamma)}. \tag{2.13}$$

Theorem 2. Assume that $F \preceq_c W$. Then for arbitrary $0 < p < 1$ and $b = W^{-1}(p)$, we have

$$\frac{F^{-1}(p) - \mu_F}{\sigma_F} \leq \bar{\delta}_W(b), \quad (2.14)$$

and the equality in (2.14) holds for

$$F_2(x) = \begin{cases} W\left(\bar{\mu}_W(b) + \bar{\sigma}_W(b) \frac{x - \mu_F}{\sigma_F}\right), & \text{if } \frac{x - \mu_F}{\sigma_F} < \bar{\delta}_W(b), \\ 1, & \text{if } \frac{x - \mu_F}{\sigma_F} \geq \bar{\delta}_W(b). \end{cases} \quad (2.15)$$

Formula (2.15) defines the distribution function of affinely transformed W -distributed random variable right truncated at its p -th quantile. This becomes a life distribution if $\mu_F = \sigma_F[\bar{\mu}_W(b) - a_W]/\bar{\sigma}_W(b)$. Bounds (2.14) continuously increase from 0 at $p = 0$ to $(d_W - \mu_W)/\sigma_W$ at $p = 1$, which is finite if W has a finite support. For large p , these are close to respective quantiles of parent W . However, they are positive for all quantiles in contrast to negative values of standardized quantiles of small orders for all $F \succeq_c W$ (including W itself).

Proposition 3. If F has an increasing density, then

$$\frac{F^{-1}(p) - \mu_F}{\sigma_F} \leq \left[\frac{3p}{4 - 3p} \right]^{1/2}, \quad 0 < p < 1. \quad (2.16)$$

Inequality (2.16) becomes equality for a mixture of the uniform distribution on the interval $[\mu_F - \sigma_F\sqrt{3}(2-p)[p(4-3p)]^{-1/2}, \mu_F + \sigma_F(3p)^{1/2}(4-3p)^{-1/2}]$ and the degenerate measure concentrated at $\mu_F + \sigma_F(3p)^{1/2}(4-3p)^{-1/2}$ with probabilities p and $1-p$, respectively.

Proposition 4. If F has an increasing failure rate, then

$$\frac{F^{-1}(p) - \mu_F}{\sigma_F} \leq \frac{-\ln(1-p) - p}{[p(2-p) + 2(1-p)\ln(1-p)]^{1/2}}, \quad 0 < p < 1. \quad (2.17)$$

Equality is attained by the distribution of $\mu_F + \sigma_F \min\{X, -\ln(1-p) - p\}[p(2-p) + 2(1-p)\ln(1-p)]^{-1/2}$ for a standard exponentially distributed random variable X .

3. Distributions determined by star order. For $b = W^{-1}(p)$, write

$$\eta_W(b) = b - a - \int_b^d (x - a) w(x) dx = b - pa - (1-p)\mu_W(b), \quad (3.1)$$

$$\begin{aligned} \vartheta_W^2(b) &= \int_b^d (x - a)^2 w(x) dx - \left[\int_b^d (x - a) w(x) dx \right]^2 \\ &= (1-p)\sigma_W^2(b) - p(1-p)a[2\mu_W(b) - a] \end{aligned} \quad (3.2)$$

(cf. (2.1), (2.2)). Note that (3.1) is strictly increasing from $\eta_W(a_W) = a_W - \mu_W < 0$ to $\eta_W(d_W) = d_W - a_W > 0$, and (3.2) is positive for all $b \in [a_W, d_W]$.

Theorem 3. Let $F \succeq_* W$. If $\eta_W(b) \leq 0$, then $F^{-1}(p) \leq \mu_F$. Otherwise

$$\frac{F^{-1}(p) - \mu_F}{\sigma_F} \leq A = A_3(p) = \frac{\eta_W(b)}{\vartheta_W(b)}, \quad (3.3)$$

and bound (3.3) is attained by

$$F_3(x) = \begin{cases} 0, & \text{if } \frac{x - \mu_F}{\sigma_F} < -\frac{(1-p)\mu_W(b)}{\vartheta_W(b)}, \\ p, & \text{if } -\frac{(1-p)\mu_W(b)}{\vartheta_W(b)} \leq \frac{x - \mu_F}{\sigma_F} \leq A, \\ W\left(b - \eta_W(b) + \vartheta_W(b) \frac{x - \mu_F}{\sigma_F}\right), & \text{if } \frac{x - \mu_F}{\sigma_F} \geq A. \end{cases} \quad (3.4)$$

Obviously, the bounds of Theorem 3 dominate those of Theorem 1, because $F \succeq_c W$ implies $F \succeq_* W$. It follows in particular that (3.3) tends to ∞ as $p \nearrow 1$. The main difference between (2.8) and (3.4) is that the atom of the latter is distant from the smooth part and has a greater weight.

Proposition 5. Suppose that F has a decreasing density in average. Then

$$\frac{F^{-1}(p) - \mu_F}{\sigma_F} \leq \frac{\max\{\sqrt{3}(p^2 + 2p - 1), 0\}}{(1 + 6p^2 - 4p^3 - 3p^4)^{1/2}}, \quad 0 < p < 1. \quad (3.5)$$

If $p > \sqrt{2} - 1$, then positive bound in (3.5) is attained by the combination of an atom at $\mu_F - \sigma_F \sqrt{3}(1 - p^2)(1 + 6p^2 - 4p^3 - 3p^4)^{-1/2}$ with probability p , and the uniform distribution on $[\mu_F + \sigma_F \sqrt{3}(p^2 + 2p - 1)(1 + 6p^2 - 4p^3 - 3p^4)^{-1/2}, \mu_F + \sigma_F \sqrt{3}(1 + p^2)(1 + 6p^2 - 4p^3 - 3p^4)^{-1/2}]$ with probability $1 - p$.

Proposition 6. If F has a decreasing failure rate in average, then

$$\frac{F^{-1}(p) - \mu_F}{\sigma_F} \leq \frac{\max\{p[1 - \ln(1 - p)] - 1, 0\}}{\{(1 - p)p[1 - \ln(1 - p)]^2 + 1 - p\}^{1/2}}, \quad 0 < p < 1. \quad (3.6)$$

Let $p_0 \approx 0.553567$ be the unique zero of the strictly increasing function in the numerator of the right-hand side of (3.6). If $p > p_0$, then the equality in (3.6) is attained by the combination of jump distribution at $\mu_F - \sigma_F(1 - p)[1 - \ln(1 - p)]\{(1 - p)p[1 - \ln(1 - p)]^2 + 1 - p\}^{-1/2}$ and the exponential distribution with location $\mu_F + \sigma_F(p[1 - \ln(1 - p)] - 1)\{(1 - p)p[1 - \ln(1 - p)]^2 + 1 - p\}^{-1/2}$ and scale $\sigma_F(1 - p)\{(1 - p)p[1 - \ln(1 - p)]^2 + 1 - p\}^{-1/2}$, with probabilities p and $1 - p$, respectively.

Theorem 4 asserts that general bound (1.1) cannot be improved in classes of distributions defined by $F \preceq_* W$. Moreover, we conclude analogous results for all F preceding a fixed W in partial order more general than the star one, e.g., superadditive and Laplace transform orders (see, e.g., [8]).

Theorem 4. If density function $w(x)$ is bounded on a neighborhood of a_W , then for arbitrary $0 < p < 1$ there exist sequences of absolutely

continuous distribution functions $F_k \preceq_* W$ with arbitrary common mean μ and variance σ^2 such that

$$\lim_{k \rightarrow \infty} \frac{F_k^{-1}(p) - \mu}{\sigma} = \sqrt{\frac{p}{1-p}}.$$

In particular, the claim holds for distributions with increasing density and failure rate in average. Postponing the proof to Section 4, we present here some informal arguments. To fix the ideas, we consider the approximation of two-point distribution (1.2) by $F_k \preceq_* U$, i.e., ones with nondecreasing $F_k(x)/(x - a_k)$ on respective supports $[a_k, d_k)$. Letting $a_k \rightarrow -\infty$, we relax conditions on the increase rate of F_k itself. Accordingly, such F_k can be starshaped and approximate well any two-point distribution at and besides jump points. The formal proof of Theorem 4 is constructive. It is also possible to provide special constructions of sequences satisfying the statement for specific W with unbounded densities about respective left support ends.

4. Proofs. A common approach to proving Theorems 1–4 is based on projecting linear functionals onto convex cones in Hilbert spaces. The method was formulated and applied in [3] for determining sharp moment bounds on expectations of order statistics of dependent samples with identical marginal distributions which have a decreasing density and/or a decreasing failure rate. We briefly summarize main ideas here. If a family of distributions can be represented as a convex cone $\mathcal{C}$ in a real Hilbert space $\mathcal{H}$ (e.g., by means of centered quantile functions $F^{-1} - \mu_F \in L^2([0, 1), dx)$), and a statistical functional can be represented as a continuous linear functional $T(F^{-1} - \mu_F) = (F^{-1} - \mu_F, h)$, $h \in \mathcal{H}$, then the best upper bound on standardized functional $T(F^{-1} - \mu_F)/\sigma_F$ (note that $\sigma_F = \|F^{-1} - \mu_F\|$) is the norm $\|Ph\|$ of the projection Ph of functional h onto cone $\mathcal{C}$. Moreover, the bound is attained by $F^{-1} - \mu_F = \sigma_F Ph/\|Ph\| \in \mathcal{C}$, which allows one to identify the distribution function providing the extreme value of the standardized functional. Evaluating quantiles, we take into account limits of sequences of continuous linear functionals

$$\begin{aligned} T_p(F^{-1} - \mu_F) &= \lim_{q \searrow p} T_{pq}(F^{-1} - \mu_F) \\ &= \lim_{q \searrow p} \int_0^1 [F^{-1}(x) - \mu_F] \left[\frac{1}{q-p} \mathbf{1}_{[p,q)}(x) - 1 \right] dx \\ &= \lim_{q \searrow p} (F^{-1} - \mu_F, h_{pq} - 1) \end{aligned} \quad (4.1)$$

with $h_{pq} = (q-p)^{-1} \mathbf{1}_{[p,q)}$. We here subtract 1 for technical reasons, and this does not change the value of the functional, because $F^{-1} - \mu_F$ integrates to 0. Since we defined right continuous versions of quantile functions, the functional values tend monotonously to $F^{-1}(p) - \mu_F$. Studying distributions related to given W in convex and star orders, it is convenient to change

variables and analyze compositions $F^{-1}W - \mu_F \in L^2([a_W, d_W], w(x) dx)$, which belong to the cones

$$\mathcal{C}_1 = \{g: g(x) \text{ — nondecreasing, convex}\}, \quad (4.2)$$

$$\mathcal{C}_2 = \{g: g(x) \text{ — nondecreasing, concave}\}, \quad (4.3)$$

$$\mathcal{C}_3 = \left\{g: g(x), \frac{g(x) - g(a_W)}{x - a_W} \text{ — nondecreasing}\right\}, \quad (4.4)$$

$$\mathcal{C}_4 = \left\{g: g(x) \text{ — nondecreasing, } \frac{g(x) - g(a_W)}{x - a_W} \text{ — nonincreasing}\right\}, \quad (4.5)$$

if $F \succeq_c W$, $F \preceq_c W$, $F \succeq_* W$, and $F \preceq_* W$, respectively. The respective modification of (4.1) has the form

$$\begin{aligned} T_b(F^{-1}W - \mu_F) &= \lim_{c \searrow b} T_{bc}(F^{-1}W - \mu_F) \\ &= \lim_{c \searrow b} \int_a^d [F^{-1}W(x) - \mu_F][h_{pq}(x) - 1] w(x) dx, \end{aligned} \quad (4.6)$$

where

$$h_{bc}(x) = \frac{1}{W(c) - W(b)} \mathbf{1}_{[b,c)}(x), \quad b = W^{-1}(p), \quad c = W^{-1}(q). \quad (4.7)$$

Formally, condition $(g, \mathbf{1}) = \int_a^d g(x) w(x) dx = 0$ should be added to definitions (4.2)–(4.5) in order to fully characterize all possible $F^{-1}W - \mu_F$. However, since all the cones $\mathcal{C}_i$ are translation invariant, respective projections P_i preserve values of weighted integrals of arguments (cf. [7, Lemma 1]), and so

$$\int_a^d P_i(h_{bc}(x) - 1) w(x) dx = \int_a^d [h_{bc}(x) - 1] w(x) dx = 0, \quad i = 1, 2, 3, 4. \quad (4.8)$$

Therefore we do not need to bother about the integral condition, and confine ourselves to projecting onto (4.2)–(4.5) which have natural graphical interpretations. Moreover, translation invariance of all $\mathcal{C}_i$ implies that

$$P_i(h_{bc} - 1) = P_i h_{bc} - 1,$$

and so it suffices to project h_{bc} and subtract 1 from the results. We also have $\|P_i h_{bc} - 1\|^2 = \|P_i h_{bc}\|^2 - 1$. We shall observe that in contrast to original functionals h_{bc} , respective projections $P_i h_{bc}$, $c \searrow b$, have proper limits $P_i h_b \in \mathcal{C}_i \subset L^2([a_W, d_W], w(x) dx)$.

Our procedure of determining the projections consists in choosing an arbitrary starting point in the cone and constructing consecutive approximations improving the previous ones. We first try to describe the shape of projection function up to several parameters using geometric arguments. Then we determine optimal parameters analytically. First we tackle the problem of projecting (4.7) onto (4.2).

Lemma 1. *If*

$$\int_a^d xw(x) dx \geq \frac{\int_b^c xw(x) dx}{W(c) - W(b)}, \quad (4.9)$$

then $P_1 h_{bc} = \mathbf{1}$.

Otherwise there exist a unique $\beta < b$ that solves

$$\frac{\int_{\max\{\beta, a\}}^d (x - \beta)^2 w(x) dx}{\int_{\max\{\beta, a\}}^d (x - \beta) w(x) dx} = \frac{\int_b^c (x - \beta) w(x) dx}{\int_b^c w(x) dx} \quad (4.10)$$

and

$$\alpha = \alpha(\beta) = \left[\int_{\max\{\beta, a\}}^d (x - \beta) w(x) dx \right]^{-1} > 0 \quad (4.11)$$

such that

$$P_1 h_{bc}(x) = \alpha(x - \beta)_+ = \frac{(x - \beta)_+}{\int_{\max\{\beta, a\}}^d (y - \beta) w(y) dy}. \quad (4.12)$$

Precisely, if relation $\ll \geq \gg$ holds in (4.10) with β replaced by a_W , then $\beta \leq a_W$, and (4.12) is linear on $[a_W, d_W]$. In the opposite case, $\beta > a_W$ and the projection is actually a two-piece broken line.

P r o o f. We first show that for every $g \in \mathcal{C}_1$ there exists

$$g_{\alpha\beta}(x) = \alpha(x - \beta)_+, \quad \alpha \geq 0, \beta < b, \quad (4.13)$$

with $g_{\alpha\beta}(b) = \alpha(b - \beta) < h_{bc}(b) = [W(c) - W(b)]^{-1}$ such that $\|h - g_{\alpha\beta}\| \leq \|h - g\|$. The two-parametric class consists of constants, increasing linear functions and broken lines with two pieces: horizontal and increasing ones.

Note first that if $g(a_W) < 0$, then $\max\{g, 0\} \in \mathcal{C}_1$ lies closer to h_{bc} than original g . If $g(a) \geq 0$ and $g(b) \geq h_{bc}(b)$, then $g \geq h_{bc}$ and $(g, \mathbf{1}) > (h_{bc}, \mathbf{1})$. Since $\|g + C - h_{bc}\|$ is minimized in the class of translations $g + C$ of a given function g by the real C satisfying $(g + C, \mathbf{1}) = (h_{bc}, \mathbf{1})$ (cf. (4.8)), we improve approximation of h_{bc} by adding a negative constant and truncating at level 0. The modifications lead us to functions satisfying $0 \leq g(a) \leq g(b) < h_{bc}(b)$. Function $g = \mathbf{0}$ can be replaced by a positive constant. A correction of a nonnegative nonzero function $g \in \mathcal{C}_1$ that vanishes at c consists in setting 0 in $[c, d)$ and adding a positive constant. Therefore we can further assume that $g(c) > 0$ and define $\gamma = \inf\{x \in (b, c]: g(x) \geq h_{bc}(x)\}$. We easily check that function $\max\{[g(\gamma) - g(b)/(\gamma - b)](x - b) + g(b), 0\}$ lies closer to h_{bc} than g , and can be alternatively parametrized as (4.13) with respective restrictions on parameters.

Now we aim at minimizing

$$D_1(\alpha, \beta) = \|g_{\alpha\beta} - h_{bc}\|^2 = \alpha^2 \int_{\max\{\beta, a\}}^d (x - \beta)^2 w(x) dx - \frac{2\alpha}{W(c) - W(b)} \int_b^c (x - \beta) w(x) dx + \frac{1}{W(c) - W(b)} \quad (4.14)$$

with respect to α and β . For fixed $\beta < b$, we easily find optimal

$$\alpha = \alpha(\beta) = \frac{\int_b^c (x - \beta) w(x) dx}{[W(c) - W(b)] \int_{\max\{\beta, a\}}^d (x - \beta)^2 w(x) dx} > 0, \quad (4.15)$$

which plugged in (4.14) yields

$$D_1(\alpha(\beta), \beta) = \frac{1}{W(c) - W(b)} - \frac{[\int_b^c (x - \beta) w(x) dx]^2}{[W(c) - W(b)]^2 \int_{\max\{\beta, a\}}^d (x - \beta)^2 w(x) dx}. \quad (4.16)$$

By differentiation,

$$\begin{aligned} \frac{dD(\alpha_*(\beta), \beta)}{d\beta} &= \frac{2 \int_b^c (x - \beta) w(x) dx}{[W(c) - W(b)]^2 \int_{\max\{\beta, a\}}^d (x - \beta)^2 w(x) dx} \\ &\times \left\{ [W(c) - W(b)] \int_{\max\{\beta, a\}}^d (x - \beta)^2 w(x) dx - \int_{\max\{\beta, a\}}^d (x - \beta) w(x) dx \int_b^c (x - \beta) w(x) dx \right\}. \end{aligned}$$

Since the factor in the first line is positive, the sign of the derivative is identical with that of the last two which will be denoted by $K(\beta)$. Observe that

$$\begin{aligned} \frac{K(b)}{[1 - W(b)][W(c) - W(b)]} &= \mathbf{E}_W((X - b)^2 \mid X > b) \\ &\quad - \mathbf{E}_W(X - b \mid X > b) \mathbf{E}_W(X - b \mid b < X < c) \\ &\geq \mathbf{D}_W(X \mid X > b) > 0 \end{aligned} \quad (4.17)$$

for a W -distributed random variable X . Furthermore,

$$\begin{aligned} K'(\beta) &= [1 - W(\max\{\beta, a\})] \int_b^c (x - \beta) w(x) dx \\ &\quad - [W(c) - W(b)] \int_{\max\{\beta, a\}}^d (x - \beta) w(x) dx \end{aligned}$$

evaluated at $\beta = b$ satisfies

$$\begin{aligned} &\frac{K'(b)}{[1 - W(b)][W(c) - W(b)]} \\ &= \mathbf{E}_W(X \mid b < X < c) - \mathbf{E}_W(X \mid X > b) < 0. \end{aligned} \quad (4.18)$$

Finally,

$$K''(\beta) = -w(\beta) \int_b^c (x - \beta) w(x) dx \leq 0 \quad (4.19)$$

under the convention that $w(\beta) = 0$ for $\beta < a_W$. By (4.19) and (4.18), $K'(\beta)$ is constant for $\beta < a$ and nonincreasing for $a \leq \beta < b$ to $K'(b) < 0$.

If $K'(a_W) \leq 0$, i.e., (4.9) holds, then $K'(\beta) \leq 0$ and $K(\beta)$ is nonincreasing and, by (4.17), positive for all $\beta \leq b$. Therefore (4.16) is minimized at $\beta = -\infty$, which implies that the projection is a constant. This amounts to $(h_{bc}, \mathbf{1}) = 1$, which proves the first assertion.

If $K'(a_W) > 0$, then $K(\beta)$ changes its sign once from minus to plus at some $\beta < b$, at which (4.16) is minimized. Equations (4.10) and $K(\beta) = 0$ are equivalent, and allow us to rewrite (4.15) as (4.11). Note that $K(a_W) \geq 0$ implies that the solution to (4.10) satisfies $\beta \leq a$ and linearity of projection. In the opposite case a broken line is obtained. This proves the final claim of Lemma 1.

P r o o f o f T h e o r e m 1. Assume that $b = W^{-1}(p) > \mu_W$. Then for arbitrary $c = W^{-1}(q) > b$ condition (4.9) does not hold, and, by (4.10), there exists $\beta_c < b$ such that

$$\sigma_W^2(\beta_c) = [\mu_W(b, c) - \mu_W(\beta_c)][\mu_W(\beta_c) - \beta_c] \quad (4.20)$$

with $\mu_W(b, c) = \mathbf{E}_W(X|b < X < c)$. By (4.12) and (4.20), the squared mean-variance bound on T_{bc} can be written as

$$\begin{aligned} A_1^2(b, c) &= \|P_1 h_{bc}\|^2 - 1 = \frac{\int_{\max\{\beta_c, a\}}^d (x - \beta_c)^2 w(x) dx}{[\int_{\max\{\beta_c, a\}}^d (x - \beta_c) w(x) dx]^2} \\ &= \frac{\sigma_W^2(\beta_c)}{[1 - W(\beta_c)][\mu_W(\beta_c) - \beta_c]^2} + \frac{W(\beta_c)}{1 - W(\beta_c)} \\ &= \frac{[\mu_W(b, c) - \mu_W(\beta_c)]^2}{[1 - W(\beta_c)]\sigma_W^2(\beta_c)} + \frac{W(\beta_c)}{1 - W(\beta_c)}. \end{aligned} \quad (4.21)$$

Let now $q \searrow p$. Writing

$$\mu_W(b, c) = \frac{\mathbf{E}_W(X(X - \beta_c) | X > \beta_c)}{\mathbf{E}_W(X - \beta_c | X > \beta_c)}$$

(cf. (4.10)), and referring to (2.9), (2.10), we observe that $c \searrow b$ implies $\beta_c \searrow \beta$ defined in (4.10), $[\mu_W(b, c) - \mu_W(\beta_c)]/\sigma_W(\beta_c) \searrow \delta_W(\beta)$, and (4.21) decreases to $A_1^2(p)$, which provides the bound for p -th quantile. Observe that for p satisfying $\mu_W < W^{-1}(p) \leq \mu_W + \sigma_W^2/(\mu_W - a_W)$, we have $\beta \leq a_W$ with $W(\beta) = 0$, and then $A_1(p) = \delta_W(a_W) = [W^{-1}(p) - \mu_W]/\sigma_W$. If $W^{-1}(p) \searrow \mu_W$, then $A_1(p) \searrow 0$ which, by monotonicity of $A_1(p)$, gives the bound in the first case.

It is obvious that equality in (2.4) is attained by (2.5). Since (2.8) is strictly increasing above its quantile of order $W(\beta) < p$, it suffices to evaluate it at $\mu_F + \sigma_F A_1(p)$. By the definition, and (2.7) and (2.6), we have

$$\begin{aligned} F_1(\mu_F + \sigma_F A_1(p)) &= W\left(\beta + [\mu_W(\beta) - \beta][\delta_W^2(\beta) + 1]\right) \\ &= W\left(\beta + [\mu_W(\beta) - \beta]\left[\frac{W^{-1}(p) - \mu_W(\beta)}{\mu_W(\beta) - \beta} + 1\right]\right) = p, \end{aligned}$$

which ends the proof.

Lemma 2. *Projection of*

$$h_{bc} = \frac{\mathbf{1}_{[b,c]}}{W(c) - W(b)} \in L^2([a_W, d_W], w(x) \, dx)$$

onto (4.3) has the form

$$P_2 h_{bc}(x) = \alpha(\gamma) [\min\{x, \gamma\} - \beta(\gamma)] + 1 \tag{4.22}$$

with

$$\alpha(\gamma) = \frac{[W(c) - W(b)]^{-1} \int_b^c \min\{x, \gamma\} w(x) \, dx - \beta(\gamma)}{\int_a^d \min^2\{x, \gamma\} w(x) \, dx - \beta^2(\gamma)}, \tag{4.23}$$

$$\beta(\gamma) = \int_a^d \min\{x, \gamma\} w(x) \, dx, \tag{4.24}$$

and some $\gamma \in [b, c]$.

P r o o f. Observe first that

$$\begin{aligned} \beta(\gamma) &= \mathbf{E}_W \min\{X, \gamma\} \in (a_W, \gamma), \\ \alpha(\gamma) &= \frac{\mathbf{E}_W(\min\{X, \gamma\} \mid X > b) - \mathbf{E}_W \min\{X, \gamma\}}{\mathbf{D}_W \min\{X, \gamma\}} > 0, \end{aligned}$$

and so actually $P_2 h_{bc} \in \mathcal{C}_2$. Now we prove that for any $g \in \mathcal{C}_2$ there exists an element of $\mathcal{C}_2$ of form (4.22) that is less distant from h_{bc} than the original g . We start with eliminating all $g \in \mathcal{C}_2$ satisfying $g(c) \leq 0$. Obviously, $g \leq \max\{g, 0\} \leq h_{bc} \neq \max\{g, 0\}$. The first two relations show that $\max\{g, 0\}$ provides a better approximation. The latter two imply that this can be further improved by adding a constant $C > 0$ such that $(\max\{g, 0\} + C, 1) = (h_{bc}, 1)$. We can repeat the truncation and translation operations several times until we eventually obtain a function positive at c .

Then we assume that $g(c) > 0$, and separately study three cases: $g(b) \leq 0$, $g(a) \leq 0 < g(b)$, and $g(a) > 0$. In the first one, we construct the line passing through $(b, 0)$ and tangent to the graph of g in $(b, c]$, and truncate it at level $g(c)$. The resulting function has the desired form, runs above g and below h_{bc} in $[a, c)$, and the reversed relations hold in $[c, d)$. In the second case, we take the linear function secant to g at $\beta \in [a, b)$ satisfying $g(\beta) = 0$

and b . The truncation defined above provides the conclusions of the former case. Finally, if $g(a) > 0$, then it suffices to replace point $(\beta, 0)$ by $(a, 0)$ in the construction of the second case. Determination of optimal parameters (4.23)–(4.24) for fixed $\gamma \in [b, c]$ is a simple matter, because the squared distance

$$D_2(\alpha, \beta, \gamma) = \int_{aw}^{dw} [\alpha(\min\{x, \gamma\} - \beta) + 1 - h_{bc}(x)]^2 w(x) dx$$

is a convex quadratic function in both α and β . This completes the proof.

P r o o f o f T h e o r e m 2. We do not need to calculate precisely parameter γ in Lemma 2, because we merely look for $P_2 h_b = \lim_{c \searrow b} P_2 h_{bc}$. Note that (4.23) and (4.24) tend to

$$\alpha(b) = \frac{b - \beta(b)}{\int_a^d \min^2\{x, b\} w(x) dx - \beta^2(b)} = \frac{W^{-1}(p) - \bar{\mu}_W(b)}{\bar{\sigma}_W^2(b)}, \quad (4.25)$$

and $\beta(b) = \bar{\mu}_W(b)$, respectively, uniformly in all $\gamma \in [b, c]$, and therefore $P_2 h_{bc}$ tend to

$$P_2 h_b(x) = \alpha(b) [\min\{x, b\} - \beta(b)] + 1 \quad (4.26)$$

in the L^2 -norm. Referring to the projection method, we establish (2.14) by calculating

$$A_2^2(p) = \|P_2 h_b - 1\|^2 = \alpha^2(b) \bar{\sigma}_W^2(b) = \bar{\delta}_W^2(b) \quad (4.27)$$

(cf. (2.11)–(2.13) and (4.25)–(4.26)). Writing

$$\frac{F^{-1}W(x) - \mu_F}{\sigma_F} = \frac{P_2 h_b(x) - 1}{A_2(p)} = \frac{\min\{x, b\} - \bar{\mu}_W(b)}{\bar{\sigma}_W(b)}$$

(cf. (4.25)–(4.27)), we easily determine distribution function (2.15) that has p -th quantile at $A_2(p)$. Theorem 2 is proved.

Lemma 3. *If*

$$\frac{\int_b^c (x - a) w(x) dx}{W(c) - W(b)} \leq \int_b^d (x - a) w(x) dx, \quad (4.28)$$

then $P_3 h_{bc} = 1$ is the projection of h_{bc} onto convex cone (4.4).

Otherwise

$$P_3 h_{bc}(x) = \beta + \alpha(x - a) \mathbf{1}_{[b, d)}(x) \quad (4.29)$$

for

$$\alpha = \frac{\frac{\int_b^c (x - a) w(x) dx}{W(c) - W(b)} - \int_b^d (x - a) w(x) dx}{\int_b^d (x - a)^2 w(x) dx - [\int_b^d (x - a) w(x) dx]^2}, \quad (4.30)$$

$$\beta = \frac{\int_b^d (x - a)^2 w(x) dx - \frac{\int_b^c (x - a) w(x) dx}{W(c) - W(b)} \int_b^d (x - a) w(x) dx}{\int_b^d (x - a)^2 w(x) dx - [\int_b^d (x - a) w(x) dx]^2}. \quad (4.31)$$

P r o o f. First notice that the denominator of (4.30) is positive by the Schwarz inequality with $\int_b^d w(x) dx < 1$, and so is the numerator if (4.28) does not hold. Therefore (4.29) is actually a nondecreasing starshaped function.

We claim that the projection has actually parametric form (4.29) with $0 = h_{bc}(a) \leq \beta < h_{bc}(b) = 1/[W(c) - W(b)]$ and $\alpha \geq 0$, which admits constant functions as well. The first steps of reasoning are similar to those of Lemma 1. If $g \in \mathcal{C}_3$, so does $\max\{g, 0\}$. Moreover, the latter lies closer to h_{bc} . If $g(c) \leq 0$, then zero function better approximates h_{bc} , and can be further improved by constant $1 = (h_{bc}, 1)$. If $g(b) \geq h_{bc}(b) = \max h_{bc}$, then a downward translation of g gives a better approximation. Therefore we can confine ourselves to functions $g \in \mathcal{C}_3$ such that $0 \leq g(a) \leq g(b) < h_{bc}(b)$ and $g(c) > 0$. Let $\gamma \in (b, c]$ be the smallest point at which g exceeds h_{bc} . Put

$$g_\gamma(x) = g(a) + \frac{g(\gamma) - g(a)}{\gamma - a} (x - a) \mathbf{1}_{[b, d]}.$$

This is the closest to h_{bc} in $[a, b]$ among all nondecreasing starshaped functions starting from level $g(a) \geq 0$. Moreover, $g \in \mathcal{C}_3$ implies that $g \leq g_\gamma \leq h_{bc}$ in $[b, \gamma)$ and $g \geq g_\gamma \geq h_{bc}$ in $[\gamma, d]$. This gives the desired conclusion.

We now proceed to minimization of

$$\begin{aligned} D_3(\alpha, \beta) &= \int_a^d \left[\beta + \alpha(x - a) \mathbf{1}_{[b, d]}(x) - \frac{\mathbf{1}_{[b, c]}(x)}{W(c) - W(b)} \right]^2 w(x) dx \\ &= \beta^2 - 2\beta - \frac{1}{W(c) - W(b)} + \alpha^2 \int_b^d (x - a)^2 w(x) dx \\ &\quad - 2\alpha \left[\frac{\int_b^c (x - a) w(x) dx}{W(c) - W(b)} - \beta \int_b^d (x - a) w(x) dx \right] \end{aligned} \quad (4.32)$$

with respect to parameters $\alpha \geq 0$ and $0 \leq \beta < 1/[W(c) - W(b)]$. Fixing α , we minimize (4.32) at

$$\beta(\alpha) = 1 - \alpha \int_b^d (x - a) w(x) dx. \quad (4.33)$$

Plugging it into (4.32), we obtain

$$\begin{aligned} D_3(\alpha, \beta(\alpha)) &= \alpha^2 \left\{ \int_b^d (x - a)^2 w(x) dx - \left[\int_b^d (x - a) w(x) dx \right]^2 \right\} \\ &\quad - 2\alpha \left[\frac{\int_b^c (x - a) w(x) dx}{W(c) - W(b)} - \int_b^d (x - a) w(x) dx \right] \\ &\quad + \frac{1}{W(c) - W(b)} - 1. \end{aligned} \quad (4.34)$$

This is a quadratic function of α , with a positive coefficient of the quadratic term. Under nonnegativity condition, (4.34) is minimized at the maximum

of 0 and (4.30). If (4.28) holds, then the best $\alpha = 0$, and, by (4.33), $\beta = 1$. This proves the first statement. Otherwise (4.34) is minimized by (4.30), for which (4.33) coincides with (4.31).

Observe finally that (4.31) is positive, because its numerator, divided by $1 - W(b)$, can be interpreted as $\mathbf{E}_W((X - a)^2 | X > b) - \mathbf{E}_W(X - a | b < X < c)\mathbf{E}_W(X - a | X > b) \geq \mathbf{D}_W(X | X > b) > 0$. Moreover, by (4.33), $\beta < 1 < [W(c) - W(b)]^{-1}$, and so (4.31) satisfies both the constraints deduced from the geometric arguments presented in the first part of the proof. Lemma 3 is proved.

P r o o f o f T h e o r e m 3. Positivity of (3.1) contradicts (4.28) for any $c = W^{-1}(q) > b = W^{-1}(p)$. By arguments of the projection method, we get $\sup_{F \succeq_* W} [T_{bc}(F^{-1}W - \mu_F)/\sigma_F] = \alpha \vartheta_W(b)$, where $\alpha = \alpha(p, q)$ is defined in (4.30). Letting $q \searrow p$, we obtain (3.3). Distribution function (3.4) satisfies

$$\frac{F_3^{-1}W(x) - \mu_F}{\sigma_F} = \frac{(x - a) \mathbf{1}_{[b, d)}(x) - \int_b^d (x - a) w(x) dx}{\vartheta_W(b)},$$

which implies its starshapedness. Formula (3.4) immediately implies that $F_3^{-1}(p) = A_3(p)$. Since $A_3(p)$ continuously increases for arguments satisfying $\eta_W(W^{-1}(p)) > 0$, we cannot obtain a positive bound in the opposite case. Theorem 3 is proved.

P r o o f o f T h e o r e m 4. Due to (1.2), the general bound (1.1) is attained by the quantile functions equal to $\mu_F - \sigma_F \sqrt{(1-p)/p}$ for $x < p$ and $\mu_F + \sigma_F \sqrt{p/(1-p)}$ otherwise. Under change of variables, we have

$$\frac{F^{-1}W(x) - \mu_F}{\sigma_F} = g_p(x) = \begin{cases} -\sqrt{\frac{1-p}{p}}, & \text{if } a \leq x < b = W^{-1}(p), \\ \sqrt{\frac{p}{1-p}}, & \text{if } b \leq x < d. \end{cases} \quad (4.35)$$

Our proof consists in constructing a sequence $g_k \in \mathcal{C}_4$, $k \rightarrow \infty$, that converges to (4.35) in $L^2([a_W, d_W], w(x) dx)$. For sufficiently large k , we define piecewise linear continuous nondecreasing starshaped functions

$$\tilde{g}_k = \begin{cases} k^4(x - a) - k - \sqrt{\frac{1-p}{p}}, & \text{if } a \leq x \leq a + k^{-3}, \\ -\sqrt{\frac{1-p}{p}}, & \text{if } a + k^{-3} \leq x \leq b, \\ k \frac{x - a}{b - a} - k - \sqrt{\frac{1-p}{p}}, & \text{if } b \leq x \leq b + \frac{b - a}{k\sqrt{p(1-p)}}, \\ \sqrt{\frac{p}{1-p}}, & \text{if } b + \frac{b - a}{k\sqrt{p(1-p)}} \leq x < d. \end{cases}$$

Observe that

$$\begin{aligned} \|\tilde{g}_k - g_p\|^2 &= \int_a^{a+k^{-3}} [k^4(x-a) - k]^2 w(x) dx \\ &\quad + \int_b^{b+(b-a)/[k\sqrt{p(1-p)}]} \left[k \frac{x-a}{b-a} - k - \frac{1}{\sqrt{p(1-p)}} \right]^2 w(x) dx \\ &\leq \sup_{a \leq x \leq a+k^{-3}} \frac{w(x)}{3k} + \frac{W(b + (b-a)[k\sqrt{p(1-p)}]^{-1}) - W(b)}{p(1-p)} \longrightarrow 0 \end{aligned}$$

as $k \rightarrow \infty$. The same holds for $g_k = [\tilde{g}_k - (\tilde{g}_k, \mathbf{1})]/\|\tilde{g}_k - (\tilde{g}_k, \mathbf{1})\|$, because $(\tilde{g}_k, \mathbf{1}) \rightarrow (\tilde{g}_p, \mathbf{1}) = 0$, and $\|\tilde{g}_k - (\tilde{g}_k, \mathbf{1})\| \rightarrow \|g_p\| = 1$. Relations $[F_k^{-1}W(x) - \mu_F]/\sigma_F = g_k(x)$ define a sequence of $F_k \preceq_* W$ with common first two moments for which $F_k^{-1}(p) \nearrow \sqrt{p/(1-p)}$. Theorem 4 is proved.

5. Conclusions. Strictly speaking, most of the bounds presented above are not attained by absolutely continuous elements of the families determined by the convex and star orders. This is clear, because the convex cones defined by absolutely continuous distributions are not closed, and projections are usually border points of the cones. However, mimicking the proof of Theorem 4, we can choose inner points of the cones representing absolutely continuous elements of the other restricted classes of distributions that tend to the representant of the distribution function F with atoms which attains a bound. Note that convergence $F_k^{-1}W \rightarrow F^{-1}W$, $k \rightarrow \infty$, in $L^2([a_W, d_W], w(x) dx)$ is equivalent to that of original quantile functions in $L^2([0, 1], dx)$. Defining random variables $X = F^{-1}(U)$, $X_k = F_k^{-1}(U)$, for a random variable U uniformly distributed on $[0, 1]$, we see that $\lim_{k \rightarrow \infty} \mathbf{E}(X_k - X)^2 = 0$ is a sufficient condition on distributions of X_k for attaining bounds in limit.

The projection method enables us to determine sharp lower bounds for quantiles in the restricted families as well. It suffices to project negatives of functionals analyzed here onto convex cones (4.2)–(4.5). However, arguments leading to constructions of projections heavily depend on the form of function under study, and cannot be simply deduced from the above considerations. It is also possible to apply the method for finding bounds measured in scale units expressed by the square root of the second raw moment of the parent distribution. An interesting problem is to evaluate combinations of various quantiles, e.g., the interquartile distance.

In fact, general bound (1.1) can be obtained by projecting the quantile functional onto the convex cone of nondecreasing functions. This is the specific cone which allows general construction of projections for an arbitrary function. In the case of uniform weighting, the output is the derivative of the greatest convex minorant of the antiderivative of input (see [6]). For general weighting, some modifications are needed. We can also derive (1.4)

for symmetric distributions by projecting functionals onto the family of non-decreasing functions antisymmetric about $\frac{1}{2}$. The construction consists in combining the greatest convex minorant method with folding functionals about $\frac{1}{2}$. The latter combined with projections onto (4.2) allows us to determine optimal bounds for quantiles of symmetric unimodal distributions

$$\frac{F^{-1}(p) - \mu_F}{\sigma_F} \leq \begin{cases} 0, & \text{if } 0 < p \leq \frac{1}{2}, \\ 2\sqrt{3} \left(p - \frac{1}{2}\right), & \text{if } \frac{1}{2} \leq p \leq \frac{5}{6}, \\ \frac{1}{3} \sqrt{\frac{2}{1-p}}, & \text{if } \frac{5}{6} \leq p < 1. \end{cases} \quad (5.1)$$

The last bound, identical with (1.6) implied by the Gauss inequality, can be concluded from Lemma 1. The middle one, which improves the Gauss bound, is attained by the uniform distributions. We omit the detailed verification of (5.1) here.

Table 1. Sharp uniform mean-variance bounds on quantiles for various families of distributions

p	G	S	SUN	DD	DFR	DDA	DFRA	ID	IFR
0.05	0.22942	0	0	0	0	0	0	0.19739	0.19782
0.10	0.33333	0	0	0	0	0	0	0.28475	0.28601
0.15	0.42008	0	0	0	0	0	0	0.35603	0.35874
0.20	0.5	0	0	0	0	0	0	0.42008	0.42465
0.25	0.57735	0	0	0	0	0	0	0.48038	0.48741
0.30	0.65465	0	0	0	0	0	0	0.53882	0.54905
0.35	0.73380	0	0	0	0	0	0	0.59660	0.61097
0.40	0.81650	0	0	0	0	0	0	0.65465	0.67435
0.45	0.90453	0	0	0	0	0.13511	0	0.71375	0.74030
0.50	1	1	0	0	0	0.32163	0	0.77460	0.81005
0.55	1.10554	1.05409	0.17321	0.17321	0	0.50913	0	0.83793	0.88503
0.60	1.22474	1.11803	0.34641	0.34641	0	0.70235	0.15541	0.90453	0.96708
0.65	1.36277	1.19523	0.51962	0.51962	0.04982	0.90763	0.33725	0.97531	1.05869
0.70	1.52753	1.29099	0.69282	0.69389	0.20397	1.13406	0.53901	1.05131	1.16351
0.75	1.73205	1.41421	0.86603	0.88192	0.38629	1.39582	0.77040	1.13389	1.28721
0.80	2	1.58114	1.03923	1.10554	0.60944	1.71781	1.05060	1.22474	1.43941
0.85	2.38048	1.82574	1.21716	1.40106	0.89712	2.15063	1.42081	1.32613	1.63860
0.90	3	2.23607	1.49071	1.85592	1.30641	2.82311	1.98883	1.44115	1.92754
0.95	4.35890	3.16228	2.10819	2.80872	2.10081	4.24076	3.18532	1.57425	2.44875

In Table 1, we compare numerically bounds on standardized quantiles $[F^{-1}(p) - \mu_F]/\sigma_F$ of orders $p = 0.05(0.05)0.95$ for nine families of distributions: general (G), symmetric (S), and symmetric unimodal (SUN) ones, distributions with decreasing density (DD) and failure rate (DFR), and those in average (DDA and DFRA, respectively), and with increasing density (ID) and failure rate (IFR). Bounds for distributions with increasing density and failure rate in average are identical with the general ones. Certainly, the general bounds are merely valid for wider classes of life distributions, e.g., NBU,

HNBLUE and $\mathcal{L}$ -class defined in [5], which are generated by more general orders than the star order. On the other hand, there are still open interesting problems of accurate bounds on quantiles of distributions succeeding a given one in these general orders.

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